A NEW APPROACH TO (s,S) INVENTORY SYSTEMS

Jian-Qiang Hu*
Manufacturing Engineering
Boston University

Soracha Nananukul† and Wei-Bo Gong†
Electrical and Computer Engineering
University of Massachusetts, Amherst

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Abstract

In this paper, we consider period review (s, S) inventory systems with independent and identically distributed continuous demands and full backlogging. Using an approach recently proposed by Gong and Hu (1992), we derive an infinite system of linear equations for all moments of inventory level. Based on this infinite system, we develop two algorithms to calculate the moments of the inventory level. In the first one, we solve a finite system of linear equations whose solution converges to the moments as its dimension goes to infinity. In the second one, we in fact obtain the power series of the moments with respect to s and S. Both algorithms are based on some very simple recursive procedures. To show their efficiency and speed, we provide some numerical examples for the first algorithm.

(s, S) INVENTORY SYSTEMS; DYNAMIC RECURSIVE EQUATIONS; INFINITE LINEAR EQUATIONS; MACLAURIN SERIES

1 Introduction

Recently, Gong and Hu (1992) proposed a new approach to study single-server queueing systems and suggested that the approach might be applied to a large class of stochastic discrete event systems. Their basic idea is to obtain all moments of performance measures of interest using a set of infinite linear equations, which can often be derived based on dynamic recursive equations of sample paths of stochastic discrete event systems. In Gong and Hu (1992), infinite linear equations for moments of waiting time in a GI/G/1 is obtained based on the well-known Lindley recursive equation. They then showed that the moments can be easily obtained based on these linear equations. In fact, Gong and Hu (1992) provided a simple recursive procedure for calculating MacLaurin series of the moments of the waiting time in terms of a parameter in the service or interarrival time distribution. These results can be extremely useful in optimizing the

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[*] Postal Address: Department of Manufacturing Engineering, Boston University, 44 Cummington Street, MA 02215, USA

[†] Postal Address: Department of Electrical and Computer Engineering, University of Massachusetts, Amherst, MA 01003, USA.

performance measure with respect to the parameter of interest, since it is possible to calculate the whole response curve based on the MacLaurin series.

In this paper, we apply the approach of Gong and Hu (1992) to (s, S) inventory systems with periodic reviews. The (s, S) inventory systems have been investigated by many authors for nearly four decades since Arrow, Harris, and Marschak (1951) introduced the multi-stage periodic review inventory model. The system considered here has a single item in which the demands in all periods are independent and identically distributed (i.i.d.) and take on values from a continumum. The ordering policy in each period is specified by two numbers s and S $(0 \leq s < S)$ as follows. At the end of each period, the system is reviewed. If the inventory on hand plus that on order falls below s, a positive order is placed to bring the inventory to S. Otherwise, no action is taken. It has been shown that (s, S) policies are optimal for a class of systems; e.g., see Scarf (1960) and Iglehart (1963).

We consider the inventory position (the level of inventory on hand plus that in order). It can be used to evaluate several important measures of the system, e.g., the inventory level, the cost rate function, the fill rate, etc. The usual method of obtaining its distribution is through renewal theory (see Sahin 1990), in which the n-fold convolution of the probability distribution of demand has to be computed. We derive a different and simpler procedure to evaluate the inventory position. We notice that the evolution of the inventory position is governed by a simple recursive equation based on which we can derive an infinite system of linear equations for all the moments of the inventory position. This infinite system has many nice properties and enables us to obtain the moments of inventory position easily based on some simple recursive procedures. The main contribution of this paper is that it provides a novel method to solve a class of (s, S) inventory systems. As we shall see, the new method is simpler and more efficient compared with the existing methods. ΠEspecially, one of our two algorithms provides a possible way to calculate the whole response curve of the inventory position with respect to the parameters s and S since it gives the coefficients of the MacLaurin series with respect to $(S - s)$.

The rest of the paper is organized as follows. In Section 1, we first establish a fundamental system—an infinite system of linear equations for the moments of the inventory position. This system plays a key role in deriving two recursive procedures for calculating the moments in later sections. Our first algorithm is presented in Section 2. We show that the infinite system derived in Section 1 has a unique solution and we also obtain explicit formulas for the solution. A different way to calculate the moments is discussed in Section 3. We provide a simple recursive procedure to obtain the power series of all the moments in terms of s and S. Therefore, the whole response curves of the moments with respect to s and S might be obtained. In Section 4, we derive an analytical formula for the case in which the demand is exponentially distributed and show that our result is consistent with the previous results. Some numerical examples are provided in Section 5. Section 6 contains a conclusion.

2 A Fundamental System

Consider a period review (s, S) inventory system with i.i.d. demands. Let D_n be the demand in period n (i.i.d. for all n) and D be a generic demand in a period with distribution function $F(\cdot)$ and density function $f(\cdot)$. We shall assume that the Taylor series of $f(x)$ at $x = 0$ converges absolutely at $x = S - s$. This assumption is satisfied, say, for phase-type distribution. Let X_n be the inventory on hand plus on order at the beginning of period n. If the level is below s, an order is placed for an amount to bring the level up to S; otherwise, no order is placed. Then, the demand for the period is subtracted from X_n. Thus the recursive dynamic equation for X_n is given by

$$X_{n+1} = \begin{cases} X_n - D_n & \text{if } X_n \geq s \\ S - D_n & \text{if } X_n < s \end{cases}, \tag{1}$$

For ease of exposition, in the rest of this paper, we shall focus on the inventory position (the inventory level on hand plus on order). It should be pointed out that in many cases one may be interested in the inventory level; however, the latter can be easily calculated based on the former and the lead times of the orders. Of course, if the orders are delivered without delay, then the two are the same.

We assume the process $\{X_n\}$ is stable and ergodic, i.e., $X_n \xrightarrow{d} X$; however, this usually can be easily verified under some mild conditions by using the regenerative structure of the process; see Crane and Iglehart (1975) for details. To simplify notation and derivations in the rest of the paper, we introduce

$$\begin{aligned} Y_n &= X_n + D_{n-1} - s, \quad \text{where } D_{-1} = 0; \\ Y &= X + D - s; \\ q &= S - s. \end{aligned}$$

Then it follows from Equation (1) that

$$Y_{n+1} = \begin{cases} Y_n - D_{n-1} & \text{if } Y_n - D_{n-1} \geq 0 \\ q & \text{if } Y_n - D_{n-1} < 0 \end{cases}. \tag{2}$$

It is clear that Y_n and D_{n-1} are independent of each other and $0 \leq Y_n \leq q$ for all n. Letting $n \to \infty$ in Equation (2), we get

$$Y \overset{d}{=} \begin{cases} Y - D & \text{if } Y - D \geq 0 \\ q & \text{if } Y - D < 0 \end{cases}. \tag{3}$$

Note Y and D are independent of each other in the right-hand side of Equation (3) and $0 \leq Y \leq q$. From (3), it then follows that

$$\begin{aligned} E[Y^k] &= E[(Y-D)^k I(Y \geq D) + q^k I(Y < D)] \\ &= E[\int_0^Y (Y-x)^k f(x)dx + q^k(1-F(Y))] \\ &= E[\int_0^Y \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}(Y-x)^k x^j dx + q^k(1 - \int_0^Y \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j dx)] \\ &= E[\sum_{j=0}^{\infty} \frac{k! f^{(j)}(0)}{(k+j+1)!} Y^{k+j+1} + q^k(1 - \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{(j+1)!} Y^{j+1})] \\ &= q^k - q^k \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{(j+1)!} E[Y^{j+1}] + \sum_{j=0}^{\infty} \frac{k! f^{(j)}(0)}{(k+j+1)!} E[Y^{k+j+1}] \\ &\triangleq q^k + \sum_{j=0}^{\infty} (\alpha_{kj} E[Y^{k+j+1}] - \beta_j q^k E[Y^{j+1}]), \quad k = 1, 2, \cdots, \end{aligned} \tag{4}$$

where

$$\alpha_{kj} = \frac{k! f^{(j)}(0)}{(k+j+1)!} \quad \text{and} \quad \beta_j = \frac{f^{(j)}(0)}{(j+1)!}. \tag{5}$$

Since the power series $\sum_{j=0}^{\infty} \frac{|f^{(j)}(0)|}{j!} z^j$, $\sum_{j=0}^{\infty} |\beta_j| z^{j+1}$ and $\sum_{j=0}^{\infty} |\alpha_{kj}| z^{k+j+1}$ $(k = 1, 2, \ldots)$ all converge for any $z \in [0, q]$ and $0 \leq Y \leq q$, all the interchanges between different operations (summation, integration, and expectation) in the above derivation are guaranteed by the dominated convergence theorem. (4) is an infinite system of linear equations. As we shall see, it will play a key role in later sections.

3 Solutions for Moments of X

Writing (4) in matrix form, we have

$$\begin{bmatrix} 1 & -\alpha_{10}q & -\alpha_{11}q^2 & -\alpha_{12}q^3 & \ldots \\ 0 & 1 & -\alpha_{20}q & -\alpha_{21}q^2 & \ldots \\ 0 & 0 & 1 & -\alpha_{30}q & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} E[Y]/q \\ E[Y^2]/q^2 \\ E[Y^3]/q^3 \\ E[Y^4]/q^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1-u \\ 1-u \\ 1-u \\ 1-u \\ \vdots \end{bmatrix}, \tag{6}$$

where

$$u = \sum_{j=0}^{\infty} \beta_j E[Y^{j+1}] < \infty.$$

First we notice that $0 \leq 1-u \leq 1$ since $1-u = E[I(Y < D)] = \Pr(Y < D)$. On the other hand, if $1-u = 0$ we have $E[D] = 0$, which implies $D = 0$ with probability one because $D \geq 0$. This is a trivial case. So from now on we shall always assume $1-u > 0$.

Denoting $W_k = \frac{E[Y^k]}{q^k(1-u)}$ $(k = 1, 2, \ldots)$ and $v = \sum_{j=0}^{\infty} \beta_j W_{j+1} q^{j+1}$, we have

$$\begin{bmatrix} 1 & -\alpha_{10}q & -\alpha_{11}q^2 & -\alpha_{12}q^3 & \cdots \\ 0 & 1 & -\alpha_{20}q & -\alpha_{21}q^2 & \cdots \\ 0 & 0 & 1 & -\alpha_{30}q & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}. \tag{7}$$

Because $0 \leq E[Y^k]/q^k \leq 1$, $\{W_k; k = 1, 2, \ldots\}$ is bounded by $1/(1-u)$, therefore $v < \infty$. By the definition of W_k, we further have

$$(1-u)v = \sum_{j=0}^{\infty} \beta_j (1-u) W_{j+1} q^{j+1} = u.$$

Hence,

$$E[Y^k] = \frac{W_k q^k}{(1+v)} \tag{8}$$

We first want to show that (7) has a unique solution. We use the following result from Kantorovich and Krylov (1964).

Lemma 1. Consider the infinite system of linear equations

$$\begin{bmatrix} 1-c_{11} & -c_{12} & -c_{13} & \cdots \\ -c_{21} & 1-c_{22} & -c_{23} & \cdots \\ -c_{31} & -c_{32} & 1-c_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} b \\ b \\ b \\ \vdots \end{bmatrix},$$

where b is a real number, and c_{ij}'s are real numbers satisfying

$$\sum_{k=1}^{\infty} |c_{ik}| \leq \theta < 1, \quad (i = 1, 2, \ldots)$$

for some constant $\theta \in (0, 1)$. This system has one and only one bounded solution $\{z_i^*; i = 1, 2, \ldots\}$. Furthermore, if $\{z_i^N; i = 1, 2, \ldots, N\}$ is the solution of the finite system of linear equations

$$\begin{bmatrix} 1-c_{11} & -c_{12} & \cdots & -c_{1N} \\ -c_{21} & 1-c_{22} & \cdots & -c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{N1} & -c_{N2} & \cdots & 1-c_{NN} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix}$$

then $\{z_i^N; i = 1, 2, \ldots, N\}$ is uniformly bounded, i.e., $|z_i^N| < M$ for some constant M, and

$$z_i^* = \lim_{N \to \infty} z_i^N, \quad (i = 1, 2, \ldots, N).$$

Proof. See Kantorovich and Krylov (1964) pp. 26-31. ■

Theorem 1. The infinite system of linear equations

$$\begin{bmatrix} 1 & -\alpha_{10}q & -\alpha_{11}q^2 & -\alpha_{12}q^3 & \cdots \\ 0 & 1 & -\alpha_{20}q & -\alpha_{21}q^2 & \cdots \\ 0 & 0 & 1 & -\alpha_{30}q & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}, \tag{9}$$

where α_{kj} are defined in (5) and q is any nonnegative real number, has one and only one bounded solution $\{z_i^*; i = 1, 2, \ldots\}$. Furthermore, if $\{z_i^N; i = 1, 2, \ldots, N\}$ is the solution of the finite system

$$\begin{bmatrix} 1 & -\alpha_{10}q & \cdots & -\alpha_{1(N-2)}q^{N-1} \\ 0 & 1 & \cdots & -\alpha_{2(N-3)}q^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1^N \\ z_2^N \\ \vdots \\ z_N^N \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{10}$$

then $\{z_i^N; i = 1, 2, \ldots, N\}$ is uniformly bounded, and

$$\lim_{N\to\infty} z_k^N = z_k^* \text{ for all } k = 1, 2, \ldots, N.$$

Proof. First, we have

$$\begin{aligned} \sum_{j=0}^{\infty} |\alpha_{kj}q^{j+1}| &= \sum_{j=0}^{\infty} \left| \frac{k! f^{(j)}(0)}{(k+j+1)!} q^{j+1} \right| \\ &\le q \sum_{j=0}^{\infty} \frac{1}{k+j+1} \left| \frac{f^{(j)}(0)q^j}{j!} \right| \\ &\le \frac{qL}{k+1}, \end{aligned}$$

where $L = \sum_{j=0}^{\infty} \left| \frac{f^{(j)}(0)q^j}{j!} \right| < \infty$. Clearly, for any $q \ge 0$, there exists a positive integer K such that for all $k > K$

$$\sum_{j=0}^{\infty} |\alpha_{kj}q^{j+1}| \le \theta < 1,$$

where $\theta \in (0, 1)$. From Lemma 1, we know that the following system

$$\begin{bmatrix} 1 & -\alpha_{(K+1)0}q & -\alpha_{(K+1)1}q^2 & -\alpha_{(K+1)2}q^3 & \cdots \\ 0 & 1 & -\alpha_{(K+2)0}q & -\alpha_{(K+2)1}q^2 & \cdots \\ 0 & 0 & 1 & -\alpha_{(K+3)0}q & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} z_{K+1} \\ z_{K+2} \\ z_{K+3} \\ z_{K+4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

has one and only one bounded solution $\{z_i^*; i = K + 1, K + 2, \ldots\}$. This leads to the conclusion that

$$\begin{bmatrix} 1 & -\alpha_{10}q & \cdots & -\alpha_{1(K-2)}q^{K-1} \\ 0 & 1 & \cdots & -\alpha_{2(K-3)}q^{K-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_K \end{bmatrix} = \begin{bmatrix} 1 + \sum_{j=K+1}^{\infty} \alpha_{1j}q^{j+1}z_j^* \\ 1 + \sum_{j=K+1}^{\infty} \alpha_{2j}q^{j+1}z_j^* \\ \vdots \\ 1 + \sum_{j=K+1}^{\infty} \alpha_{Kj}q^{j+1}z_j^* \end{bmatrix} \tag{11}$$

has a unique bounded solution (note that all the series on the right-hand side converge). Therefore, the system of equations (9) has one and only one bounded solution.

Again, based on Lemma 1, we know that the solution $\{z_k^N; k = K+1, K+2, \ldots, N\}$ of the system of equations

$$\begin{bmatrix} 1 & -\alpha_{(K+1)0}q & \cdots & -\alpha_{(K+1)(N-K-2)}q^{N-K-1} \\ 0 & 1 & \cdots & -\alpha_{(K+2)(N-K-3)}q^{N-K-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_{K+1}^N \\ z_{K+2}^N \\ \vdots \\ z_N^N \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{12}$$

is uniformly bounded and

$$\lim_{N\to\infty} z_k^N = z_k^*, \quad (k = K+1, K+2, \ldots, N). \tag{13}$$

Consider the series on the right-hand side of (11). Since $\{z_k^N; k = K+1, K+2, \ldots, N\}$ is uniformly bounded and for each $k = 1, 2, \ldots, K$ $\sum_{j=K+1}^{\infty} \alpha_{kj} q^{j+1} z$ converges uniformly with respect to z in any bounded interval. Therefore, for each $k = 1, 2, \ldots, K$,

$$\sum_{j=K+1}^{\infty} \alpha_{kj} q^{j+1} z_j^* = \lim_{N\to\infty} \sum_{j=K+1}^{N} \alpha_{kj} q^{j+1} z_j^N.$$

(Note that $z_i^N = 0$ for all $i > N$.) This in term implies that if $\{z_k^N; k = 1, 2, \ldots, K\}$ is the solution of

$$\begin{bmatrix} 1 & -\alpha_{10}q & \cdots & -\alpha_{1(K-2)}q^{K-1} \\ 0 & 1 & \cdots & -\alpha_{2(K-3)}q^{K-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1^N \\ z_2^N \\ \vdots \\ z_K^N \end{bmatrix} = \begin{bmatrix} 1 + \sum_{j=K+1}^{N} \alpha_{1j} q^{j+1} z_j^N \\ 1 + \sum_{j=K+1}^{N} \alpha_{2j} q^{j+1} z_j^N \\ \vdots \\ 1 + \sum_{j=K+1}^{N} \alpha_{Kj} q^{j+1} z_j^N \end{bmatrix}, \tag{14}$$

then

$$\lim_{N\to\infty} z_k^N = z_k^*, \quad (k = 1, 2, \ldots, K). \tag{15}$$

Note that $\{z_k^N; k = 1, 2, \ldots, K\}$ is also uniformly bounded, so is $\{z_k^N; k = 1, 2, \ldots, N\}$.

Combining (12), (13), (14) and (15), we complete the proof. ■

In the following theorem, we provide explicit formulas for $E[Y^k]$ as well as $E[X^k]$.

Theorem 2. $E[Y^k]$ and $E[X^k]$ are given by

$$E[Y^k] = \frac{\sum_{m=0}^{\infty} a_{km} q^{k+m}}{\sum_{m=0}^{\infty} d_m q^m} \tag{16}$$

and

$$E[X^k] = \sum_{j=0}^{k} \sum_{i=0}^{j} \frac{(-1)^i s^{j-i} k!}{(k-j)!(j-i)!i!} E[Y^{k-j}] E[D^i] \tag{17}$$

$(k = 1, 2, \ldots)$, where

$$a_{k0} = 1 \quad \text{and} \quad a_{km} = \sum_{j=0}^{m-1} \alpha_{kj} a_{(k+j+1)(m-j-1)} \tag{18}$$

and

$$d_0 = 1 \quad \text{and} \quad d_m = \sum_{j=0}^{m-1} \beta_j a_{(j+1)(m-j-1)}. \tag{19}$$

Proof. Based on Theorem 1, we know that $\{W_k; k = 1, 2, \ldots\}$ has one and only one bounded solution which is given by $\{z_k^*; k = 1, 2, \ldots\}$. Denote $\{z_k^N; k = 1, 2, \ldots, N\}$ in Theorem 1 as $\{W_k^N; k = 1, 2, \ldots, N\}$. Recall that $v = \sum_{j=0}^{\infty} \beta_j W_{j+1} q^{j+1}$. Let

$$v_N = \sum_{j=0}^{N-1} \beta_j W_{j+1}^N q^{j+1}.$$

Since $\{W_k^N; k = 1, 2, \ldots, N\}$ is uniformly bounded and $\sum_{j=0}^{\infty} \beta_j q^{j+1} z$ converges uniformly with respect to z in any bounded interval, we have

$$\lim_{N \to \infty} v_N = v.$$

Combining this and (8), we obtain

$$E[Y^k] = \lim_{N \to \infty} \frac{W_k^N q^k}{(1 + v_N)}. \tag{20}$$

W_k^N can be solved directly from (10):

$$W_k^N = \sum_{m=0}^{N-k} a_{km} q^m, \quad (k = 1, 2, \ldots, N) \tag{21}$$

where a_{km}'s are defined by the following recursive equation

$$a_{km} = \sum_{j=0}^{m-1} \alpha_{kj} a_{(k+j+1)(m-j-1)},$$

with $a_{k0} = 1$. Using (21), we can write v_N as

$$\begin{aligned} v_N &= \sum_{j=0}^{N-1} \beta_j q^{j+1} \sum_{i=0}^{N-j-1} a_{(j+1)i} q^i \\ &= \sum_{j=0}^{N-1} \sum_{i=0}^{N-j-1} \beta_j a_{(j+1)i} q^{i+j+1} \\ &= \sum_{m=1}^{N} d_m q^m, \end{aligned} \tag{22}$$

where

$$d_0 = 1 \text{ and } d_m = \sum_{j=0}^{m-1} \beta_j a_{(j+1)(m-j-1)}.$$

Substituting (21) and (22) into (20), we then obtain (16). Notice Y and D are independent of each other in $X = Y - D + s$. Therefore, (17) follows. ■

To illustrate how a_{km} in (16) can be calculated based on the recursive equation (18), we provide the following table. The numbers in parentheses indicate the order in which the corresponding calculations are performed.

$(1)a_{10} = 1$	$(3)a_{11} = \alpha_{10}a_{20}$	$(6)a_{12} = \alpha_{10}a_{21} + \alpha_{11}a_{30}$	$(10)a_{13} = \alpha_{10}a_{22} + \alpha_{11}a_{31} + \alpha_{12}a_{40}$	$\cdots$
$(2)a_{20} = 1$	$(5)a_{21} = \alpha_{20}a_{30}$	$(9)a_{22} = \alpha_{20}a_{31} + \alpha 21a_{40}$	$(14)a_{23} = \alpha_{20}a_{32} + \alpha_{21}a_{41} + \alpha_{22}a_{50}$	$\cdots$
$(4)a_{30} = 1$	$(8)a_{31} = \alpha_{30}a_{40}$	$(13)a_{32} = \alpha_{30}a_{41} + \alpha_{31}a_{50}$	$(19)a_{33} = \alpha_{30}a_{42} + \alpha_{31}a_{51} + \alpha_{32}a_{60}$	$\cdots$
$(7)a_{40} = 1$	$(12)a_{41} = \alpha_{40}a_{50}$	$(18)a_{42} = \alpha_{40}a_{51} + \alpha_{41}a_{60}$	$(25)a_{43} = \alpha_{40}a_{52} + \alpha_{41}a_{61} + \alpha_{42}a_{70}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Calculation of Coefficient a_{km}

4 MacLaurin Series

In this section, we discuss a different way to compute $E[Y^k]$ based on (4). Instead of solving (4) directly to obtain $E[Y^k]$ as we did in Section 2, we now use it to derive the MacLaurin series of $E[Y^k]$ with respect to q. The MacLaurin series provides us with some possibilities to obtain the whole response curve of $E[Y^k]$ with respect to q. So it is extremely useful in performance optimization, say, with respect to s and S.

Assume $E[Y^k]$ has all derivatives at $q = 0$. We write

$$E[Y^k] = \sum_{m=1}^{\infty} y_{km} q^i, \tag{23}$$

which is the MacLaurin series of $E[Y^k]$. Note that this series may not converge for all values of q. Substituting (23) into (4) and comparing the coefficients of $q, q^2, \ldots, q^m$, we obtain

$$y_{km} = \begin{cases} 0, & m < k; \\ 1, & m = k; \\ \sum_{j=0}^{m-k-1} (\alpha_{kj} y_{(k+j+1)m} - \beta_j y_{(j+1)(m-k)}), & m > k, \end{cases} \tag{24}$$

where $k = 1, 2, \cdots$.

Again, we provide the following table to demonstrate how y_{km} can be calculated based on (24). Similar to Table 1, the numbers in parentheses indicate the order in which the corresponding calculations are performed.

Moments	q	q^2	q^3	$\cdots$	q^m
EY	$(1) y_{11} = 1$	$(3) y_{12} = \alpha_{10} y_{22} - \beta_0 y_{22}$	$(6) y_{13} = \alpha_{10} y_{23} + \alpha_{11} y_{33} - \beta_0 y_{12} - \beta_1 y_{22}$	$\cdots$	$(m(m+1)/2)$
EY^2	0	$(2) y22 = 1$	$(5) y_{23} = \alpha_{20} y_{33} - \beta_0 y_{11}$	$\cdots$	$(m(m+1)/2 - 1)$
EY^3	0	0	$(4) y_{33} = 1$	$\cdots$	$(m(m+1)/2 - 2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2: Calculation of MacLaurin Series Coefficient y_{km}

The MacLaurin series we have obtained above might be used to obtain the whole response of the moments of the inventory with respect to s and S. However, we should point out that the MacLaurin series (23) might have a convergence radius which could be finite, in which case the series diverges if q falls outside the convergent region. One possible way to overcome this difficulty is to use analytical continuation, and another way is to use Padé approximation.

5 Demands with Exponential Distribution

We now consider a special case in which D is exponential distributed. First, it follows from (4) that

$$
\begin{aligned}
E[Y^k] &= q^k(1-u) + \sum_{j=0}^{\infty} \frac{k! f^{(j)}(0)}{(k+j+1)!} E[Y^{k+j+1}] \\
&= q^k(1-u) + \frac{k! f(0)}{(k+1)!} E[Y^{k+1}] + \sum_{j=1}^{\infty} \frac{k! f^{(j)}(0)}{(k+j+1)!} E[Y^{k+j+1}] \\
&= q^k(1-u) + \frac{f(0)}{k+1} (q^{k+1}(1-u) + \sum_{j=0}^{\infty} \frac{k! f^{(j)}(0)}{(k+j+2)!} E[Y^{k+j+2}]) \\
&\quad + \sum_{j=0}^{\infty} \frac{k! f^{(j+1)}(0)}{(k+j+2)!} E[Y^{k+j+2}]) \\
&= q^k(1-u) + \frac{f(0)}{k+1} q^{k+1}(1-u) \\
&\quad + \sum_{j=0}^{\infty} \frac{k!}{(k+j+2)!} (f(0) f^{(j)}(0) + f^{(j+1)}(0)) E[Y^{k+j+2}]. \qquad (25)
\end{aligned}
$$

If D is exponentially distributed with mean $1/\lambda$, i.e., $f^{(j)}(0) = \lambda(-\lambda)^j$, then (25) can be simplified as

$$E[Y^k] = (1-u)(q^k + \frac{\lambda}{k+1}q^{k+1}). \tag{26}$$

Substituting (26) into

$$u = \sum_{j=0}^{\infty} \beta_j E[Y^{j+1}],$$

we have

$$\begin{aligned} u &= (1-u)\sum_{j=0}^{\infty} \frac{\lambda(-\lambda)^j}{(j+1)!}(q^{j+1} + \frac{\lambda}{j+2}q^{j+2}) \\ &= (1-u)(-\sum_{j=0}^{\infty} \frac{(-\lambda q)^{j+1}}{(j+1)!} + \sum_{j=0}^{\infty} \frac{(-\lambda q)^{j+2}}{(j+2)!}) \\ &= (1-u)\lambda q. \end{aligned} \tag{27}$$

Hence,

$$u = \frac{\lambda q}{1+\lambda q}. \tag{28}$$

Substituting (28) into (26), we finally get

$$E[Y^k] = \frac{(k+1+\lambda q)q^k}{(k+1)(1+\lambda q)}. \tag{29}$$

(Note that this equation can also be obtained from 16 directly.)

With (29), we now calculate the Laplace transforms of X and Y. First we have

$$\begin{aligned} E[e^{zY}] &= E\left[\sum_{k=0}^{\infty} \frac{(zY)^k}{k!}\right] \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} E[Y^k] \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{(k+1+\lambda q)q^k}{(k+1)(1+\lambda q)} \\ &= \frac{1}{1+\lambda q}(e^{qz} + \frac{\lambda}{z}(e^{qz}-1)). \end{aligned} \tag{30}$$

Notice that $X = Y - D + s$, where Y and D are independent of each other. We then have

$$\begin{aligned} E[e^{zX}] &= E[e^{zY}]E[e^{-zD}]e^{zs} \\ &= \frac{\lambda e^{sz}}{(1+\lambda q)(z+\lambda)}(e^{qz} + \frac{\lambda}{z}(e^{qz}-1)). \end{aligned} \tag{31}$$

On the other hand, it is known that the probability density function of X is given by

$$f_X(x) = \frac{1}{1+R(q)} \begin{cases} 0 & \text{if } x > s+q \\ r(s+q-x) & \text{if } s \le x \le s+q \\ v_q(s-x) & \text{if } x \le s \end{cases}, \tag{32}$$

where

$$\begin{aligned} R(x) &= \sum_{n=1}^{\infty} F_n(x), \text{ where } F_n \text{ is the n-fold convolution of } F \text{ with itself,} \\ r(x) &= \frac{dR(x)}{dx}, \\ v_q(x) &= f(x+q) + \int_0^q r(u) f(q+x-u) du. \end{aligned} \tag{33}$$

Now since $F(x) = 1 - e^{-\lambda x}$, we get

$$\begin{aligned} R(x) &= \lambda x, \\ r(x) &= \lambda, \\ v_q(x) &= e^{-\lambda x}. \end{aligned} \tag{34}$$

(see Sahin 1982.) From Equations (32) and (34), we can obtain

$$\begin{aligned} E[e^{zX}] &= \frac{1}{1+\lambda q}\left[\int_s^{s+q} \lambda e^{zx} dx + \int_{-\infty}^{s} \lambda e^{zx} e^{-\lambda(s-x)} dx\right] \\ &= \frac{1}{1+\lambda q}\left[\frac{\lambda}{z}(e^{qz}-1) + \frac{\lambda}{z+\lambda} e^{sz}\right] \\ &= \frac{\lambda e^{sz}}{(1+\lambda q)(z+\lambda)}(e^{qz} + \frac{\lambda}{z}(e^{qz}-1)), \end{aligned} \tag{35}$$

which matches our result given by Equation (31).

6 Numerical Examples

In this section, we apply the algorithm developed in Section 2 to two examples. In the first example we use the k-stage Erlang distribution with mean $1/\lambda$ for the demand distribution while in the second example we use the k-stage hyperexponential distribution with the density

$$f(x) = \sum_{i=1}^{k} p_i \lambda_i e^{-\lambda_i x} \quad x \geq 0.$$

Numerical results are given in Tables 3 and 4 along with simulation results. We compare the first two moments of Y calculated by our method with those obtained from simulation. The errors of simulation results are within 0.5%. N in the last column of the two tables is the least N in (10) and (20) needed to ensure the convergence of numerical values in the calculations (precision of 16 digits). For $N = 120$, the computation time is about 3.1 seconds on a VAX 6310. Clearly, the results are quite accurate and the speed of the algorithm is very fast.

k	λ	q	Calculation		Simulation		N
			$E[Y]$	$E[Y^2]$	$E[Y]$	$E[Y^2]$	
4	0.5	5	4.287	20.631	4.286	20.623	40
		10	7.004	61.597	7.006	61.627	50
		15	9.624	119.491	9.624	119.485	60
	1.0	5	3.502	15.399	3.503	15.407	50
		10	6.100	48.567	6.098	48.537	70
		15	8.643	98.500	8.640	98.468	90
	1.2	5	3.357	14.356	3.357	14.358	60
		10	5.934	46.204	5.938	46.249	80
		15	8.470	94.825	8.470	94.830	110
6	0.5	5	4.837	24.051	4.837	24.051	40
		10	7.911	74.665	7.907	74.651	50
		15	10.379	137.192	10.379	137.174	60
	1.0	5	3.956	18.666	3.955	18.667	50
		10	6.512	54.707	6.518	54.767	70
		15	9.076	107.945	9.076	107.925	90
	1.2	5	3.694	16.928	3.693	16.932	60
		10	6.288	51.409	6.288	51.417	80
		15	8.834	102.711	8.834	102.758	110

Table 3: Numerical Results For Demands with Erlang Distribution

k	(p_1, p_2, p_3, p_4)	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	q	Calculation		Simulation		N
				$E[Y]$	$E[Y^2]$	$E[Y]$	$E[Y^2]$	
2	(0.5,0.5,-,-)	(1.0,2.0,-,-)	5	2.848	10.632	2.848	10.641	40
			10	5.379	38.366	5.384	38.368	70
			15	7.891	82.794	7.892	82.792	120
	(0.3,0.7,-,-)	(0.5,1.0,-,-)	5	3.031	11.844	3.039	11.884	30
			10	5.616	41.468	5.617	41.501	50
			15	8.151	87.911	8.150	87.933	70
4	(0.2,0.3,0.4,0.1)	(0.3,1.0,0.5,1.5)	5	3.198	12.918	3.201	12.934	40
			10	5.864	44.673	5.864	44.705	60
			15	8.441	93.554	8.461	93.834	80
	(0.3,0.2,0.2,0.3)	(0.4,0.8,1.0,1.2)	5	3.089	12.200	3.089	12.198	30
			10	5.713	42.676	5.711	42.678	50
			15	8.267	90.113	8.266	90.111	70

Table 4: Numerical Results for Demands with Hyperexponential Distribution

7 Conclusion

We have proposed a new approach to the study of the (s, S) inventory system. Our basic idea is to derive an infinite system of linear equations for the moments of the inventory position based on the recursive equation which describes the dynamic of the inventory level of the system. Based on this infinite system of equations, we propose two recursive algorithms to obtain the moments of the inventory position. In the first one, we solve a finite system of linear equations whose solution converges to the moments as its dimension goes to infinity. In the second one, we calculate the coefficients of the power series of the moments with respect to s and S, therefore it is possible for us to obtain the whole response curve of the moments based on it. Both algorithms are very simple and easy to implement. For the first one, we have provided two numerical examples.

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