

Optimal Control of a Failure Prone Manufacturing System With the Possibility of Temporary Increase in Production Capacity *

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Abstract

Manufacturers have two general strategies to guard against uncertainties in the production or demand process: (a) building inventory to hedge against periods in which the production capacity is not sufficient to satisfy demand, and (b) temporarily increasing the production capacity, for example, by hiring temporary workers, adding overtime and/or extra shifts, or outsourcing part of the production. A production control problem is to strike a balance between the costs of building inventory, “buying” extra capacity, and backlogged demand. In this paper, we address this problem in the context of a single failure prone machine that produces a single part-type and show that the optimal control policy is a switching policy that is generally characterized by three switching levels.

1 Introduction

Uncertainty is one of the key ingredients that renders the planning and scheduling of manufacturing systems very difficult and highly challenging. From an aggregate point of view uncertainty manifests itself in two main areas: demand for products (volume, mix, and arrival times) and capacity of production (availability of raw material, operators, and machines). Manufacturers have two general strategies to guard against these uncertain elements in the demand and production processes: (a) building inventory to hedge against periods in which the production capacity is not sufficient to satisfy demand, and (b) temporarily increasing the production capacity, for example, by hiring temporary workers, adding overtime and/or extra shifts, or outsourcing part of the production. Clearly, each of these options has an associated cost and the task of a production planner and scheduler is to strike a balance between these costs, namely, the cost of building inventory, “buying” extra capacity, and unsatisfied demand. In this paper we address and solve this problem in the context of a simplified example.

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We formulate the problem in the framework of flow control of manufacturing systems. In this framework, discrete parts are approximated by fluids and the movement of parts across the manufacturing floor is approximated by the flow of fluids among a network of capacitated machines. The changes in the demand and the production capacity are modeled by a discrete space stochastic process, often a Markov chain. The focus of most work reported in the literature has been the selection of the optimal production control policy when building inventory is the sole strategy to guard against uncertainty, i.e., it is assumed that the option of buying extra capacity is not available. In what follows we briefly review previous work on flow control of manufacturing systems; for a more detailed review and a comprehensive list of references on flow control of manufacturing systems, see two recent books by Gershwin [5] and Sethi and Zhang [22].

Olsder and Suri [19] were among the first to study the flow control problem for manufacturing systems and they proposed to use models with jump Markov disturbances based on Rishel's formalism ([20]). Kimemia and Gershwin [16] showed that the optimal control for such systems has a special structure called the hedging point policy. In such a policy, a nonnegative production surplus of part-types is maintained during times when excess capacity is available to hedge against future capacity shortages. The value of the optimal hedging point for systems with one machine and one part-type was first obtained by Akella and Kumar for the discounted cost problem [1] and by Bielecki and Kumar [2] for the average cost problem.

Sharifnia [23] showed how the optimum hedging points in the case of one part-type multiple machine states can be calculated. The problem of the complete evaluation of the optimal production policy in multiple part-types multiple machines is difficult because it requires solving either systems of partial differential equations or large dynamic programming problems that easily run into the "curse of dimensionality." Therefore, several approximation procedures have been proposed to obtain near-optimal controllers (e.g., [7, 3, 4]).

In all the work mentioned above, it has been assumed that the machine failure rates are independent of the production rates and are constant as long as the system is in one of its discrete capacity states, or in other words, the underlying Markov chain is homogeneous. Recently, several researchers [21, 10, 12, 13, 14, 18, 17, 24] have studied the problem of flow control for systems whose underlying state processes are not homogeneous Markov chains and established various structural properties for the optimal control policy. Hu and Xiang [11, 13, 8, 9] showed how some of these structural properties can be used to obtain the optimal control policy based on some known results in the area of stochastic process and queueing theory.

In this paper, we consider a problem in which it is possible to temporarily increase the production capacity at some additional cost. Being able to buy extra capacity is the main difference between our problem and those previously studied in the literature. This problem was first proposed by Gershwin [6]. A similar, but much simplified, problem was also considered by Hu [8]

where it is assumed that backlog is not permitted, i.e., extra capacity has to be bought whenever the inventory level becomes zero. We will completely solve the problem in this paper. We prove that the optimal control policy is a policy with three thresholds: one corresponds to an inventory hedging level and the other two are levels below which extra production capacity should be bought to prevent further demand shortages (one for each machine state: up or down). Also, the optimal thresholds are obtained. It is interesting to notice that the optimal control policy is very much similar to the (S, s) inventory policy studied in inventory theory. In the (S, s) policy whenever the inventory level is below s , an order has to be placed to bring the inventory level back to S ; in our problem, extra production capacity needs to be bought when the inventory level hits certain thresholds.

The rest of this paper is organized as follows. The problem formulation and some preliminaries are given in Section 2. In Section 3, a class of hedging point policies is defined and some of its properties are derived. The main results of the paper are given in Section 4. We conclude in Section 5.

2 Problem formulation and preliminaries

To put the problem we address in this paper in perspective, we begin with the description of a general version of it. Consider a production system that produces a number of product-types. Let $\mathbf{x} = (x_1, \dots, x_n)$ denote the vector of inventory levels or backlogged demands for product-types (x_i is a continuous variable that represents the inventory level of product-type i if $x_i > 0$, or the backlogged demand for this product-type if $x_i < 0$). Let $\alpha = \{\alpha(t); t \geq 0\}$ denote the stochastic process representing random changes in the production capacity and random fluctuations in the demand rates for product-types (in general, α is a vector process). $(\mathbf{x}, \alpha) = \{(\mathbf{x}(t), \alpha(t)); t \geq 0\}$ is the state process of this system.

The production capacity at time t , denoted by $C_\alpha(t)$, is a random set and the vector of demand rates, denoted by $\mathbf{d}_\alpha(t)$ is a random vector. The control variables of production rates and rates of buying extra capacity for product-types at time t are denoted, respectively, by $\mathbf{u}(\mathbf{x}, \alpha)(t)$ and $\mathbf{v}(\mathbf{x}, \alpha)(t)$. The vector of feasible production rates at time t belongs to the random set $C_\alpha(t)$ and we assume that the vector of the rates of buying extra capacity belongs to a fixed set C . The dynamics of \mathbf{x} , the inventory-backlog vector, is given by

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{u}(\mathbf{x}, \alpha)(t) + \mathbf{v}(\mathbf{x}, \alpha)(t) - \mathbf{d}_\alpha(t). \quad (1)$$

Let $g(\mathbf{x})$ be the cost per unit time of holding \mathbf{x} units of inventory or backlog, and let $h(\mathbf{v})$ be the cost per unit time of buying extra capacity at rate \mathbf{v} . Let π belong to a feasible set of appropriately defined control policies Π , then we define the cost associated with policy π to be the long-run

average cost given by

$$J^\pi = \lim_{T \rightarrow \infty} \frac{1}{T} E\left[\int_0^T (g(\mathbf{x}) + h(\mathbf{v}))dt \mid \mathbf{x}(0), \alpha(0)\right]. \quad (2)$$

An optimal control policy in this setting is a feasible control policy $\pi^* \in \Pi$ that yields the minimum cost, i.e., such that

$$J^{\pi^*} \leq J^\pi \quad \text{for all } \pi \in \Pi. \quad (3)$$

Having defined a general version of the problem, we now turn to specifying the assumptions that define the problem we address in this paper:

- $n = 1$, i.e., a single product-type is produced and x is a scalar.
- α is a time-homogeneous Markov chain on the set $\{1, 2\}$. (We assume that the sojourn time of α in state i is exponentially distributed with rate q_i .) We assume that $\alpha = 2$ represents a state with “low” production capacity and $\alpha = 1$ represents a state with “high” production capacity; examples include the case where the production capacity C_α is deterministic and remains unchanged and the demand for the product alternates between periods of high demand ($\alpha = 2$) and low demand ($\alpha = 1$) according to a Markov chain α , or the case where the demand rate d_α is fixed and the production capacity alternates between periods of high capacity ($\alpha = 1$) and low capacity ($\alpha = 2$). To simplify the presentation and highlight the connection between the problem we consider here and others considered in the literature, we assume that the demand is fixed ($d_\alpha = d$) and that $\alpha = 2$ corresponds to the case where production capacity is equal to zero (down state). It is worth pointing out that solving this problem is equivalent to solving the others mentioned above.
- $C_1 = [0, r_1]$; we assume that $r_1 \geq d$, in other words, during the “up” periods the maximum possible production rate is higher than the demand rate;
 $C_2 = \{0\}$;
 $C = [0, r_2]$; we assume that $r_2 \geq d$.
- $g(x) = c^+ x^+ + c^- x^-$ where $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, i.e., the inventory/backlog cost is piecewise linear;
 $h(v) = c_v v$, i.e., the cost of purchasing extra capacity is linear in the rate of purchase.

We limit ourselves to the class of stationary feedback control policies, where $\pi(x, \alpha, t) = \pi(x, \alpha) = (u(x, \alpha), v(x, \alpha))$.

Define the differential cost function $V_i^\pi(x)$ (value functions for simplicity) by

$$V_i^\pi(x) = \lim_{T \rightarrow \infty} \frac{1}{T} E\left[\int_0^T [(g(\mathbf{x}) + h(\mathbf{v}) - J^\pi)]dt \mid \mathbf{x}(0) = x, \alpha(0) = i\right] \quad (4)$$

We use $V_i^*(x)$ ($i = 1, 2$) to denote optimal value functions. If we assume that they are differentiable, then the well-known Hamilton-Jacobi-Bellman equation for the optimal control is given by

$$\begin{aligned} \min_{0 \leq u \leq r_1, 0 \leq v \leq r_2} \left\{ (u + v - d) \frac{dV_1^*(x)}{dx} + c_v v \right\} &= q_1(V_1^*(x) - V_2^*(x)) - g(x) + J^{\pi^*} \\ \min_{0 \leq v \leq r_2} \left\{ (v - d) \frac{dV_2^*(x)}{dx} + c_v v \right\} &= q_2(V_1^*(x) - V_2^*(x)) - g(x) + J^{\pi^*}; \end{aligned} \quad (5)$$

The following theorem tells us that the Hamilton-Jacobi-Bellman equation is a sufficient condition for a control to be optimal

Theorem 1 *If $(u, v) = \pi^*(x, \alpha)$, $V_1^{\pi^*}(x)$ and $V_2^{\pi^*}(x)$ satisfy Equation (5), then π^* is an optimal control policy.*

The proof of Theorem 1 is essentially the same as the one given in [2] which uses the Dynkin formula. Therefore, we shall not repeat it here.

3 Hedging point policies

In this section, we define a set of policies called *hedging point policies*. We will show subsequently that the optimal control policy belongs to this class of policies.

Each hedging point policy $\pi^{\mathbf{z}}$ is characterized by a vector of three parameters $\mathbf{z} = (z_1, z_2, z_3)$ where $z_3 < z_2 \leq z_1$, and is defined as follows:

$$\begin{aligned} \pi^{\mathbf{z}}(x, 1) = (u, v) &= \begin{cases} (0, 0) & x > z_1 \\ (d, 0) & x = z_1 \\ (r_1, 0) & z_3 < x < z_1 \\ (r_1, r_2) & x \leq z_3, \end{cases} \\ \pi^{\mathbf{z}}(x, 0) = (u, v) &= \begin{cases} (0, 0) & x > z_2 \\ (0, d) & x = z_2 \\ (0, r_2) & x < z_2. \end{cases} \end{aligned} \quad (6)$$

Note that, under the above policy, once the inventory/backlog is within the interval $[z_2, z_1]$, it remains in this interval for ever. In other words, all inventory/backlog levels in the intervals $(-\infty, z_2)$ and (z_1, ∞) are transient and are reached only a finite number of times. Thus $J^{\mathbf{z}}$, the cost associated with policy $\pi^{\mathbf{z}}$, is independent of z_3 and is only a function of z_1 and z_2 . To make this fact explicit, we occasionally write $J(z_1, z_2)$ for $J^{\mathbf{z}}$.

The following lemma implies that in search for the optimal hedging point policy among the class of hedging point policies, we only need to consider those policies for which $z_1 \geq 0$ and $z_2 \leq 0$.

Lemma 1 *If $0 < z_2 \leq z_1$, then $J(z_1 - z_2, 0) < J(z_1, z_2)$; if $z_2 \leq z_1 < 0$ then $J(0, z_2 - z_1) < J(z_1, z_2)$.*

Proof. Let π and π' be the hedging point policies defined by (z_1, z_2) , where $0 < z_2 \leq z_1$, and by $(z_1 - z_2, 0)$, respectively (choose the third parameter of the hedging point policies arbitrarily). Let $X^\pi(0) = z_1$ and $X^{\pi'}(0) = z_1 - z_2$ and assume that the stochastic processes of the capacities are coupled, in the sense that the up and down periods of both processes coincide. Under these assumptions it is easy to verify that

$$X^{\pi'}(t) = X^\pi(t) - z_2.$$

Clearly there is no cost of backlog for either policy; moreover, under the above assumptions the durations of time when the policies purchase extra capacity and the corresponding rates of the purchase coincide and hence the associated costs are identical. However, in light of the above identity, the cost of inventory for policy π' is lower than that of policy π ; hence $J^{\pi'} = J(z_1 - z_2, 0) < J^\pi = J(z_1, z_2)$.

A similar argument can be used to prove the result for the case where $z_2 \leq z_1 < 0$: Let π' be defined by $(0, z_2 - z_1)$, assume $X^\pi(0) = z_2$ and $X^{\pi'}(0) = z_2 - z_1$, and also couple the processes of the capacities as above. Then

$$X^{\pi'}(t) = X^\pi(t) - z_1.$$

In this case, there are no inventory costs for either policy, the costs of purchasing extra capacity are identical, and the cost of backlog is lower for policy π' , hence $J^{\pi'} = J(0, z_2 - z_1) \leq J^\pi = J(z_1, z_2)$. \square

In light of the above lemma, from now on, we limit ourselves to the class of hedging point policies for which $z_1 \geq 0 \geq z_2$. We denote this class by \mathcal{H} .

Our main results, given in the next section, are based on explicit evaluation of the average cost and the differential cost functions associated with a number of candidate hedging point policies. In preparation for what follows, we derive the average cost of an arbitrary policy $\pi^z \in \mathcal{H}$ and a set of differential equations that characterize the differential cost functions associated with. We begin with evaluating the average cost.

3.1 Average cost

Let $\pi^z \in \mathcal{H}$ be a hedging point policy. We define the following notation:

$$\begin{aligned} \beta_1 &= \frac{q_1}{r_1 - d}, \\ \beta_2 &= \frac{q_2}{d}, \\ \lambda_0 &= \beta_1 - \beta_2, \\ \rho &= \frac{\beta_2}{\beta_1}, \\ p_1 &= \frac{q_2}{q_1 + q_2}, \\ p_2 &= \frac{q_1}{q_1 + q_2}. \end{aligned}$$

Let $\{X^z(t); t \geq 0\}$ be the process of inventory/backlog of the system, then we have

Proposition 1 $X^z(t)$ converges weakly to the random variable of the steady state inventory/backlog denoted by $X^z(\infty)$ with probability density

$$\begin{aligned} f(x) &= \frac{r_1}{d} p_1 \frac{\beta_1}{\beta_1 - \beta_2 e^{-\lambda_0(z_1 - z_2)}} \lambda_0 e^{-\lambda_0(x - z_2)} \quad z_2 < x < z_1, \\ P_{z_1} = P(X^z(\infty) = z_1) &= p_1 \frac{1}{\beta_1 - \beta_2 e^{-\lambda_0(z_1 - z_2)}} \lambda_0 e^{-\lambda_0(z_1 - z_2)}, \\ P_{z_2} = P(X^z(\infty) = z_2) &= p_2 \frac{1}{\beta_1 - \beta_2 e^{-\lambda_0(z_1 - z_2)}} \lambda_0. \end{aligned}$$

Proof. The same results are obtained in [8] for the case $z_2 = 0$. In [8] it is shown that $X^z(t)$ is related to the workload process of an M/M/1 queue with limited workload, based on which the density function of $X^z(t)$ can then be obtained by using the existing results in queueing theory (also see [11]). For $z_2 < 0$, we can apply the results in [8] to the process $X^z(t) - z_2$, then shift everything by z_2 . \square

Given the above steady state density, it is straightforward to evaluate the average cost associated with policy π^z . Let J_1^z = cost of holding inventory, J_2^z = cost of having backlogged demand, and J_3^z = cost of purchasing extra capacity. Then

$$\begin{aligned} J_1^z &= c^+ p_1 \frac{1}{\beta_1 - \beta_2 e^{-\lambda_0(z_1 - z_2)}} \left\{ (\lambda_0 - \frac{r_1}{d} \beta_1) z_1 e^{-\lambda_0(z_1 - z_2)} - \frac{r_1}{d} \beta_1 \lambda_0 e^{-\lambda_0(z_1 - z_2)} + \frac{r_1}{d} \beta_1 \frac{1}{\lambda_0} e^{\lambda_0 z_2} \right\}, \\ J_2^z &= c^- \frac{1}{\beta_1 - \beta_2 e^{-\lambda_0(z_1 - z_2)}} \left\{ (\frac{r_1}{d} \beta_1 p_1 + p_2 \lambda_0) z_2 - \frac{r_1}{d} \beta_1 p_1 \frac{1}{\lambda_0} e^{\lambda_0 z_2} + \frac{r_1}{d} \beta_1 p_1 \frac{1}{\lambda_0} \right\}, \\ J_3^z &= c_0 d p_2 \frac{\lambda_0}{\beta_1 - \beta_2 e^{-\lambda_0(z_1 - z_2)}}. \end{aligned}$$

The total cost, $J^z = J(z_1, z_2) = J_1^z + J_2^z + J_3^z$. It is worth noting that $J(z_1, z_2)$ is, clearly, a continuously differentiable function of z_1 and z_2 on the region $\{z_1 \geq 0, z_2 \leq 0\}$.

3.2 Value function

Let $V^z(x) = (V_1^z(x), V_2^z(x))'$ be the vector of value functions corresponding to policy π^z (\prime denotes transpose). Also let

$$\begin{aligned} A_1 &= \begin{bmatrix} -\frac{q_1}{d} & \frac{q_1}{d} \\ \frac{q_2}{d} & -\frac{q_2}{d} \end{bmatrix} & b_1 &= \begin{bmatrix} \frac{1}{d} \\ \frac{1}{d} \end{bmatrix} & A_0 &= \begin{bmatrix} \frac{q_1}{r_1 - d} & -\frac{q_1}{r_1 - d} \\ \frac{q_2}{d} & -\frac{q_2}{d} \end{bmatrix} & b_0 &= \begin{bmatrix} -\frac{1}{r_1 - d} \\ \frac{1}{d} \end{bmatrix} \\ A_2 &= \begin{bmatrix} \frac{q_1}{r_1 - d} & -\frac{q_1}{r_1 - d} \\ -\frac{q_2}{r_2 - d} & \frac{q_2}{r_2 - d} \end{bmatrix} & b_2 &= \begin{bmatrix} -\frac{1}{r_1 - d} \\ -\frac{1}{r_2 - d} \end{bmatrix} & b'_2 &= \begin{bmatrix} 0 \\ -\frac{1}{r_2 - d} \end{bmatrix} \\ A_3 &= \begin{bmatrix} \frac{q_1}{r_2 + r_1 - d} & -\frac{q_1}{r_2 + r_1 - d} \\ -\frac{q_2}{r_2 - d} & \frac{q_2}{r_2 - d} \end{bmatrix} & b_3 &= \begin{bmatrix} -\frac{1}{r_1 + r_2 - d} \\ -\frac{1}{r_2 - d} \end{bmatrix}. \end{aligned}$$

We have

Proposition 2 *The value function V^z satisfies the following set of piecewise linear differential equations:*

$$\begin{aligned}
\frac{dV^z(x)}{dx} &= A_1 V^z(x) - b_1(J^z - c^+ x); & z_1 \leq x \\
\frac{dV^z(x)}{dx} &= A_0 V^z(x) - b_0(J^z - c^+ x); & 0 \leq x < z_1 \\
\frac{dV^z(x)}{dx} &= A_0 V^z(x) - b_0(J^z + c^- x); & z_2 \leq x < 0 \\
\frac{dV^z(x)}{dx} &= A_2 V^z(x) - b_2(J^z + c^- x) + b'_2 c_v r_2; & z_3 \leq x < z_2 \\
\frac{dV^z(x)}{dx} &= A_3 V^z(x) - b_3(J^z + c^- x) + b_3 c_v r_2; & x < z_3
\end{aligned} \tag{7}$$

with the following boundary conditions:

$$\begin{aligned}
V_1^z(z_1) - V_2^z(z_1) &= \frac{c^+ z_1 - J^z}{q_1} \\
V_1^z(z_2) - V_2^z(z_2) &= \frac{c^- z_2 + J^z - c_v d}{q_2}
\end{aligned} \tag{8}$$

Proof. It is very straightforward, e.g., see [2].

4 Optimal Control Policies

In this section we give the main results of this paper and show that the optimal control policies belong to the class of hedging point policies. Four cases are identified that correspond to: (1) the hedging levels for inventory and backlog are both non-zero, (2) the hedging level for inventory is zero but the backlog hedging level is nonzero, (3) the hedging level for backlog is zero but the inventory hedging level is nonzero, and, finally, (4) both hedging levels are zero. In each case we first identify necessary conditions for the existence of a hedging point policy that is optimal among an appropriate subset of hedging point policies. This yields a candidate policy. We then identify sufficient conditions under which the candidate policy is optimal among all control policies.

4.1 Case 1: Nonzero inventory, nonzero backlog ($z_1^* > 0$, $z_2^* < 0$)

A necessary condition for the existence of a hedging point policy π^{z^*} , with $z_1^* > 0$, $z_2^* < 0$, that is optimal among all feasible policies is that the policy be optimal among all hedging point policies. Given the fact that the cost function $J(z_1, z_2)$ is differentiable, a set of necessary conditions for the latter to be true is the existence of two hedging levels $z_1^* > 0$, $z_2^* < 0$, such that

$$\frac{\partial J}{\partial z_1}(z_1^*) = 0, \quad \frac{\partial J}{\partial z_2}(z_2^*) = 0. \tag{9}$$

Taking derivatives, setting them equal to zero and rearranging terms yields the following set of equations that is equivalent to (9) in the sense that the existence of $z_1^* > 0$, $z_2^* < 0$ satisfying one set of equations implies that $z_1^* > 0$ and $z_2^* < 0$ satisfy the other,

$$\begin{aligned} f_1(z_1^*) &= c^+ \rho e^{-\lambda_0 z_1^*} + c^- e^{-\frac{\lambda_0}{c^-}(k - c^+ z_1^*)} - \frac{r_1}{d} p_1 (c^+ + c^-) = 0; \\ f_2(z_2^*) &= c^+ \rho e^{-\frac{\lambda_0}{c^+}(k - c^- z_2^*)} + c^- e^{-\lambda_0 z_2^*} - \frac{r_1}{d} p_1 (c^+ + c^-) = 0, \end{aligned} \quad (10)$$

where $k = dp_2[\rho(\frac{c^-}{q_2} - c_v) - (\frac{c^+}{q_1} - c_v)]$.

Proposition 3 *A set of necessary and sufficient conditions for the existence of a pair of values $z_1^* > 0$, $z_2^* < 0$ that satisfy equation(9) is $f_1(0) < 0$ and $f_2(0) < 0$. The vector (z_1^*, z_2^*) is unique when it exists.*

Proof. Note that f_1 and f_2 are linear combinations of convex functions with non-negative coefficients, therefore they are convex (both are of the form $a_1 e^{b_1 z} + a_2 e^{-b_2 z} + a_3$); moreover, $f_i(\infty) = f_i(-\infty) = \infty$; $i = 1, 2$ (note that $b_1, b_2, a_1, a_2 > 0$). The minimizers of the functions f_1 and f_2 , denoted by z_1^m and z_2^m respectively, are $z_1^m = \frac{c^- \ln \rho + \lambda_0 k}{\lambda_0 (c^+ + c^-)}$, and $z_2^m = \frac{c^+ \ln \rho - \lambda_0 k}{\lambda_0 (c^+ + c^-)}$; the minimums of these functions are respectively $f_1(z_1^m) = (c^+ + c^-)[\rho e^{-\lambda_0 z_1^m} - \frac{r_1}{d} p_1]$, $f_2(z_2^m) = (c^+ + c^-)[e^{-\lambda_0 z_2^m} - \frac{r_1}{d} p_1]$.

Let $f_1(0) < 0$ and $f_2(0) < 0$, then given the above properties of the two functions f_1 and f_2 , it can be easily verified that f_1 has a unique root $z_1^* > 0$ and f_2 has a unique root $z_2^* < 0$.

On the other hand, assume that $z_1^* > 0$. If $z_1^m \leq 0$, then $f_1(z_1^m) < 0$ (recall $f_1(z_1^*) = 0$) and clearly $f_1(0) < 0$; Assume $z_1^m > 0$; further assume that $\rho > 1$, in this case $\lambda_0 < 0$ and $\lambda_0 z_1^m < 0$, and $\rho e^{-\lambda_0 z_1^m} > \rho$, therefore

$$f_1(z_1^m) = (c^+ + c^-)[\rho e^{-\lambda_0 z_1^m} - \frac{r_1}{d} p_1] > (c^+ + c^-)[\rho - \frac{r_1}{d} p_1] = (c^+ + c^-) p_1 (\rho - 1) > 0$$

which contradicts the existence of $z_1^* > 0$; hence $z_1^* > 0$ and $z_1^m > 0$, implies $\rho < 1$. $\rho < 1$ is equivalent to $\lambda_0 > 0$; this latter condition and $z_1^m > 0$ yield $c^- \ln \rho + \lambda_0 k > 0$, therefore $\ln \rho > -\frac{\lambda_0 k}{c^-}$; hence

$$f_1(0) = c^+ \rho + c^- e^{-\frac{\lambda_0 k}{c^-}} - \frac{r_1}{d} p_1 (c^+ + c^-) < c^+ \rho + c^- \rho - \frac{r_1}{d} p_1 (c^+ + c^-) = (c^+ + c^-) p_1 (\rho - 1) < 0.$$

Now assume that $z_2^* < 0$. If $z_2^m \geq 0$, then $f_2(z_2^m) < 0$ (recall $f_2(z_2^*) = 0$) and clearly $f_2(0) < 0$; Assume $z_2^m < 0$; further assume that $\rho < 1$, in this case $\lambda_0 > 0$ and $-\lambda_0 z_2^m > 0$, and $\rho e^{-\lambda_0 z_2^m} > \rho$, therefore,

$$f_2(z_2^m) = (c^+ + c^-)[\rho e^{-\lambda_0 z_2^m} - \frac{r_1}{d} p_1] > (c^+ + c^-)[\rho - \frac{r_1}{d} p_1] = (c^+ + c^-) p_1 (\rho - 1) > 0$$

which contradicts the existence of $z_2^* < 0$; hence $z_2^* < 0$ and $z_2^m < 0$, implies $\rho > 1$. $\rho > 1$ is equivalent to $\lambda_0 < 0$; this latter condition and $z_2^m < 0$ yield $-c^+ \ln \rho + \lambda_0 k > 0$, therefore

$-\ln \rho > -\frac{\lambda_0 k}{c^+}$; hence

$$f_2(0) = c^+ \rho e^{-\frac{\lambda_0 k}{c^+}} + c^- - \frac{r_1}{d} p_1 (c^+ + c^-) < c^+ + c^- - \frac{r_1}{d} p_1 (c^+ + c^-) = (c^+ + c^-) p_1 (1 - \rho) < 0.$$

□

The following corollary is one of the main results of this subsection:

Theorem 2 *A set of necessary conditions for the existence of a hedging point policy π^{z^*} that is optimal among all feasible policies and such that $z_1^* > 0, z_2^* < 0$ is*

$$f_1(0) < 0, f_2(0) < 0. \quad (11)$$

We now turn to deriving sufficient conditions for the existence of such a hedging point policy π^{z^*} . We will show that the above conditions are also sufficient. In order to accomplish this, we first need to specify the policy π^{z^*} completely. Note that while the set of conditions (9) uniquely defines z_1^* and z_2^* , it does not specify the hedging point policy uniquely. This is because the third hedging point z_3^* is undefined. To define z_3^* , we begin with an arbitrary $z_3 < z_2^*$ and consider the hedging point policy associated with (z_1^*, z_2^*, z_3) . We denote the cost associated with this policy by J^* . Note that J^* is independent of z_3 and is uniquely defined by z_1^* and z_2^* . Using equations (9), J^* can be written in terms of either z_1^* or z_2^* as

$$\begin{aligned} J^* &= c^+ z_1^* + c^+ \frac{d}{q_1 + q_2}, \\ J^* &= c^- (-z_2^*) + c^- \frac{r_1 - d}{q_1 + q_2} + c_v \left(d - \frac{q_2 r_1}{q_1 + q_2} \right). \end{aligned} \quad (12)$$

To simplify the presentation and with a slight abuse of notation, we denote the value functions associated with the above hedging point policy by V_1^* and V_2^* .

Using the set of piecewise linear differential equations (7), the boundary conditions (8), and the above identities, we can determine $\frac{dV_1^*(x)}{dx}$ and $\frac{dV_2^*(x)}{dx}$. Before specifying these functions, we introduce the following notation:

$$\begin{aligned} \lambda_1 &= \frac{q_1}{d} + \frac{q_2}{d}; \\ \lambda_0 &= \frac{q_1}{r_1 - d} - \frac{q_2}{d}; \\ \lambda_2 &= \frac{q_1}{r_1 - d} + \frac{q_2}{r_2 - d}; \\ \lambda_3 &= \frac{q_1}{r_1 + r_2 - d} + \frac{q_2}{r_2 - d}. \end{aligned}$$

Let $R_1 = \{x; z_1^* \leq x\}$, $R_0 = \{x; z_2^* \leq x < z_1^*\}$, $R_2 = \{x; z_3 \leq x < z_2^*\}$, and $R_3 = \{x; x \leq z_3\}$. Then $\frac{1}{\lambda_i}$ = average change in the inventory/backlog per one up and down period in region R_i , assuming the trajectory remains in R_i during this time. Given this notation, we have

For $z_1^* \leq x$:

$$\begin{aligned} \frac{dV_1^*(x)}{dx} &= \frac{c^+(q_1 + q_2)}{d^2} \left[\frac{1}{\lambda_1} (x - z_1^*) - \frac{1}{\lambda_1^2} (1 - e^{-\lambda_1(x-z_1^*)}) \right]; \\ \frac{dV_2^*(x)}{dx} &= \frac{c^+(q_1 + q_2)}{d^2} \left[\frac{1}{\lambda_1} (x - z_1^*) + \frac{q_1}{q_2} \frac{1}{\lambda_1^2} (1 - e^{-\lambda_1(x-z_1^*)}) \right] - \frac{c^+}{q_1}. \end{aligned}$$

For $0 \leq x < z_1^*$:

$$\begin{aligned}\frac{dV_1^*(x)}{dx} &= \frac{c^+(q_1 + q_2)}{d(r_1 - d)} \left[-\frac{1}{\lambda_0}(z_1^* - x) + \frac{1}{\lambda_0^2}(1 - e^{-\lambda_0(z_1^* - x)}) \right]; \\ \frac{dV_2^*(x)}{dx} &= \frac{c^+(q_1 + q_2)}{d(r_1 - d)} \left[-\frac{1}{\lambda_0}(z_1^* - x) + \frac{\rho}{\lambda_0^2}(1 - e^{-\lambda_0(z_1^* - x)}) \right] - \frac{c^+}{q_1}.\end{aligned}$$

For $z_2^* \leq x < 0$:

$$\begin{aligned}\frac{dV_1^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{d(r_1 - d)} \left[\frac{1}{\lambda_0}(x - z_2^*) - \frac{1}{\rho\lambda_0^2}(1 - e^{-\lambda_0(x - z_2^*)}) \right] - c_v + \frac{c^-}{q_2}; \\ \frac{dV_2^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{d(r_1 - d)} \left[-\frac{1}{\lambda_0}(x - z_2^*) - \frac{1}{\lambda_0^2}(1 - e^{-\lambda_0(x - z_2^*)}) \right] - c_v.\end{aligned}$$

For $z_3 \leq x < z_2^*$:

$$\begin{aligned}\frac{dV_1^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d)} \left[-\frac{1}{\lambda_2}(z_2^* - x) - \frac{q_1(r_2 - d)}{q_2(r_1 - d)} \frac{1}{\lambda_2^2}(1 - e^{-\lambda_2(z_2^* - x)}) \right] - c_v + \frac{c^-}{q_2}; \\ \frac{dV_2^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d)} \left[-\frac{1}{\lambda_2}(z_2^* - x) - \frac{1}{\lambda_2^2}(1 - e^{-\lambda_2(z_2^* - x)}) \right] - c_v.\end{aligned}$$

We are now prepared to define z_3^* . Taking derivative of $\frac{dV_1^*(x)}{dx}$ over the interval $z_3 \leq x < z_2^*$, we have

$$\frac{d^2V_1^*(x)}{dx^2} = \frac{c^-(q_1 + q_2)}{d(r_1 - d)} \left[\frac{1}{\lambda_2} + \frac{q_1}{q_2} \frac{1}{\lambda_2} (1 - e^{-\lambda_2(z_2^* - x)}) \right].$$

Given that $\lambda_2 = \frac{q_1}{r_1 - d} + \frac{q_2}{r_2 - d} > 0$, it follows that $\frac{d^2V_1^*(x)}{dx^2} > 0$ on the interval $[z_3, z_2^*]$; hence, on this interval, $\frac{dV_1^*(x)}{dx}$ is continuous and strictly decreasing. Moreover, we have $\frac{dV_1^*(z_2^*)}{dx} = -c_v + \frac{c^-}{q_2}$ and $\lim_{x \rightarrow -\infty} \frac{dV_1^*(x)}{dx} = -\infty$ (since z_3 is arbitrary, assume $z_3 \rightarrow -\infty$ as $x \rightarrow -\infty$). Therefore, there exists a unique value at which the derivative of V_1^* is equal to $-c_v$. We define z_3^* to be this unique value, i.e., z_3^* is defined uniquely via the identity

$$\frac{dV_1^*(z_3^*)}{dx} = -c_v.$$

We have now completely specified the policy π^{z^*} which we claim is optimal among all feasible policies. We denote the value functions associated with this policy also by V_1^* and V_2^* . On the interval (z_3^*, ∞) the derivatives of these functions are the same as the ones calculated above for the hedging point policy with parameters (z_1^*, z_2^*, z_3) , (replace z_3 by z_3^*). On the last region, R_3 , the derivatives are given by:

For $x \leq z_3^*$:

$$\frac{dV_1^*(x)}{dx} = \frac{c^-(q_1 + q_2)}{(r_1 + r_2 - d)(r_2 - d)} \left[-\frac{1}{\lambda_3}(z_3^* - x) + \frac{r_1 q_1}{(q_1 + q_2)(r_1 + r_2 - d)} \frac{1}{\lambda_3^2}(1 - e^{-\lambda_3(z_3^* - x)}) \right]$$

$$\begin{aligned}
& + \frac{c^-(q_1 + q_2)}{(r_1 + r_2 - d)(r_2 - d)} \left[-\frac{1}{\lambda_3}(z_2^* - z_3^*) + \frac{r_1 - d}{q_1 + q_2} \right] (1 - e^{-\lambda_3(z_3^* - x)}) - c_v, \\
\frac{dV_2^*(x)}{dx} & = \frac{c^-(q_1 + q_2)}{(r_1 + r_2 - d)(r_2 - d)} \left[-\frac{1}{\lambda_3}(z_3^* - x) + \frac{r_1 q_2}{(q_1 + q_2)(r_2 - d)} \frac{1}{\lambda_3^2} (1 - e^{-\lambda_3(z_3^* - x)}) \right] \\
& + \frac{c^-(q_1 + q_2) q_2}{(r_2 - d)^2 q_1} \left[-\frac{1}{\lambda_3}(z_2^* - z_3^*) + \frac{r_1 - d}{q_1 + q_2} \right] (1 - e^{-\lambda_3(z_3^* - x)}) \\
& - \frac{c^-(q_1 + q_2)}{q_1(r_2 - d)} [(z_2^* - z_3^*) - (r_1 - d)] - c_v.
\end{aligned}$$

Proposition 4 Let V_i^* ($i = 1, 2$) be the value functions associated with the hedging point policy defined by z_1^*, z_2^* and z_3^* given above; then V_i^* is convex and continuously differentiable ($i = 1, 2$).

Proof. The continuity of the derivatives at all points except at $x = 0$ and $x = z_3^*$ is obvious; the continuity at $x = 0$ follows from the identities $f_1(z_1^*) = 0$ and $f_2(z_2^*) = 0$ while the continuity at $x = z_3^*$ can be easily verified by using $dV_1^*(z_3^*)/dx = -c_v$. The proof of convexity is straightforward for $x > z_3^*$. So we focus on the convexity for $x \leq z_3^*$. First for $x \leq z_3^*$ we have

$$\begin{aligned}
\frac{d^2 V_1^*(x)}{dx^2} & = \frac{c^-}{(r_1 + r_2 - d)(r_2 - d)} ((q_1 + q_2)(z_2^* - z_3^*) + (r_2 - r_1)) e^{-\lambda_3(z_3^* - x)} \\
& + \frac{c^-(q_1 + q_2)}{\lambda_3(r_1 + r_2 - d)(r_2 - d)} (1 - e^{-\lambda_3(z_3^* - x)}) \\
\frac{d^2 V_2^*(x)}{dx^2} & = \frac{c^-}{q_1(r_1 + r_2 - d)(r_2 - d)} (-q_2(q_1 + q_2)(z_2^* - z_3^*) + q_1(r_2 - d) + q_2(r_1 - d)) e^{-\lambda_3(z_3^* - x)} \\
& + \frac{c^-(q_1 + q_2)}{\lambda_3(r_1 + r_2 - d)(r_2 - d)} (1 - e^{-\lambda_3(z_3^* - x)})
\end{aligned}$$

To prove $d^2 V_1^*(x)/dx^2 \geq 0$ and $d^2 V_2^*(x)/dx^2 \geq 0$, it is sufficient to show that $(q_1 + q_2)(z_2^* - z_3^*) + (r_2 - r_1) \geq 0$ and $-q_2(q_1 + q_2)(z_2^* - z_3^*) + q_1(r_2 - d) + q_2(r_1 - d) \geq 0$. To verify these two inequalities, we consider the interval $[z_3^*, z_2^*]$. Since

$$\frac{dV_1^*(z_2^*)}{dx} = \frac{c^-}{q_2} - c_v \quad \text{and} \quad \frac{dV_1^*(z_3^*)}{dx} = -c_v$$

we have

$$\begin{aligned}
\frac{c^-}{q_2} & = \int_{z_3^*}^{z_2^*} \frac{d^2 V_1^*(x)}{dx^2} dx \\
& = \frac{c^- q_1 (q_1 + q_2)}{\lambda_2^2 q_2 (r_1 - d)^2} (1 - e^{-\lambda_2(z_2^* - z_3^*)}) + \frac{c^-(q_1 + q_2)(z_2^* - z_3^*)}{q_1(r_2 - d) + q_2(r_1 - d)}. \tag{13}
\end{aligned}$$

Based on (13), we have

$$\begin{aligned}
\frac{c^-}{q_2} & = \frac{c^- q_1 (q_1 + q_2)}{\lambda_2^2 q_2 (r_1 - d)^2} (1 - \lambda_2(z_2^* - z_3^*) - e^{-\lambda_2(z_2^* - z_3^*)}) + \frac{c^-(q_1 + q_2)(z_2^* - z_3^*)}{q_2(r_1 - d)} \\
& \leq \frac{c^-(q_1 + q_2)(z_2^* - z_3^*)}{q_2(r_1 - d)},
\end{aligned}$$

hence, $(q_1 + q_2)(z_2^* - z_3^*) + (r_2 - r_1) \geq 0$. On the other hand, (13) also leads to

$$\frac{c^-}{q_2} \geq \frac{c^-(q_1 + q_2)(z_2^* - z_3^*)}{q_1(r_2 - d) + q_2(r_1 - d)},$$

which gives $-q_2(q_1 + q_2)(z_2^* - z_3^*) + q_1(r_2 - d) + q_2(r_1 - d) \geq 0$. This completes our proof. \square

We are now prepared to state the other main result of this subsection:

Theorem 3 *The set of conditions (11) is sufficient for the existence of a hedging point policy with nonzero hedging levels that is optimal among all feasible policies.*

Proof. Assume that conditions (11) are satisfied. Then by proposition (3) there exist unique hedging levels $z_1^* > 0$ and $z_2^* < 0$ that satisfy equations (9). According to the preceding discussion, we can then define z_3^* uniquely. The proof of the theorem is complete if we show that the hedging point policy defined by (z_1^*, z_2^*, z_3^*) satisfies the Hamilton-Jacoby-Bellman (HJB) equations.

For $\alpha = 1$, i.e., when the machine is up, using formulas for the derivatives of the value functions, and Proposition (4), it is easy to verify that $\frac{dV_1^*(x)}{dx}$ switches sign from negative to positive at $x = z_1^*$, and $\frac{dV_1^*(x)}{dx} + c_v$ switches sign from negative to positive at z_3^* . Therefore, the policy $u(x, 1) = r_1$, $x < z_1^*$ and $u(x, 1) = 0$, $x > z_1^*$ and $v(x, 1) = r_2$, $x \leq z_3^*$ and $v(x, 1) = 0$, $x > z_3^*$ solves the following minimization problem:

$$\min_{0 \leq u \leq r_1, 0 \leq v \leq r_2} \left\{ (u + v - d) \frac{dV_1^*(x)}{dx} + c_v v \right\}.$$

It is also easy to verify that $\frac{dV_1^*(x)}{dx} + c_v$ switches sign from negative to positive at z_2^* ; hence the policy $v(x, 2) = r_2$, $x < z_2^*$ and $v(x, 2) = 0$, $x > z_2^*$ solves the following minimization problem:

$$\min_{0 \leq v \leq r_2} \left\{ (v - d) \frac{dV_2^*(x)}{dx} + c_v v \right\}.$$

The minimizing policy is defined everywhere except at $u(z_1^*, 1)$ and at $v(z_2^*, 2)$. To avoid chattering, we set $u(z_1^*, 1) = v(z_2^*, 2) = d$. This policy is exactly the hedging point policy with parameters (z_1^*, z_2^*, z_3^*) . The HJB equation is therefore satisfied by this hedging point policy. \square

Before closing this subsection, we provide some discussion on the conditions $f_1(0) < 0$ and $f_2(0) < 0$. First, we present the following result:

Proposition 5 *The conditions $f_1(0) < 0$ and $f_2(0) < 0$ imply $c_v > c^+/q_1$ and $c_v > c^-/q_2$, respectively.*

Proof. We will only give the proof for $c_v > c^+/q_1$; the proof for $c_v > c^-/q_2$ is essentially the same. Using the inequality $1 + x \leq e^x$, we have

$$c^+ \rho + c^- \left(1 - \frac{\lambda_0 k}{c^-}\right) - \frac{r_1}{d} p_1 (c^+ + c^-) \leq f_1(0) < 0.$$

With some simplification, this leads to

$$\frac{(1 - \rho)^2 dq_1}{(r_1 - d)(q_1 + q_2)}(c^+ - c_v q_1) < 0,$$

which implies $c_v > c^+/q_1$. \square

Based on the above proposition, it is clear that $c_v > c^+/q_1$ and $c_v > c^-/q_2$ are necessary conditions for the existence of a hedging point policy π^{z^*} with $z_1^* > 0, z_2^* < 0$ that is optimal among all feasible policies. In the following subsections we will show that the relationships between c_v and $c^+/q_1, c^-/q_2$ also give necessary and sufficient conditions for the other three cases.

4.2 Zero inventory, nonzero backlog ($z_1^* = 0, z_2^* < 0$)

We now turn to the case where the optimal inventory hedging level is zero and that of backlog is strictly negative. A necessary condition for the existence of a hedging point policy with these properties is that it be optimal among all hedging point policies with zero inventory hedging levels; a necessary condition for the latter to be true is

$$\frac{\partial J(0, z_2)}{\partial z_2} \Big|_{z_2=z_2^*} = 0.$$

Taking derivative of $J(0, z_2)$ with respect to z_2 , simplifying, and setting it equal to zero, yields the following equation:

$$f_3(z_2) = \frac{1}{\lambda_0} [c^- \lambda_0 z_2 + \frac{c^-}{\rho} (e^{-\lambda_0 z_2} - 1) + \frac{d(r_1 - d)}{q_1 + q_2} \lambda_0^2 (\frac{c^-}{q_2} - c_v)] = 0.$$

Proposition 6 *A necessary and sufficient condition for the existence of a negative root for $f_3(z_2) = 0$ is*

$$c_v > \frac{c^-}{q_2};$$

such a negative root is unique.

Proof. Let $g_1(z_2) = \lambda_0 f_3(z_2)$. The roots of f_3 and g_1 coincide, so we can focus on the function g_1 . Note that $\frac{d^2 g_1}{dz_2^2}(z_2) = \frac{c^-}{\rho} \lambda_0^2 e^{-\lambda_0 z_2} > 0$, hence g_1 is convex. Moreover, it is easy to verify that the minimizer of g_1 is at $z_2^m = -\frac{\ln \rho}{\lambda_0}$; note that $\rho < 1 \Leftrightarrow \lambda_0 < 0$, therefore $z_2^m > 0$. Hence, a necessary and sufficient condition for the existence of a negative root for g_1 is $g_1(0) = \frac{d(r_1 - d)}{q_1 + q_2} \lambda_0^2 (\frac{c^-}{q_2} - c_v) < 0$, which is equivalent to $c_v > \frac{c^-}{q_2}$. Clearly, in this case, the negative root of g_1 , and hence of f_3 , is unique. \square

We can now give one of the main results of this subsection which is a corollary of the above proposition.

Theorem 4 *A necessary condition for the existence of a hedging point policy π^{z^*} with $z_1^* = 0, z_2^* < 0$, that is optimal among all feasible policies, is*

$$c_v > \frac{c^-}{q_2}. \quad (14)$$

Before looking at the value functions, we prove a result about the hedging level z_2^* which will be needed in what follows.

Lemma 2 *Assume $c_v > \frac{c^-}{q_2}$, and let z_2^* be as defined above. Then*

$$\frac{q_1 + q_2}{q_2(r_1 - d)} c^- z_2^* > \frac{c^-}{q_2} - c_v. \quad (15)$$

Proof. Assume $\lambda_0 > 0$. Using the inequality $(e^{-\lambda_0 x} - 1) > -\lambda_0 x$ in the equation $f_3(z_2^*) = 0$, we get

$$c^- z_2^* - \frac{1}{\rho} c^- z_2^* < -\frac{d(r_1 - d)}{q_1 + q_2} \lambda_0 \left(\frac{c^-}{q_2} - c_v \right).$$

The result follows immediately from the above inequality. The case of $\lambda_0 < 0$ can be proved similarly. \square

Let 0 and z_2^* denote the optimal hedging levels (when they exist) and let J^* denote the corresponding cost; J^* is given by

$$J^* = c^-(-z_2^*) + c^- \frac{r_1 - d}{q_1 + q_2} + c_v d \left(1 - \frac{q_2 r_1}{d(q_1 + q_2)} \right). \quad (16)$$

Note that the above cost function J^* and that for the case $z_1^* > 0, z_2^* < 0$ (equation (12), J^* expressed in terms of z_2^*) are identical in form (the optimal z_2^* for the two cases are in general different). Note also that the value functions are obtained from solving the set of differential equations (7) after substituting J^* for J^z (J^* expressed in terms of z_2^* is substituted on the interval $(-\infty, 0)$). Hence, given the similarity in form of J^* in the two cases, the value functions V_1^* and V_2^* are also similar in form on the interval $(-\infty, 0)$. As a result, following the same steps as in the previous section, we can define z_3^* , and the same arguments as those given in the previous section can be used to show that V_1^* and V_2^* are convex and continuously differentiable on the interval $(-\infty, 0)$. We now focus on the interval $[0, \infty)$.

For $0 \leq x$:

$$\begin{aligned} \frac{dV_1^*(x)}{dx} &= \frac{c^+}{d} x - \frac{J^*}{d} (1 - e^{-\lambda_1 x}); \\ \frac{dV_2^*(x)}{dx} &= \frac{c^+}{d} x - \frac{q_2}{q_1} \frac{J^*}{d} (1 - e^{-\lambda_1 x}) - \frac{J^*}{d} \left(1 + \frac{q_2}{q_1} \right). \end{aligned}$$

Lemma 3 *The value functions V_1^* and V_2^* are continuously differentiable.*

Proof. The only point at which the continuity of the derivatives of V_1^* , and V_2^* is not established is at $x = 0$. This can be verified using the identity $f_3(z_2^*) = 0$. \square

Proposition 7 *The set of conditions*

$$\frac{c^-}{q_2} < c_v \leq \frac{c^+}{q_1} \quad (17)$$

is sufficient for the convexity of V_1^ , and V_2^* .*

Proof. The convexity of V_1^* , and V_2^* on $(-\infty, 0)$ is already established. On the interval $[0, \infty)$ we have :

$$\begin{aligned} \frac{d^2 V_1(x)}{dx^2} &= \lambda_1 \frac{J^*}{d} (1 - e^{-\lambda_1 x}) + \frac{1}{d} [c^+ - c_v q_1 + \frac{r_1 - d}{d} q_2 (c_v - \frac{c^-}{q_2} (1 - \frac{q_1 + q_2}{r_1 - d} z_2^*))]; \\ \frac{d^2 V_2(x)}{dx^2} &= \frac{q_2 J^*}{q_1 d} \lambda_1 e^{-\lambda_1 x} + \frac{c^+}{d}. \end{aligned}$$

It is easy to see that $\frac{d^2 V_2(x)}{dx^2}$ is positive. As for $\frac{d^2 V_1(x)}{dx^2}$, note that the first term is clearly positive. We also know that $c_v < \frac{c^+}{q_1}$ and $\frac{c^-}{q_2} < c_v$ imply $c_v \geq \frac{c^-}{q_2} (1 - \frac{q_1 + q_2}{r_1 - d} z_2^*)$ (Lemma 2). Therefore the second term is also positive. Hence $\frac{d^2 V_1(x)}{dx^2} > 0$ on $[0, \infty)$ and the proof is complete. \square

Theorem 5 *Under conditions (17) there exists a hedging point policy with zero inventory level ($z_1^* = 0$) and negative backlog level ($z_2^* < 0$) that is optimal among all feasible policies.*

Proof. From our previous discussions it follows that under conditions (17) there exist a hedging point policy for which $z_1^* = 0$, $z_2^* < 0$, and the associated value functions are convex and continuously differentiable. To show that this policy is optimal among all feasible policies it is sufficient to show that it satisfies the HJB equations. The proof is almost identical to the proof of Theorem 3 and is based on the fact that $\frac{dV_1^*}{dx}$ changes sign from negative to positive at $x = z_1^* = 0$, $\frac{dV_1^*}{dx} - c_v$ changes sign from negative to positive at $x = z_3^*$, and $\frac{dV_2^*}{dx} - c_v$ changes sign from negative to positive at $x = z_2^*$. \square

We point out that the sufficient condition $c_v \leq c^+/q_1$ used here for the optimality of the hedging point policy with $z_1^* = 0$) and $z_2^* < 0$ can be replaced two weaker conditions

$$f_1(0) \geq 0 \quad (18)$$

$$dp_2[\rho(\frac{c^-}{q_2} - c_v) - (\frac{c^+}{q_1} - c_v)] \leq 0. \quad (19)$$

(That is to say $c_v \leq c^+/q_1$ implies (18) and (19).) In fact, it can be shown that (19) is also a necessary condition for the convexity.

4.3 Zero backlog, nonzero inventory ($z_1^* > 0, z_2^* = 0$)

The arguments used for establishing results in this case are very similar to those used in the previous case. A necessary condition for the existence of an optimal hedging point policy with non-negative inventory hedging level and zero backlog is that this hedging point policy be optimal among all hedging point policies of the same type. A necessary condition for the latter to be true is

$$\frac{\partial J(z_1, 0)}{\partial z_1} \Big|_{z_1=z_1^*} = 0.$$

Taking derivative of $J(z_1, 0)$ with respect to z_1 , simplifying, and setting it equal to zero yields the following equation:

$$f_4(z_1) = \frac{1}{\lambda_0} [c^+ \lambda_0 z_1 + c^+ \rho (e^{-\lambda_0 z_1} - 1) + \frac{d(r_1 - d)}{q_1 + q_2} \lambda_0^2 (\frac{c^+}{q_1} - c_v)] = 0.$$

Proposition 8 *A necessary and sufficient condition for the existence of a positive root for $f_4(z_1) = 0$ is*

$$c_v > \frac{c^+}{q_1};$$

such a negative root is unique.

Proof. Let $g_2(z_1) = \lambda_0 f_4(z_1)$. The roots of f_4 and g_2 coincide so we can focus on the function g_2 . Note that $\frac{d^2 g_2}{dz_1^2}(z_1) = c^+ \rho \lambda_0^2 e^{-\lambda_0 z_1} > 0$, hence g_2 is convex. Moreover, it is easy to verify that the minimizer of g_2 is at $z_1^m = \frac{\ln \rho}{\lambda_0}$; note that $\rho < 1 \Leftrightarrow \lambda_0 < 0$, therefore $z_1^m > 0$. Hence a necessary and sufficient condition for the existence of a negative root for g_2 is $g_2(0) = \frac{d(r_1 - d)}{q_1 + q_2} \lambda_0^2 (\frac{c^+}{q_1} - c_v) < 0$, which is equivalent to $c_v > \frac{c^+}{q_1}$. Clearly, in this case, the negative root of g_2 , and hence of f_4 , is unique. \square

The following corollary is one of the main results of this section:

Theorem 6 *A necessary condition for the existence of a hedging point policy π^{z^*} with $z_1^* > 0, z_2^* = 0$, that is optimal among all feasible policies, is*

$$c_v > \frac{c^+}{q_1}. \tag{20}$$

A useful result about the hedging level z_1^* which will be needed in what follows is:

Lemma 4 *Assume $c_v > \frac{c^+}{q_1}$, and let z_1^* be as defined above. Then*

$$\frac{q_1 + q_2}{q_1 d} c^+ z_1^* < c_v - \frac{c^+}{q_1}. \tag{21}$$

Proof. Assume $\lambda_0 > 0$. Using the inequality $(e^{-\lambda_0 x} - 1) > -\lambda_0 x$ in the equation $f_4(z_1^*) = 0$, we get

$$c^+(1 - \rho)z_1^* < \frac{d(r_1 - d)}{q_1 + q_2} \lambda_0 (c_v - \frac{c^+}{q_2}).$$

The result follows immediately from the above inequality. The case of $\lambda_0 < 0$ can be proved similarly. \square

Let z_1^* and 0 denote the optimal hedging levels (when they exist) and let J^* denote the corresponding cost. J^* is given by

$$J^* = c^+ z_1^* + \frac{dc^+}{q_1 + q_2} \quad (22)$$

The above cost function J^* and that for the case $z_1^* > 0, z_2^* < 0$ (equation (12)), J^* expressed in terms of z_1^* are identical in form. Hence, using an argument similar to what we gave in the previous section, it can be shown that the value functions V_1^* and V_2^* are also similar in form on the interval $[0, \infty)$. Therefore, V_1^* and V_2^* are convex and continuously differentiable on the interval $[0, \infty)$. We focus on the interval $(-\infty, 0)$.

For $z_3 \leq x \leq 0$

$$\begin{aligned} \frac{dV_1^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d)} \frac{q_1}{q_2} x + \frac{q_1}{r_1 - d} \left[\frac{J^* - c_v d}{q_2} + \frac{\delta_2 J^*}{\lambda_2} + \frac{\delta_2 c^-}{\lambda_2^2} \right] e^{\lambda_2 x} - \frac{q_1}{r_1 - d} \left[\frac{\delta_2 J^*}{\lambda_2} + \frac{\delta_2 c^-}{\lambda_2^2} \right] \\ &\quad + \frac{q_1}{r_1 - d} \frac{c_v r_2}{\lambda_2 (r_2 - d)} \left[e^{\lambda_2 x} - 1 \right] + \frac{J^*}{r_1 - d}; \\ \frac{dV_2^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d)} \frac{q_1}{q_2} x - \frac{q_2}{r_2 - d} \left[\frac{J^* - c_v d}{q_2} + \frac{\delta_2 J^*}{\lambda_2} + \frac{\delta_2 c^-}{\lambda_2^2} \right] e^{\lambda_2 x} + \frac{q_2}{r_2 - d} \left[\frac{\delta_2 J^*}{\lambda_2} + \frac{\delta_2 c^-}{\lambda_2^2} \right] \\ &\quad - \frac{q_2}{r_2 - d} \frac{c_v r_2}{\lambda_2 (r_2 - d)} \left[e^{\lambda_2 x} - 1 \right] + \frac{J^*}{r_2 - d} - \frac{c_v r_2}{r_2 - d}, \end{aligned}$$

where $\delta_2 = 1/(r_1 - d) + 1/(r_2 - d)$. As will be shown shortly, under conditions

$$\frac{c^+}{q_1} < c_v \leq \frac{c^-}{q_2} \quad (23)$$

we can uniquely specify an appropriate z_3^* and hence completely specify the policy π^{z^*} . Then we have

Proposition 9 *Under conditions (23) the value functions V_1^* and V_2^* are convex and continuously differentiable.*

Proof. We first complete the definition of the policy π^{z^*} . On the interval $[z_3, 0)$, we have

$$\frac{d^2 V_1^*(x)}{dx^2} = \frac{c^-(q_1 + q_2)}{(r_1 - d)q_2 + (r_2 - d)q_1} (1 - e^{\lambda_2 x})$$

$$\begin{aligned} & + \frac{q_1}{r_1 - d} \frac{c^+}{dq_1(1 - \rho)} (r_1(1 - e^{\lambda_2 z_1^*}) + d(1 - \rho)e^{\lambda_2 z_1^*}) e^{\lambda_2 x} + \frac{c^-}{r_1 - d} e^{\lambda_2 x} \\ \frac{d^2 V_2^*(x)}{dx^2} & = \frac{q_2}{(r_2 - d)\rho} (c_v - (\frac{c^+}{q_1} + \frac{q_1 + q_2}{dq_1} c^+ z_1^*)) e^{\lambda_2 x} + \frac{q_2}{r_2 - d} (\frac{c^-}{q_2} - c_v) e^{\lambda_2 x} \end{aligned}$$

It is easy to see that $\frac{d^2 V_1^*(x)}{dx^2} > 0$ and under conditions (23) $\frac{d^2 V_2^*(x)}{dx^2} > 0$. Hence, the value functions are convex on $[z_3, 0)$ and therefore $\frac{dV_1^*(x)}{dx}$ is strictly increasing. Note that $\lim_{x \rightarrow -\infty} \frac{dV_1^*(x)}{dx} = -\infty$. On the other hand it can be shown that $\frac{dV_1^*(0)}{dx} > -c_v$, hence there exists a unique z_3^* such that $\frac{dV_1^*(0)}{dx} = -c_v$. We have now completely defined the policy $\pi^{\mathbf{z}^*}$. To verify that the value functions are continuously differentiable for this policy we only need to consider the point $x = 0$; as in the other cases, the continuity of the derivatives follows from the characterizing equation for the optimal hedging levels ($f_4(z^*) = 0$ in this case). Convexity of the value functions has already been established on $[z_3^*, \infty)$; convexity on $(-\infty, z_3^*)$ follows from arguments similar to those given in section 4.1. \square

Theorem 7 *Under conditions (23) there exists a hedging point policy with positive inventory level ($z_1^* > 0$) and zero backlog level ($z_2^* = 0$) that is optimal among all feasible policies.*

Proof. The proof is identical to the corresponding theorem in the previous section. \square

Similar to the second case, the condition $c_v \leq c^-/q_2$ used here for the optimality of the hedging point policy with $z_1^* > 0$ and $z_2^* = 0$ can be replaced by two weaker conditions

$$f_2(0) \geq 0, \tag{24}$$

$$dp_2[\rho(\frac{c^-}{q_2} - c_v) - (\frac{c^+}{q_1} - c_v)] \geq 0. \tag{25}$$

(i.e., $c_v \leq c^-/q_2$ implies (24) and (25).) Furthermore, it can be shown that (25) is also a necessary condition for the convexity.

4.4 Zero inventory, zero backlog ($z_1^* = 0, z_2^* = 0$)

With no inventory and no backlog, the only cost in this case is the cost of purchasing extra capacity. This cost is given by

$$J^* = c_v d \frac{q_1}{q_1 + q_2}.$$

We can prove the first basic result of this subsection immediately.

Theorem 8 *The set of conditions*

$$c_v \leq \frac{c^+}{q_1}, \quad c_v \leq \frac{c^-}{q_2} \tag{26}$$

is necessary for the hedging point policy with $z_1^ = 0, z_2^* = 0$ to be optimal among all feasible policies.*

Proof. For this policy to be optimal it is necessary that $J(0,0) \leq J(z_1,0)$ for all $z_1 > 0$ and $J(0,0) \leq J(0,z_2)$ for all $z_2 < 0$. Therefore, in light of Propositions 6, it is necessary that f_3 have no negative roots, and, in light of Propositions 7, it is necessary that f_4 have no positive roots; these latter conditions are equivalent to (as proved in Propositions 6 and 7) $c_v \leq \frac{c^+}{q_1}$ and $c_v \leq \frac{c^-}{q_2}$. \square

The value functions associated with the above policy can be calculated as follows.

For $0 \leq x$

$$\begin{aligned}\frac{dV_1^*(x)}{dx} &= \frac{q_1 + q_2}{d^2} \left[c^+ \frac{1}{\lambda_1} x - c_v q_1 \frac{1}{\lambda_1^2} (1 - e^{-\lambda_1 x}) \right]; \\ \frac{dV_2^*(x)}{dx} &= \frac{q_1 + q_2}{d^2} \left[c^+ \frac{1}{\lambda_1} x + c_v q_2 \frac{1}{\lambda_1^2} (1 - e^{-\lambda_1 x}) \right].\end{aligned}$$

Taking derivative we have

$$\begin{aligned}\frac{d^2 V_1^*(x)}{dx^2} &= \frac{q_1}{d} \left[\left(\frac{c^+}{q_1} - c_v \right) + c_v (1 - e^{-\lambda_1 x}) \right]; \\ \frac{d^2 V_2^*(x)}{dx^2} &= \frac{q_2}{d} \left[c_v e^{-\lambda_1 x} + \frac{c^+}{q_2} \right].\end{aligned}$$

From the above equations it can be simply verified that $\frac{d^2 V_2^*(x)}{dx^2}$ is positive and that a sufficient condition for $\frac{d^2 V_1^*(x)}{dx^2}$ to be positive is

$$c_v < \frac{c^+}{q_1}. \quad (27)$$

For $z_3 \leq x < 0$

$$\begin{aligned}\frac{dV_1^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d)} \left[\frac{1}{\lambda_2} x + \frac{1}{\lambda_2^2} (1 - e^{\lambda_2 x}) \right] - \frac{c_v q_1 + c^-}{(r_1 - d)} \frac{1}{\lambda_2} (1 - e^{\lambda_2 x}) \\ \frac{dV_2^*(x)}{dx} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d)} \frac{1}{\lambda_2} x - \frac{c^-(q_1 + q_2) q_2}{(r_2 - d)^2} \frac{1}{q_1 \lambda_2^2} (1 - e^{-\lambda_2 x}) + \frac{q_2 c_v q_1 + c^-}{q_1 (r_2 - d)} \frac{1}{\lambda_2} (1 - e^{-\lambda_2 x}) - c_v\end{aligned}$$

Taking derivative we have

$$\begin{aligned}\frac{d^2 V_1^*(x)}{dx^2} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d) \lambda_2} (1 - e^{\lambda_2 x}) + \frac{c_v q_1 + c^-}{(r_1 - d)} e^{\lambda_2 x}; \\ \frac{d^2 V_2^*(x)}{dx^2} &= \frac{c^-(q_1 + q_2)}{(r_1 - d)(r_2 - d) \lambda_2} (1 - e^{\lambda_2 x}) + \frac{q_2}{(r_2 - d)} \left(\frac{c^-}{q_2} - c_v \right) e^{\lambda_2 x}.\end{aligned}$$

From the above identities it can be simply verified that $\frac{d^2 V_1^*(x)}{dx^2}$ is positive and that a sufficient condition for $\frac{d^2 V_2^*(x)}{dx^2}$ to be positive is

$$c_v < \frac{c^-}{q_2}. \quad (28)$$

If the above two conditions (27) and (28) are satisfied, as in the previous sections, we can define a z_3^* such that $\frac{dV_1^*(z_3^*)}{dx} = -c_v$. For the resulting policy we have:

Proposition 10 *The set of conditions (26) is necessary and sufficient for the value functions V_1^* , and V_2^* to be convex and continuously differentiable.*

Proof. Continuity of the derivatives follows directly from the above identities. We already showed that under the conditions of the proposition, the value functions are convex on $[z_3^*, \infty)$. The proof of convexity for values below z_3^* is similar to that given in section 4.1 for the same region.

We now can give the main result of this subsection:

Theorem 9 *Under conditions (26) there exists a hedging point policy with zero inventory level ($z_1^* = 0$) and zero backlog level ($z_2^* = 0$) that is optimal among all feasible policies.*

Proof. The proof is identical to the corresponding theorem in section 4.2. \square

5 Conclusion

We studied a flow control problem for a system in which it is possible to obtain extra capacity at additional cost. We set up the problem in a general setting. For systems with one part-type and two machine states, we studied the hedging point policies with three hedging levels. We derived necessary and sufficient conditions under which such a hedging point policy is optimal for four different cases, depending on whether the hedging levels are zero or not. Also, the optimal values of the three hedging levels were obtained for these cases. However, the conditions we established do not cover all parameter ranges, and can be improved. This is a future research direction.

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