Higher Order Approximations for Tandem Queueing Networks*

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In this paper a higher order approximation for single server queues and tandem queueing networks is proposed and studied. Different from the most popular two-moment based approximations in the literature, the higher order approximation uses the higher moments of the interarrival and service distributions in evaluating the performance measures for queueing networks. It is built upon the MacLaurin series analysis, a method that is recently developed to analyze single-node queues, along with the idea of decomposition using higher orders of the moments matched to a distribution. The approximation is computationally flexible in that it can use as many moments of the interarrival and service distributions as desired and produce the corresponding moments for the waiting and interdeparture times. Therefore it can also be used to study several interesting issues that arise in the study of queueing network approximations, such as the effects of higher moments and correlations. Numerical results for single server queues and tandem queueing networks show that this approximation is better than the two-moment based approximations in most cases.

Keywords: approximations, GI/G/1 queue, MacLaurin series, tandem queueing networks.

1 Introduction

Many important systems developed in modern technology, including computer-communication networks, complex manufacturing facilities, automated teller machines, and transportation systems, have a common feature that they consist of jobs requesting service from various resources and thus can be ideally modeled as queueing networks. Unfortunately, it is extremely difficult (perhaps will never be possible) to obtain exact solutions for most queueing networks except a few limited classes (e.g., see [2, 16]). Therefore, approximation methods become crucial to the analysis of queueing networks.

Most queueing network approximations developed in the literature are two-moment based approximations, i.e., they only use the first two moments of the service and inter-

*Queueing Systems: Theory and Applications, Vol. 22, 249-276, 1996

arrival distributions (e.g., see [32] and references therein). Though these approximations work reasonably well in many applications, it has been shown that in some cases they can perform poorly. For example, it was shown in [31] and [33] that for the GI/GI/1 queue the errors in two-moment based approximations increase when the squared coefficients of variation (defined as the variance divided by the square of the mean) of the interarrival and/or the service times increases and the traffic intensity decreases. It is shown in [7] that the first two moments do not capture the degree of clustering in the arrival processes completely, especially in light traffic. Furthermore, the two-moment based approximations can only give at most the first two moments of performance measures of interest. However, in some applications it might be interesting to obtain other statistics; for example, in communication networks it is often of importance to know the probability that the waiting time of a message exceeds a given limit. Also, it was shown in [36] that given the first two moments of the hyperexponential interarrival and service distributions for the GI/G/1 queue, the specification of additional parameters of the interarrival distribution are more important than those of the service distribution.

On the other hand, both theoretical and empirical evidence has been established in the literature to support the superiority of three-moment based approximations over two-moment based approximations. Therefore, several researchers have suggested using higher moments of service and interarrival distributions to improve the accuracy of queueing approximations, e.g., see [19, 20] and [33]. The major difficulty in using higher moments in queueing network approximations is the scarcity of results available on simple queues involving higher moments.

Recently, a new method has been developed in [10] and [15] to study the GI/GI/1 queue based on MacLaurin series analysis (MSA). The chief advantage of their method is that it can work with as many moments of the service and interarrival times as desired to produce the corresponding moments of the waiting time and interdeparture time. In this paper, we propose a higher order approximation for tandem queueing networks based on this method. Similar to many other queueing network approximations (see [22], [23], [24], [29], [30], [32] and [37]), our approximation also uses the idea of parametric decomposition to separate queueing networks into single-node queues, to which the MSA method can be applied. The decomposition in our approximation is achieved by matching high orders of moments of the interdeparture time to a mixed Erlang distribution which forms the approximating interarrival time distribution to the downstream node. Since the number of higher moments that can be incorporated in our approximation is flexible, it also provides a valuable tool to study the effects of higher moments as well as correlations on queueing approximations. In fact, our simulation results suggest that relatively large improvements can be made in many cases if the third moment is used in the decomposition and in some cases if the fourth moment is used while the improvement becomes less significant when more than the first four moments are used. Also, we observe that for tandem queueing networks correlations in departures are often not very significant and thus can be ignored.

We present a number of numerical examples in this paper. Three examples of GI/G/1 queue are studied in which various interarrival and service distributions are considered. Then, we present three examples of tandem queueing networks with two nodes in which the previous three examples act as the first queue. Two cases of tandem networks with 8 nodes and one case with 9 nodes are studied. Finally, we study two examples of a two-node network in which the correlations in the departure process from the first nodes are quite significant.

The rest of this paper is organized as follows. In Section 2, we first introduce the MacLaurin series analysis method developed in [10] and [15], and then in Section 3 we discuss how it can be used to develop our higher order queueing approximation. Some numerical examples of the GI/G/1 queue are also presented in this section. In Section 4, we apply the approximation to tandem queueing networks, and we investigate its accuracy and compare it with the Queueing Network Analyzer (QNA). The paper is concluded with a conclusion and some discussions on future research in Section 5.

2 MacLaurin Series Analysis

In this section, we introduce the MSA method developed in [10] and [15] and summarize the results there. Consider the GI/GI/1 queue with infinite buffer size, first-come-first-served service discipline, and independent and identically distributed (i.i.d.) service and interarrival times. First, let us introduce the following notation.

$$
\begin{aligned}
W &= \text{steady-state waiting time of a job} \\
T &= \text{steady-state system time of a job} \\
A &= \text{interarrival time between two adjacent jobs} \\
S &= \text{service time of a job} \\
D &= \text{steady-state interdeparture time between two adjacent jobs} \\
f(\cdot) &= \text{probability density function (p.d.f.) of } A \\
\beta_k &= E[S^k]/k! \\
\gamma_k &= E[A^k]/k! \\
\alpha_k &= f^{(k)}(0^+) = k\text{-th right hand derivative of } f(\cdot) \text{ at } 0
\end{aligned}
$$

We now introduce a scale parameter θ into the service time, i.e., we consider a GI/GI/1 queue with interarrival time A and service time θS. It is clear that this parameterized queue reduces to the original queue when $\theta = 1$. For convenience, we still use the same notation introduced for the original queue for this new queue; however we should note that all the quantities except the interarrival time are now functions of θ.

In [14] it is proven that $E[W^k]$, $[T^k]$, and $E[D^k]$ $(k = 1, 2, \ldots)$ are analytic at $\theta = 0$ if the following two conditions are satisfied.

(A1). All the moments of S are finite, and $E[S^n] \leq n!(C_s)^n$ for $n = 0, 1, 2, \ldots$, where $C_s > 0$ is a constant;

(A2). $f(x)$ is an analytic function over $[0, \infty)$, and

$$f(x) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} x^n \quad \text{for } x \geq 0,$$

where α_n satisfies $|\alpha_n| \leq (C_f)^n$ $(n = 1, 2, \ldots)$, where $C_f > 0$ is a constant.

We note that Condition (A1) is satisfied by almost all distributions used in queueing theory. On the other hand, (A2) is satisfied by all phase-type distributions, but it excludes all c.d.f.'s whose support is not $[0,\infty)$, such as uniform, triangle, and shift exponential distributions. In fact, two examples are provided in [14] which show that if A has either a uniform distribution or a shifted exponential distribution, then the moments are not analytic at $\theta = 0$. Throughout the rest of this paper, we assume that Conditions (A1) and (A2) are satisfied.

Since $E[W^k]$, $[T^k]$, and $E[D^k]$ are analytic at $\theta = 0$, we can write

$$\frac{E[W^k]}{k!} = \sum_{m=0}^{\infty} w_{km}\theta^m, \tag{1}$$

$$\frac{E[T^k]}{k!} = \sum_{m=0}^{\infty} t_{km}\theta^m, \tag{2}$$

$$\frac{E[D^k]}{k!} = \sum_{m=0}^{\infty} d_{km}\theta^m. \tag{3}$$

Then it is shown in [10] and [15] that w_{km}, t_{km}, and d_{km} can be calculated based on the following recursive equations

$$t_{km} = \begin{cases} \beta_k, & m = k \\ \sum_{i=1}^{k} \beta_{k-i} w_{i(m-k+i)}, & m > k \\ 0, & m < k \end{cases} \tag{4}$$

$$w_{km} = \begin{cases} \sum_{i=0}^{m-k-1} \alpha_i t_{(k+1+i)m}, & m > k \\ 0, & m \le k \end{cases} \tag{5}$$

$$d_{km} = \sum_{j=\max(1,k-m)}^{k} (-1)^j \beta_{k-j} \left(\sum_{i=1}^{j} (-1)^{j-i} \gamma_{j-i} t_{i(m-k+j)} - w_{j(m-k+j)} \right) + \begin{cases} \beta_m \gamma_{k-m}, & m \le k \\ 0, & m > k \end{cases} \tag{6}$$

In the next section, we will discuss how these MacLaurin series can be used to approximate the moments of the waiting time and the interdeparture time, based on which higher order approximations can be developed.

3 Higher Order Approximations

In this section we discuss how the MacLaurin series of the moments of the waiting time and interarrival time obtained in the previous section can be used to develop higher order approximations for queueing systems.

3.1 Rational Approximation Based on MacLaurin Series

The simplest way of approximating the moments of the waiting time and the interdeparture time based on their MacLaurin series is to use polynomial functions. However, it is well known that the convergence rates of polynomial approximations are usually very slow; even worse, they may not converge at all in some cases, especially for queues in heavy traffic. This is illustrated in [10] and [11] for some simple queues such as the $E_2/M/1$ queue with two-stage Erlang interarrival time and exponential service time. This fact necessitates the need to look at more robust approximation techniques. One choice is to use rational approximation. A detailed account of rational approximation based on MacLaurin series can be found in [1] and [27]. Recently, the use of rational approximation to estimate performance measures for a class of discrete event stochastic systems was proposed in [11], which included the mean system time of the GI/G/1 queue also. Here we also use rational approximation to estimate the moments of the interdeparture time, waiting time and the system time. In what follows, we summarize the rational approximation procedure based on MacLaurin series.

Let Y be the variable to be approximated, which can represent D, W or T. Let the MacLaurin series coefficients of $E[Y^k]/k!$ be denoted by y_{kj}. Suppose we use the following rational approximant to approximate $E[Y^k]/k!$

$$\frac{E[Y^k]}{k!} \approx \frac{\sum_{i=0}^{m} q_{ki}\theta^i}{\sum_{i=0}^{n} r_{ki}\theta^i},$$

where q_{ki} and r_{ki} are coefficients to be determined (without loss of generality, we can select $r_{k0} = 1$). The order of the rational approximant (m,n) needs to be specified in advance. Usually, the larger m and n, the better the approximation, but of course, more computation is involved. To obtain the coefficients q_{ki} and r_{ki}, the first $m+n+1$ MacLaurin coefficients $y_{k0}, y_{k1}, \ldots, y_{k(m+n)}$ are needed. The coefficients q_{ki} and r_{ki} can be determined as follows. First solve for r_{ki} for $i = 1, 2, ..., n$ from the following set of linear equations:

$$\sum_{i=0}^{n-1} y_{k(m-n+j+i)} r_{k(n-i)} = -y_{k(m+j)} \text{ for } j = 1, ..., n. \tag{7}$$

Then, evaluate

$$q_{kj} = y_{kj} + \sum_{i=1}^{\min(j,n)} r_{ki} y_{k(j-i)} \text{ for } j = 1, ..., m. \tag{8}$$

The MacLaurin series obtained in the previous section can then be used to approximate $E[W^k]$, $E[T^k]$, and $E[D^k]$ based on the above rational approximation procedure. Note that the coefficients w_{ki} and t_{ki} depend on $(\beta_1, \ldots, \beta_i)$ and $(\alpha_0, \ldots, \alpha_{i-k-1})$ and in addition the coefficient d_{ki} also depends on $(\gamma_1, \ldots, \gamma_i)$, hence the order of rational approximation can be chosen in such a way as to depend on a specific number of parameters of the input distributions. To summarize, the following three steps have to be followed in order to use the higher order approximation procedure:

1. Select the parameters for the approximation: k, m and n.

2. Calculate the MacLaurin series coefficients iteratively using Equations (4)-(6).

3. Approximate the moments using Equations (7) and (8).

It is worth pointing out that for estimating higher moments of the waiting time using low orders of rational approximation, the method mentioned above may not be very good in some cases. This is due to the fact that for some queues the first several coefficients of the waiting time moments (w_{ij}) may be equal to zero and hence a low order approximation does not yield good estimates. This can be illustrated when the arrival distribution is mixed Erlang with b branches where branch i has probability P_i, number of stages n_i, and rate λ_i. Let $n^* = \min(n_1, n_2, \ldots, n_b)$. The derivatives of the arrival density function at 0 can then be expressed as

$$\alpha_k = f^{(k)}(0^+) = \sum_{\substack{i=1 \\ n_i \le k+1}}^{b} (-1)^{k+1-n_i} \binom{k}{n_i - 1} P_i \lambda_i^{k+1} \tag{9}$$

Then, it can be easily verified that $\alpha_k = 0$ for $k = 0, 1, \ldots, n^* - 1$. This means that $w_{ij} = 0$ for $j = 0, 1, \ldots, i + n^* - 1$. On the other hand, the first coefficient of the departure moments (d_{i0}) is always positive and the number of zero coefficients immediately following are very few. Hence, even a low order rational approximant yields a very good estimate of the moments of the departure process. This leads us to develop an alternate method to estimate the moments of the waiting time based on the moments of the departure process which is presented in the next section. However, we want to point out that if a very high order rational approximant is used, then both these methods will give very close results.

3.2 An Alternate Method for Estimating $E[W^k]$

In this section, we develop an alternate procedure to obtain the moments of the waiting time based on the moments of the departure process. We first note

$$D = \max(A - T, 0) + S$$

which leads to

$$E[\frac{D^k}{k!}] = \beta_k + \sum_{j=1}^{k} \beta_{k-j} (-1)^j \frac{[(T-A)^j - W^j]}{j!}$$

(for detailed derivation of the above equation see [15]). On the other hand

$$\begin{aligned} E[\frac{(T-A)^j - W^j}{j!}] &= \sum_{i=0}^{j} (-1)^{j-i} \gamma_{j-i} E[\frac{T^i}{i!}] - E[\frac{W^j}{j!}] \\ = \quad E[\frac{W^{j-1}}{(j-1)!}](\beta_1 - \gamma_1) &+ \sum_{i=0}^{j-2} [(\beta_{j-i} - \gamma_1 \beta_{j-1-i}) E[\frac{W^i}{i!}] + (-1)^{j-i} \gamma_{j-i} \sum_{l=0}^{i} \beta_{i-l} E[\frac{W^l}{l!}]] \end{aligned}$$

After considerable simplifications in the above equations we obtain

$$W_k = (-1)^{k+1} \frac{C_{k+1} + B_{k+1} - d_{k+1} + \beta_{k+1}}{\gamma_1 - \beta_1} \tag{10}$$

where

$$W_k = E[\frac{W^k}{k!}]$$

$$\begin{aligned} d_k &= E[\frac{D^k}{k!}] \\ C_k &= \sum_{j=1}^{k-1}(-1)^j \beta_{k-j} W_{j-1}(\beta_1 - \gamma_1) \\ B_k &= \sum_{j=1}^{k}(-1)^j \beta_{k-j} a_j \\ a_j &= \sum_{i=0}^{j-2}[(\beta_{j-i} - \gamma_1 \beta_{j-1-i})W_i + (-1)^{j-i}\gamma_{j-i} e_i] \\ e_i &= \sum_{l=0}^{i} \beta_{i-l} W_l \end{aligned}$$

We can observe that the k^{th} moment of the waiting time depends only on the $(k+1)^{th}$ moment of the interdeparture time and up to $k-1$ moments of the waiting time. For $k = 1$ we have

$$E[W] = \beta_1 + \frac{2\beta_2 + \gamma_2 - \beta_1\gamma_1 - E[D^2]/2}{\gamma_1 - \beta_1} \tag{11}$$

It is interesting to note that Equation (11) is used in [26] to estimate the second moment of the interdeparture time based on the estimation of $E[W]$.

Before closing this section, we provide some numerical results to illustrate the superiority of the alternate method we discussed above (Method 2) over the direct rational approximation given in the previous section (Method 1) for estimating the moments of the waiting time. For the GI/GI/1 queue used in our numerical study, the interarrival time is mixed Erlang with two branches and $(P_1, P_2) = (0.3, 0.7), (n_1, n_2) = (3, 3), (\lambda_1, \lambda_2) = (3.94, 1.18)$, and the service time is two-stage Erlang (E_2), hyperexponential with two branches (H_2), or uniform (U). The traffic intensity ($\rho = E[S]/E[A]$) was set to three different values: 0.2, 0.5, and 0.8. The parameters for the H_2 distributions are (P_1, P_2) = (0.6, 0.4) and (λ_1, λ_2) = (2.0, 4.0), (1.2, 0.8) and (1.0, 0.4), corresponding to $\rho = .2$, .5 and .8, respectively. The uniform distributions are between [0.0, 0.8], [0.0, 2.0] and [0.0, 3.2], corresponding to $\rho = .2$, .5 and .8, respectively. In the rational approximation, we took $m = n = 10$, and simulation results were obtained based on 40 replications, each with 50000 customers and a warm-up period of 10000 customers. All the simulation results in this paper are presented as simulation estimate $\pm$ standard deviation. Note that c_a^2 and c_s^2 denote the coefficients of variation of the interarrival and service distributions, respectively. The numerical results are presented in Table 1, which clearly indicate that Method 2 works much better in many cases, especially for cases in which ρ is large. (Another reason why Method 2 works better than Method 1 for large ρ is because when ρ approaches one, $E[W^k]$ approaches infinity while $E[D^k]$ is finite, i.e., $E[W^k]$ is normalized by a factor $1 - \rho$ via Equation (10).) So in the rest of this paper, we will use Method 2 for estimating the moments of the waiting time.

3.3 Comparison with Two-Moment Based Approximation

In this section, we compare our higher order approximation method with Whitt's two-moment based approximation, QNA, for the GI/G/1 queue. Further comparison between the two methods will be made in the next section for tandem queueing networks. Refer to [32] for the formulas to estimate the first two moments of the waiting time and the interdeparture time for the GI/G/1 queue. In what follows we present numerical results for three examples. In all the three examples, rational approximants with $n = m = 16$ were used, and simulation results were obtained based on 40 replications, each with 50000 customers and a warm-up period of 10000 customers. In the tables, c_a^2 and c_s^2 denote the coefficients of variation of the interarrival and service time distributions, respectively. We note that QNA can only be used to estimate the first two moments of W and D.

Example 1. The interarrival distribution is two-stage Erlang and the service distribution is either two-stage Erlang or two-branch hyperexponential. The parameters for the hyperexponential distributions are $(P_1, P_2) = (0.6, 0.4)$ and

$$(\lambda_1, \lambda_2) = \begin{cases} (2.0, 4.0), & \text{which corresponds to } \rho = 0.2 \\ (1.2, 0.8), & \text{which corresponds to } \rho = 0.5 \\ (1.0, 0.4), & \text{which corresponds to } \rho = 0.8. \end{cases}$$

The resulting queue is studied under three traffic conditions ($\rho = 0.2, 0.5, 0.8$). Numerical results are presented in Table 2. We notice that our method performs better in almost every single case. However, the improvement of our method over QNA is not significant in this example. This is because the coefficients of variation of the interarrival and service distributions are small ($c_a^2 = 0.5$ and $c_s^2 = 0.50, 1.50, 2.08, 1.42$), in which case QNA usually works reasonably well.

Example 2. In this example the interarrival and service distributions are both hyperexponential with four branches. The parameters for the interarrival time are given by

$$\begin{aligned} (P_1, P_2, P_3, P_4) &= (0.1, 0.2, 0.3, 0.4) \\ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= (0.3, 0.2, 3, 2), \end{aligned}$$

which yield $c_a^2 = 3.68$. For the service distribution

$$\begin{aligned} (P_1, P_2, P_3, P_4) &= (0.3, 0.1, 0.35, 0.25) \\ (\mu_1, \mu_2, \mu_3, \mu_4) &= \begin{cases} (13.2, 16, 1.2, 36), & \text{which corresponds to } \rho = 0.2 \\ (5.28, 6.4, 1.2, 14.4), & \text{which corresponds to } \rho = 0.5 \\ (3.3, 4, 0.3, 9), & \text{which corresponds to } \rho = 0.8. \end{cases} \end{aligned}$$

which yield $c_s^2 = 3.57$. Numerical results are given in Table 3. In this example, since c_a^2 and c_s^2 are much larger than one, QNA does not perform as well for the light and medium traffic cases, while our method produces very good approximations in all cases.

Example 3. In this example, mixed Erlang distributions are used for the interarrival and service distributions. The parameters for the interarrival distribution are given by

$$\begin{aligned} (P_1, P_2, P_3, P_4) &= (0.1, 0.2, 0.3, 0.4) \\ (n_1, n_2, n_3, n_4) &= (2, 3, 4, 5) \\ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= (0.5, 0.6, 0.7, 0.92), \end{aligned}$$

and for the service distribution $(P_1, P_2) = (0.6, 0.4)$, $(n_1, n_2) = (3, 2)$, and (μ_1, μ_2) is again set to three different values: (3, 2), (1.2, 0.8), (0.75, 0.5), which correspond to ρ

= 0.2, 0.5, 0.8, respectively. Numerical results are given in Table 4. We observe that our method gives much better approximations for $\rho = 0.2$ while for $\rho = 0.5$ and 0.8 the results based on both methods are very good. We should point out that in this example $c_a^2 = .313$ and $c_s^2 = .4$.

4 Queues in Tandem

In this section, we consider tandem queueing networks. Almost all of the methods reported in the literature have been based on second order approximation which considers only the first two moments of the interarrival and service distributions (see [32, 35] and the references therein). In this section, we apply our high order approximation method to tandem queueing networks. The main idea is to characterize the arrival process to each queue by using the moments of the departure process from the previous queue. The first queue in the network, which has the external arrival process, can be analyzed easily using the procedures described in the previous section. Then, the moments of the departure process from this queue can be matched to that of a mixed Erlang distribution, and this approximating distribution becomes the interarrival distribution for the next queue.

The issues involved are what distribution is selected by the moment matching procedure, how many moments to match, the presence of dependence among interdeparture times and how these affect the results, and how this method of approximation works for a network having a large number of queues. In this paper, we match moments to mixed Erlang distributions only using the results of [17, 21]. It should be noted that the quality of the approximations will depend on the type of moment matching procedure used. The numerical results presented later in this section show that this procedure produces very good results for tandem queueing networks. We also construct two examples of a two-node tandem network in which the correlations in the departure process form the first node are significant rendering MSA to be inaccurate since it does not take dependencies into account.

4.1 Matching Moments to Phase-type Distributions

The departure process from a GI/G/1 queue is a complicated point process which is not a renewal process in general. If the arrival process to the next queue is assumed to be a renewal process, its distribution can then be approximated by matching the moments of the departure process from the previous queue. We propose to select a mixed Erlang distribution. A number of methods have been proposed and studied in the literature to match moments to a mixed Erlang distribution (e.g., see [17], [18], [19], [21] and [28]). Here we resort to two procedures, which were developed in [17] and [21], respectively.

The first procedure is to analytically determine the distribution parameters and hence is very easy to use. In this procedure the first three moments are matched to a mixed Erlang distribution of common order with two branches. We briefly state the result here; for more details see [17]. Let M_i be the i^{th} non-central moment of the approximated distribution. To match the first three moments to a mixture of two Erlang distributions of common order (n), the parameters to be determined are n, p, λ_1 and λ_2. Suppose n^*

is the smallest integer that satisfies the following two inequalities:

$$n^* \geq \frac{M_1^2}{M_2 - M_1^2} \tag{12}$$

$$n^* \geq \frac{2(M_2 - M_1^2)^2 + M_1^2 M_2 - M_1 M_3}{M_1 M_3 - (M_2 - M_1^2)(M_2 - 2M_1^2)} \tag{13}$$

Choose any $n \geq n^*$. The other parameters of the mixed-Erlang distribution can then be determined based on the following equations:

$$\lambda_1 = \frac{2D_1}{D_2 + \sqrt{D_2^2 - 4D_1 D_3}} \tag{14}$$

$$\lambda_2 = \frac{2D_1}{D_2 - \sqrt{D_2^2 - 4D_1 D_3}} \tag{15}$$

$$p = \lambda_1 \frac{1 - \lambda_2 A}{\lambda_1 - \lambda_2}, \tag{16}$$

where

$$A = \frac{M_1}{n}$$

$$B = \frac{M_2}{n(n+1)}$$

$$C = \frac{M_3}{n(n+1)(n+2)}$$

$$D_1 = A^2 - B$$

$$D_2 = AB - C$$

$$D_3 = B^2 - AC$$

The second procedure is to use the nonlinear programming approach which matches 4 to 6 moments to a mixed Erlang distribution. The moments are specified in the constraints or in the objective function. The program MEFIT developed in [21] can be used to achieve this. After selecting the number of branches and the number of stages in each branch, one has to specify the initial mixing probabilities and the rates. The program solves a nonlinear problem which approximates the gradients by finite differences. In the numerical examples that follow we have used this procedure to study the improvement in the performance measures by matching upto 6 moments of the departure process.

4.2 Two Queues in Tandem

Three examples of a network with two queues in tandem are considered in this section. In these three examples, the first queues are the same as the GI/GI/1 queues considered in Examples 1-3 in Section 3.3. The second queue is analyzed based on MSA by treating it independently and determining its interarrival distribution by matching the moments of the departure process from the first queue. Let η denote the number of departure process moments matched. The approximating interarrival distributions to the second queue, obtained by matching the moments of the departure processes, which are mixed Erlang distributions, and their parameters can be found in [8] for all the three examples.

Example 4. In this example we use the $E_2/E_2/1$ queue studied in Example 1 as its first queue. The service distribution of the second queue is either E_2 or H_2. Note that the c_a^2 and c_s^2 values are the same as in Example 1. The moments of the waiting time and the interdeparture time for the second queue are shown in Tables 5, 6 and 7. QNA performs reasonably well in this example, but MSA is better in most cases and the improvement is prominent for the higher moments. We also note that increasing η improves the MSA estimates in low traffic.

Example 5. In this example we use the GI/GI/1 queue studied in Example 2 as its first queue. The service distribution of the second queue is also the same as the service distribution of the first queue (hyperexponential with four branches). Note that the c_a^2 and c_s^2 values are the same as in Example 2. This example captures the case with large coefficients of variation in the arrival and service distributions. Tables 8, 9 and 10 list the performance measure estimates for the second queue. For this example, the MSA estimates are extremely accurate and are significantly better than that of QNA which does not perform well when the arrival distribution has high variability. By increasing η the improvement of MSA is not significant, even though the values converge toward that of simulation for the most part.

Example 6. In this example we use the GI/GI/1 queue studied in Example 3 as its first queue. The service distribution of the second queue is again the same as that of the first queue (mixed Erlang). Note that the c_a^2 and c_s^2 values are the same as in Example 3. The performance measure estimates are provided in Tables 11, 12 and 13. Similar to the previous example, MSA also performs better than QNA for this example, especially in light and medium traffic cases, and increasing η in general improves the MSA estimates, but its effect is not significant.

To summarize, we point out that the numerical results obtained in the above examples indicate that the higher order approximation (MSA) yields better estimates of the performance measures in general in comparison with the two-moment based approximation procedure. MSA with $\eta = 3$ gives excellent accuracy for most cases. In some cases, reasonable improvements can be achieved by setting $\eta = 4$, but the extra computation of increasing η further is not rewarding enough. Though not included here, higher moments of the waiting and interdeparture times can also be approximated with remarkable accuracy using MSA. We shall see that similar conclusions can be drawn for the more complicated examples considered in the next section.

4.3 Long Tandem Networks

We numerically investigate the effect of MSA on long tandem queueing networks here by studying three examples. First we consider a tandem network with 8 nodes (Examples 7 and 8). Numerical results of two cases are presented which differ in the external arrival distributions. The estimates of QNA and MSA with matching 3 to 6 moments are compared with that of computer simulation. Secondly, in Example 9, we consider a tandem network of 9 queues studied in [6]. The mean waiting time estimated by MSA (with $\eta = 3, 4$), QNA and Sequential Bottleneck Decomposition method (SBD with $n = 4$) are compared with that of simulation. Note that in Tables 14, 16 and 18, c_a^2 and c_s^2 denote the squared coefficient of variation of the external arrival process and the service distributions, respectively.

Example 7. The external interarrival distribution and the service distributions at queues 1, 5, 6, and 8 are two-branch hyperexponential distributions with balanced means, and the service distributions at queues 2, 3, 4, and 7 are two-branch mixed Erlang. Four traffic intensities ($\rho = .2, .4, .6, .8$) are randomly assigned to each queue. Numerical results are given in Tables 14 and 15. MSA performs very well for this network and the average absolute % error is significantly lower than that of QNA. We need to point out that for this network MEFIT was unable to find a mixed Erlang distribution matching the sixth moment for the last two nodes and hence the corresponding estimates based on six moments matching are omitted. Also, for the last two nodes mixed Erlang distributions found based on MEFIT do not match the fourth and fifth moments well (with large errors), so the corresponding MSA estimated are shown with a $*$ in Tables 30 and 31. In fact, in some cases errors in MSA estimates partially result from errors on moment matching based on MEFIT, which could be the reason why MSA estimates based on high moment matching in some cases are worse than those based on the three moment matching.

Example 8. The external interarrival distribution and the service distributions at queues 2, 3, 7, and 8 are two-branch mixed Erlang distributions and the service distributions at queues 1, 4, 5, and 6 are hyperexponential with balanced means. Four traffic intensities ($\rho = .2, .4, .6, .8$) are randomly assigned to each queue. The numerical results are shown in Tables 16 and 17 which list the first and second moments of the waiting time, respectively. For this example MEFIT was unable to find a mixed Erlang distribution matching the sixth moment for most nodes, therefore we did not include the MSA estimations for $\eta = 6$. MSA with $\eta = 3$ performs well and some improvement can be made by increasing η. Notice that MSA outperforms QNA for this network as well. Even though not listed, higher moments of the waiting times can also be estimated with very good accuracy using MSA.

Example 9. (Section 3.3 in [6]) The external interarrival distribution is a balanced hyperexponential distribution with mean 1 and the squared coefficient of variation 8. The service distributions are exponential with mean 0.6 except the last queue with mean 0.9. The results are given in Table 18. SBD denotes the Sequential Bottleneck Decomposition method developed in [6]. Note that MSA performs better than the other two approximations for this network as well.

4.4 The Role of Correlations

Most queueing network approximations, including ours, have been based on the renewal assumption on the arrival process to each queue. Our numerical results in the previous two sections seem to indicate that this assumption is reasonable in most cases. In fact, our simulation results show that in all the examples we studied in the previous sections the correlations of the departure processes are very small (in most cases they are almost equal to zero). However, the examples provided in [25] show that in some cases correlations in arrival and service processes do affect performance measures such as the average waiting time. In this section, we want to investigate the effect of correlation of departure (arrival) process on queueing network approximations. We provide two examples, both are two-queue tandem networks, one with positive correlation in the departure process from its first queue and the other with negative correlation. However, we want to point out that we had to work very hard to find these examples with large enough correlations in their departure (arrival) processes. Most examples we had come up with in our study do have very small correlations. We also point out that MSA can also be used to obtain the correlations of the departure process from the GI/G/1 queue (see [15]).

Example 10. This is a two-queue tandem network. The arrival process to the first queue has two-branch hyperexponential distribution (H_2) and the service distributions for both queues are Erlang with 20 stages (E_{20}). We consider three traffic intensities (0.2, 0.5, and 0.8) for each queue. For the three different traffic intensities, the lag-1 correlation coefficients of the departure process from the first queue are equal to 0.015, 0.056, and 0.107, respectively. The estimates on $E[W]$ for the second queue are given in Table 19. When $\rho = 0.8$ for the first queue, MEFIT was unable to find a mixed Erlang distribution with $\eta = 6$ and hence the corresponding estimates are not given in Table 19. Note that in Table 19, c_a^2 and c_s^2 denote the squared coefficient of variation of the arrival process in the first queue and the service distributions in both queues (which are the same), respectively. Our numerical results show that in all the cases MSA gives better estimates than that of QNA. However, compared with the examples in the previous sections, it is clear MSA does not perform as well. This is mainly due to the effect of the correlations. So correlation does appear to have a significant effect.

Example 11. This is again a tandem network with two queues. The interarrival distribution of the first queue is Erlang with 20 stages (E_{20}) and both queues have two-stage Erlang (E_2) service distributions. Again three traffic intensities (0.2, 0.5, and 0.8) are considered for each queue. For the three different traffic intensities, the lag-1 correlation coefficients of the departure process from the first queue are equal to -0.224, -0.284, and -0.124, respectively. The estimates on $E[W]$ for the second queue are given in Table 20. When $\rho = 0.8$ for the first queue, MEFIT was unable to find mixed Erlang distributions with $\eta = 4, 5, 6$ and hence the corresponding estimates are omitted in Table 20. Note that in Table 20, c_a^2 and c_s^2 denote the squared coefficient of variation of the arrival process in the first queue and the service distributions in both queues (which are the same), respectively. For this example, our numerical results show that MSA only does well in light traffic case, and furthermore, MSA does not always perform better than QNA. In fact, in some cases QNA estimates are better. This again implies that correlation has significant effect on both MSA and QNA.

5 Conclusions

In this paper we proposed a higher order approximation for queueing networks. Our numerical results show that in most cases the higher order approximation performs better than the two-moment based approximation QNA; especially it provides significantly better approximations for queues with low traffic intensities or with large squared coefficients of variation for interarrival and service distributions. Our simulation results also suggest that sufficiently large improvements can be obtained by considering the third and fourth moments and the contributions of the higher moments are less prominent. Finally, since the renewal assumption is used in approximating the arrival process to each node in most queueing network approximations such as QNA and our approximation, we also studied the effect of correlations of departure/arrival processes on queueing approximations (We would like to point out that modifications to QNA have been done in [35] to take the correlations into account. But, we did not include it in the QNA approximation in this paper). Our results show that in most cases the correlations are very small and thus can be ignored, therefore, the renewal assumption is reasonable; however, in the few cases in which the correlations are large, they do have significant impact on the accuracy of queueing approximations.

The following are possible future research directions which are related to our work in this paper.

1. Since the improvement in the approximation is significant upto matching the first four moments, it would be interesting to develop some simple and efficient procedures to find distributions that match the first four moments.

2. In this paper, we have only studied tandem queueing networks. It is of great interest to investigate how the high order approximation can be applied to general queueing networks. Recently, methods for analyzing superposed phase renewal processes and the $\Sigma Ph_i/Ph/1$ queue have been developed in [3, 4]. By combining their methods with MSA we may be able to provide better ways of analyzing general queueing networks.

3. As we already mentioned, whenever the correlation of departure/arrival processes has significant impact on performance measures, they should be taken into consideration. However, very little is known about how to analyze queues with correlated arrival processes. One possibility is to use G/G/1 queues with Markov-modulated arrival and service processes. In fact, the technique of MSA to study G/G/1 queues with Markov-modulated arrival and service processes was illustrated in [9] and [38].

Acknowledgments

This research is supported in part by the National Science Foundation under grants DDM-9215368, and EID-9212122. The authors wish to thank Professor Mary Johnson for the MEFIT and MEMOM programs and Dr. Ward Whitt for some literature pertaining to QNA. The authors are grateful to the Associate Editor and two anonymous referees for their invaluable comments and suggestions.

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Table 1
Comparison of Methods 1 and 2 for the waiting time moments (E[A] = 2.0, c_a^2 = .554)

Service dist.	ρ	c_s^2	Perf. measure	Method 1	Method 2	Simulation
			$E[W]$	.029	.029	.028 ± .001
E_2	0.2	0.500	$E[W^2]$	.016	.016	.016 ± .001
			$E[W^3]$	.013	.013	.013 ± .001
			$E[W]$	.480	.427	.446 ± .009
E_2	0.5	0.500	$E[W^2]$	.920	.895	.968 ± .040
			$E[W^3]$	3.956	2.895	3.08 ± .251
			$E[W]$	-41.542	2.997	3.25 ± .153
E_2	0.8	0.500	$E[W^2]$	3.141	25.762	28.614 ± 4.190
			$E[W^3]$	-47.108	328.871	400.143 ± 196.000
			$E[W]$	.056	.056	.056 ± .002
H_2	0.2	1.500	$E[W^2]$	.060	.060	.060 ± .004
			$E[W^3]$	.098	.098	.097 ± .011
			$E[W]$	.695	.714	.716 ± .022
H_2	0.5	2.083	$E[W^2]$	2.126	2.593	2.614 ± .183
			$E[W^3]$	30.259	14.157	14.604 ± 2.020
			$E[W]$	4.213	5.940	6.136 ± .429
H_2	0.8	1.422	$E[W^2]$	.151	101.785	103.321 ± 16.200
			$E[W^3]$	-5.328	2553.106	2619.280 ± 782.000
			$E[W]$	.021	.021	.021 ± .000
U	0.2	0.333	$E[W^2]$	.008	.008	.008 ± .000
			$E[W^3]$	.004	.003	.004 ± .000
			$E[W]$	.485	.348	.368 ± .006
U	0.5	0.333	$E[W^2]$	4.270	1.041	.599 ± .021
			$E[W^3]$	.097	4.644	1.388 ± .095
			$E[W]$	-.621	2.304	2.712 ± .092
U	0.8	0.333	$E[W^2]$	-.353	17.635	18.810 ± 1.580
			$E[W^3]$	.041	182.022	195.689 ± 33.000

Table 2
Numerical results for Example 1 (E[A] = 2.0, $c_a^2 = 0.5$)

Service dist.	ρ	c_s^2	Perfor. measure	QNA	MSA	Simulation
			$E[W]$	.026	.024	.024 ± .001
			$E[W^2]$	.010	.013	.014 ± .001
E_2	0.2	0.500	$E[W^3]$	-	.011	.011 ± .001
			$E[D^2]$	6.000	6.083	6.082 ± .045
			$E[D^3]$	-	24.595	24.533 ± .323
			$E[W]$	.423	.390	.389 ± .009
			$E[W^2]$	1.075	.805	.798 ± .038
E_2	0.5	0.500	$E[W^3]$	-	2.443	2.406 ± .198
			$E[D^2]$	6.000	6.219	6.225 ± .043
			$E[D^3]$	-	25.973	25.946 ± .320
			$E[W]$	3.069	2.987	2.973 ± .125
			$E[W^2]$	25.935	23.981	23.977 ± 2.592
E_2	0.8	0.500	$E[W^3]$	-	287.884	292.822 ± 72.524
			$E[D^2]$	6.000	6.173	6.178 ± .053
			$E[D^3]$	-	25.863	25.837 ± .369
			$E[W]$	.068	.049	.048 ± .002
			$E[W^2]$	.010	.051	.050 ± .003
H_2	0.2	1.500	$E[W^3]$	-	.082	.079 ± .009
			$E[D^2]$	6.148	6.224	6.227 ± .041
			$E[D^3]$	-	25.642	25.620 ± .313
			$E[W]$	.713	.658	.657 ± .016
			$E[W^2]$	3.415	2.315	2.312 ± .125
H_2	0.5	2.083	$E[W^3]$	-	12.345	12.418 ± 1.442
			$E[D^2]$	6.583	6.851	6.857 ± .045
			$E[D^3]$	-	31.925	31.920 ± .373
			$E[W]$	6.018	5.905	5.887 ± .387
			$E[W^2]$	106.350	96.103	95.069 ± 14.188
H_2	0.8	1.422	$E[W^3]$	-	2256.049	2289.126 ± 617.019
			$E[D^2]$	8.360	8.568	8.570 ± .063
			$E[D^3]$	-	57.888	57.965 ± 1.127

Table 3
Numerical results for Example 2 (E[A] = 2.0, c_a^2 = 3.68)

ρ	c_s^2	Perfor. measure	QNA	MSA	Simulation
		$E[W]$	.298	.450	.448 ± .015
		$E[W^2]$	.925	1.146	1.133 ± .084
0.2	3.573	$E[W^3]$	-	4.398	4.312 ± .605
		$E[D^2]$	12.477	12.080	12.035 ± .226
		$E[D^3]$	-	163.597	161.516 ± 5.738
		$E[W]$	2.987	3.623	3.604 ± .118
		$E[W^2]$	34.432	41.084	40.762 ± 3.530
0.5	3.573	$E[W^3]$	-	700.413	694.402 ± 136.100
		$E[D^2]$	12.416	11.381	11.375 ± .202
		$E[D^3]$	-	141.952	141.562 ± 5.612
		$E[W]$	19.282	20.507	19.982 ± 1.631
		$E[W^2]$	927.887	973.272	916.619 ± 172.500
0.8	3.573	$E[W^3]$	-	69394.740	61883.810 ± 20905.000
		$E[D^2]$	12.302	11.511	11.537 ± .251
		$E[D^3]$	-	135.645	136.091 ± 5.730

Table 4
Numerical results for Example 3 (E[A] = 5.0, c_a^2 = 0.313)

ρ	c_s^2	Perfor. measure	QNA	MSA	Simulation
		$E[W]$	.015	.027	.026 ± .001
		$E[W^2]$	.011	.031	.030 ± .002
0.2	0.4	$E[W^3]$	-	.053	0.051 ± .006
		$E[D^2]$	32.935	33.435	33.535 ± .214
		$E[D^3]$	-	275.198	275.944 ± 2.800
		$E[W]$	.573	.579	.573 ± .011
		$E[W^2]$	2.286	2.289	2.255 ± .082
0.5	0.4	$E[W^3]$	-	13.311	13.062 ± .944
		$E[D^2]$	33.392	34.954	35.034 ± .202
		$E[D^3]$	-	307.114	307.563 ± 2.708
		$E[W]$	5.097	5.035	4.981 ± .143
		$E[W^2]$	73.609	73.450	72.851 ± 5.034
0.8	0.4	$E[W^3]$	-	1600.222	1610.193 ± 231.090
		$E[D^2]$	34.241	35.559	35.689 ± .206
		$E[D^3]$	-	327.048	328.222 ± 3.234

Table 5
Numerical results for the second queue in Example 4 with $\rho = 0.2$ at the first queue

Service dist.	ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
		$E[W]$	.026	.023	.033	.029	.030	.029 ± .001
		$E[W^2]$	.010	.013	.020	.017	.017	.016 ± .001
E_2	0.2	$E[W^3]$	-	.010	.017	.014	.014	.013 ± .001
		$E[D^2]$	6.000	6.169	6.137	6.150	6.147	6.148 ± .046
		$E[D^3]$	-	25.201	25.053	25.113	25.101	25.035 ± .325
		$E[W]$	.423	.406	.416	.415	.415	.403 ± .009
		$E[W^2]$	1.075	.848	.887	.880	.879	.825 ± .037
E_2	0.5	$E[W^3]$	-	2.612	2.780	2.746	2.744	2.467 ± .208
		$E[D^2]$	6.000	6.271	6.250	6.253	6.253	6.252 ± .047
		$E[D^3]$	-	26.341	26.283	26.283	26.289	26.226 ± .340
		$E[W]$	3.069	3.073	3.074	3.074	3.074	3.013 ± .132
		$E[W^2]$	25.935	25.237	25.273	25.271	25.259	24.476 ± 2.649
E_2	0.8	$E[W^3]$	-	309.329	310.604	310.575	310.394	300.782 ± 67.320
		$E[D^2]$	6.000	6.184	6.184	6.183	6.183	6.183 ± .049
		$E[D^3]$	-	25.959	25.991	25.984	25.983	25.956 ± .388
		$E[W]$	.068	.049	.057	.054	.055	.053 ± .001
		$E[W^2]$	.010	.052	.060	.057	.058	.056 ± .003
H_2	0.2	$E[W^3]$	-	.084	.097	.092	.093	.089 ± .008
		$E[D^2]$	6.148	6.305	6.280	6.289	6.288	6.289 ± .045
		$E[D^3]$	-	26.220	26.118	26.156	26.149	26.105 ± .334
		$E[W]$	.713	.676	.683	.682	.682	.665 ± .020
		$E[W^2]$	3.415	2.397	2.431	2.426	2.425	2.312 ± .158
H_2	0.5	$E[W^3]$	-	12.925	13.144	13.107	13.102	12.110 ± 1.664
		$E[D^2]$	6.583	6.898	6.883	6.886	6.886	6.886 ± .048
		$E[D^3]$	-	32.288	32.250	32.257	32.257	32.221 ± .402
		$E[W]$	6.018	5.977	5.978	5.977	5.976	5.838 ± .347
		$E[W^2]$	106.350	98.112	98.442	98.520	98.461	94.740 ± 13.080
H_2	0.8	$E[W^3]$	-	3629.825	2447.067	2446.522	2443.942	2311.704 ± 580.900
		$E[D^2]$	8.360	8.581	8.580	8.581	8.582	8.570 ± .072
		$E[D^3]$	-	57.583	57.937	58.056	58.020	57.624 ± 1.231

Table 6
Numerical results for the second queue in Example 4 with $\rho = 0.5$ at the first queue

Service dist.	ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
		$E[W]$	.026	.032	.028	.028	.029	.029 ± .001
		$E[W^2]$	.010	.018	.016	.016	.016	.016 ± .001
E_2	0.2	$E[W^3]$	-	.015	.013	.013	.013	.013 ± .001
		$E[D^2]$	6.000	6.278	6.291	6.289	6.288	6.286 ± .044
		$E[D^3]$	-	26.436	26.498	26.487	26.484	26.405 ± .331
		$E[W]$	.423	.439	.435	.435	.436	.416 ± .009
		$E[W^2]$	1.075	.953	.937	.941	.947	.866 ± .044
E_2	0.5	$E[W^3]$	-	3.045	2.971	2.967	3.002	2.656 ± .311
		$E[D^2]$	6.000	6.342	6.350	6.349	6.348	6.344 ± .049
		$E[D^3]$	-	27.303	27.326	27.329	27.342	27.197 ± .381
		$E[W]$	3.069	3.198	3.190	3.190	3.185	3.001 ± .121
		$E[W^2]$	25.935	27.099	27.088	27.262	27.483	23.495 ± 2.110
E_2	0.8	$E[W^3]$	-	343.372	341.798	343.482	347.013	269.163 ± 43.174
		$E[D^2]$	6.000	6.221	6.227	6.227	6.231	6.220 ± .046
		$E[D^3]$	-	26.463	26.568	26.776	27.118	26.366 ± .369
		$E[W]$	.068	.058	.055	.055	.055	.055 ± .002
		$E[W^2]$	.010	.061	.058	.059	.059	.058 ± .003
H_2	0.2	$E[W^3]$	-	.099	.094	.095	.096	.092 ± .009
		$E[D^2]$	6.148	6.414	6.424	6.422	6.422	6.421 ± .044
		$E[D^3]$	-	27.464	27.505	27.498	27.496	27.450 ± .325
		$E[W]$	.713	.708	.705	.705	.705	.686 ± .019
		$E[W^2]$	3.415	2.552	2.539	2.540	2.538	2.423 ± .145
H_2	0.5	$E[W^3]$	-	13.978	13.894	13.898	13.945	13.021 ± 1.373
		$E[D^2]$	6.583	6.970	6.976	6.975	6.976	6.972 ± .048
		$E[D^3]$	-	33.246	33.263	33.259	33.258	33.181 ± .441
		$E[W]$	6.018	6.104	6.101	6.099	6.117	5.931 ± .352
		$E[W^2]$	106.350	102.069	101.775	101.651	102.380	97.192 ± 13.184
H_2	0.8	$E[W^3]$	-	2571.569	2579.866	2720.757	2619.256	2410.459 ± 606.007
		$E[D^2]$	8.360	8.616	8.618	8.620	8.606	8.605 ± .071
		$E[D^3]$	-	58.470	58.177	58.089	58.570	58.293 ± 1.193

Table 7
Numerical results for the second queue in Example 4 with $\rho = 0.8$ at the first queue

Service dist.	ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
		$E[W]$	.026	.027	.023	.025	.026	.026 ± .001
		$E[W^2]$	.010	.015	.013	.014	.014	.014 ± .001
E_2	0.2	$E[W^3]$	-	.012	.010	.011	.011	.012 ± .001
		$E[D^2]$	6.000	6.248	6.258	6.251	6.250	6.252 ± .052
		$E[D^3]$	-	26.410	26.458	26.424	26.422	26.367 ± .354
		$E[W]$	.423	.416	.413	.415	.415	.404 ± .010
		$E[W^2]$	1.075	.880	.868	.874	.882	.835 ± .043
E_2	0.5	$E[W^3]$	-	2.741	2.690	2.720	2.739	2.531 ± .247
		$E[D^2]$	6.000	6.341	6.347	6.343	6.343	6.340 ± .053
		$E[D^3]$	-	27.426	27.443	27.427	27.452	27.345 ± .403
		$E[W]$	3.069	3.133	3.136	3.106	3.084	3.000 ± .147
		$E[W^2]$	25.935	26.113	26.146	25.687	26.145	24.000 ± 2.779
E_2	0.8	$E[W^3]$	-	325.423	325.790	321.022	324.875	286.452 ± 64.070
		$E[D^2]$	6.000	6.226	6.223	6.247	6.265	6.213 ± .055
		$E[D^3]$	-	26.571	26.565	26.454	27.326	26.366 ± .402
		$E[W]$	.068	.053	.050	.052	.052	.051 ± .002
		$E[W^2]$	.010	.055	.053	.054	.055	.053 ± .003
H_2	0.2	$E[W^3]$	-	.089	.085	.088	.088	.082 ± .009
		$E[D^2]$	6.148	6.385	6.393	6.387	6.387	6.378 ± .052
		$E[D^3]$	-	27.435	27.467	27.444	27.442	27.291 ± .390
		$E[W]$	.713	.687	.685	.687	.686	.667 ± .015
		$E[W^2]$	3.415	2.452	2.442	2.452	2.449	2.322 ± .112
H_2	0.5	$E[W^3]$	-	13.289	13.231	13.344	13.394	12.248 ± 1.158
		$E[D^2]$	6.583	6.965	6.969	6.965	6.966	6.960 ± .049
		$E[D^3]$	-	33.323	33.334	33.324	33.327	33.179 ± .464
		$E[W]$	6.018	6.046	6.045	6.032	6.048	5.831 ± .329
		$E[W^2]$	106.350	100.455	100.396	99.974	100.476	92.960 ± 12.775
H_2	0.8	$E[W^3]$	-	2514.573	2512.995	2477.485	2579.824	2194.630 ± 597.722
		$E[D^2]$	8.360	8.616	8.617	8.627	8.614	8.594 ± .071
		$E[D^3]$	-	58.548	58.497	58.267	58.533	58.055 ± 1.173

Table 8
Numerical results for the second queue in Example 5 with $\rho = 0.2$ at the first queue

ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
0.2	$E[W]$	.298	.411	.429	.439	.447	.462 $\pm$ .012
	$E[W^2]$	.924	1.015	1.074	1.107	1.133	1.181 $\pm$ .065
	$E[W^3]$	-	3.776	4.055	4.209	4.333	4.535 $\pm$ .439
	$E[D^2]$	12.466	11.774	11.729	11.699	11.683	11.617 $\pm$ .219
	$E[D^3]$	-	156.097	155.654	155.428	155.228	153.122 $\pm$ 5.747
0.5	$E[W]$	2.986	3.475	3.510	3.500	3.522	3.584 $\pm$ .130
	$E[W^2]$	34.395	38.420	39.028	38.838	39.218	40.228 $\pm$ 3.291
	$E[W^3]$	-	638.319	652.275	647.669	656.501	681.257 $\pm$ 106.100
	$E[D^2]$	12.407	11.212	11.164	11.164	11.144	11.121 $\pm$ .231
	$E[D^3]$	-	137.001	136.918	137.129	136.871	135.271 $\pm$ 5.548
0.8	$E[W]$	19.271	19.942	20.067	19.397	20.028	20.418 $\pm$ 1.718
	$E[W^2]$	926.827	926.956	932.099	900.279	934.611	966.817 $\pm$ 208.600
	$E[W^3]$	-	64619.330	65141.784	62766.134	65408.611	68311.420 $\pm$ 28926.000
	$E[D^2]$	12.298	11.468	11.428	11.779	11.453	11.398 $\pm$ .271
	$E[D^3]$	-	134.901	130.795	138.799	135.171	133.033 $\pm$ 5.770

Table 9
Numerical results for the second queue in Example 5 with $\rho = 0.5$ at the first queue

ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
0.2	$E[W]$	.297	.402	.449	.464	.466	.489 $\pm$ .013
	$E[W^2]$	.919	.987	1.141	1.189	1.195	1.288 $\pm$ .073
	$E[W^3]$	-	3.647	4.372	4.606	4.632	5.113 $\pm$ .555
	$E[D^2]$	12.466	11.097	10.975	10.936	10.932	10.925 $\pm$ .199
	$E[D^3]$	-	135.502	134.469	134.148	134.113	133.860 $\pm$ 5.411
0.5	$E[W]$	2.976	3.344	3.404	3.413	3.412	3.499 $\pm$ .134
	$E[W^2]$	34.203	36.083	37.054	37.167	37.159	39.045 $\pm$ 3.498
	$E[W^3]$	-	585.245	606.831	609.858	609.830	665.111 $\pm$ 129.679
	$E[D^2]$	12.407	10.727	10.630	10.615	10.616	10.655 $\pm$ .227
	$E[D^3]$	-	122.111	122.139	122.042	122.055	122.066 $\pm$ 5.680
0.8	$E[W]$	19.210	19.189	19.209	19.274	19.206	20.281 $\pm$ 1.784
	$E[W^2]$	921.270	862.269	863.753	864.306	860.860	973.380 $\pm$ 230.248
	$E[W^3]$	-	58132.121	58303.181	58389.838	58165.009	70915.739 $\pm$ 34133.768
	$E[D^2]$	12.298	11.255	11.242	11.200	11.243	11.189 $\pm$ .221
	$E[D^3]$	-	128.110	128.585	126.032	125.987	127.984 $\pm$ 5.239

Table 10
Numerical results for the second queue in Example 5 with $\rho = 0.8$ at the first queue

ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
0.2	$E[W]$	.295	.471	.504	.494	.494	.511 $\pm$ .012
	$E[W^2]$	.909	1.218	1.333	1.297	1.299	1.368 $\pm$ .063
	$E[W^3]$	-	4.749	5.319	5.139	5.150	5.516 $\pm$ .494
	$E[D^2]$	12.466	11.047	10.962	10.989	10.987	11.015 $\pm$ .251
	$E[D^3]$	-	127.508	126.797	127.022	127.009	127.691 $\pm$ 5.726
0.5	$E[W]$	2.959	3.512	3.594	3.530	3.521	3.649 $\pm$ .128
	$E[W^2]$	33.849	39.026	40.033	39.535	40.674	41.999 $\pm$ 3.383
	$E[W^3]$	-	651.362	670.876	638.547	699.434	734.399 $\pm$ 122.851
	$E[D^2]$	12.407	10.584	10.450	10.555	10.569	10.551 $\pm$ .227
	$E[D^3]$	-	115.017	114.220	115.561	118.688	115.152 $\pm$ 5.430
0.8	$E[W]$	19.096	19.321	27.969	18.751	16.835	20.229 $\pm$ 2.072
	$E[W^2]$	910.998	882.719	1277.561	887.548	716.708	961.435 $\pm$ 250.274
	$E[W^3]$	-	58971.415	86779.392	62178.264	49290.899	69226.617 $\pm$ 34502.708
	$E[D^2]$	12.298	11.300	5.714	11.668	12.906	11.156 $\pm$ .210
	$E[D^3]$	-	127.791	97.146	159.718	85.749	124.795 $\pm$ 4.381

Table 11
Numerical results for the second queue in Example 6 with $\rho = 0.2$ at the first queue

ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
0.2	$E[W]$	.016	.029	.034	.035	.038	.035 $\pm$ .001
	$E[W^2]$	.011	.033	.040	.041	.046	.041 $\pm$ .002
	$E[W^3]$	-	.055	.069	.070	.081	.069 $\pm$ .006
	$E[D^2]$	33.019	34.006	33.962	33.956	33.931	34.038 $\pm$ .225
	$E[D^3]$	-	284.785	284.321	284.254	284.015	284.763 $\pm$ 3.080
0.5	$E[W]$	.579	.628	.642	.644	.643	.613 $\pm$ .014
	$E[W^2]$	2.327	2.556	2.650	2.663	2.653	2.429 $\pm$ .108
	$E[W^3]$	-	15.250	15.992	16.092	16.058	14.074 $\pm$ 1.219
	$E[D^2]$	33.458	35.290	35.226	35.216	35.220	35.310 $\pm$ .244
	$E[D^3]$	-	313.446	312.922	312.844	312.848	313.086 $\pm$ 3.576
0.8	$E[W]$	5.131	5.261	5.214	5.232	5.299	5.021 $\pm$ .187
	$E[W^2]$	74.469	80.014	80.158	80.508	80.297	72.353 $\pm$ 6.253
	$E[W^3]$	-	1799.089	1812.688	1820.352	1817.291	1550.256 $\pm$ 253.123
	$E[D^2]$	34.272	35.693	35.808	35.770	35.636	35.734 $\pm$ .257
	$E[D^3]$	-	331.563	334.832	334.591	329.253	329.382 $\pm$ 3.963

Table 12
Numerical results for the second queue in Example 6 with $\rho = 0.5$ at the first queue

ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
0.2	$E[W]$	.018	.054	.040	.040	.043	.040 ± .001
	$E[W^2]$	.015	.069	.048	.048	.052	.047 ± .003
	$E[W^3]$	-	.124	.083	.082	.102	.079 ± .008
	$E[D^2]$	33.458	35.321	35.431	35.434	35.414	35.475 ± .200
	$E[D^3]$	-	314.365	315.539	315.569	315.359	315.340 ± 2.724
0.5	$E[W]$	.614	.747	.737	.737	.718	.687 ± .014
	$E[W^2]$	2.547	3.302	3.178	3.175	3.163	2.792 ± .107
	$E[W^3]$	-	21.246	20.144	20.120	19.525	16.493 ± 1.143
	$E[D^2]$	33.800	36.219	36.267	36.268	36.361	36.257 ± .215
	$E[D^3]$	-	337.050	337.016	337.022	338.666	335.500 ± 3.123
0.8	$E[W]$	5.305	5.612	5.828	5.829	5.417	5.211 ± .195
	$E[W^2]$	79.054	93.104	94.296	94.029	90.426	74.592 ± 6.019
	$E[W^3]$	-	2262.024	2295.536	2286.449	2183.664	1561.684 ± 217.824
	$E[D^2]$	34.437	36.509	36.076	36.075	36.899	36.068 ± .226
	$E[D^3]$	-	352.158	339.322	338.448	358.914	338.606 ± 3.415

Table 13
Numerical results for the second queue in Example 6 with $\rho = 0.8$ at the first queue

ρ	Perf. measure	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
0.2	$E[W]$	.024	.035	.037	.037	.042	.037 ± .001
	$E[W^2]$	.022	.041	.044	.044	.052	.044 ± .002
	$E[W^3]$	-	.070	.076	.076	.092	.076 ± .008
	$E[D^2]$	34.272	36.079	36.060	36.062	36.022	36.169 ± .201
	$E[D^3]$	-	335.875	335.672	335.686	335.258	336.465 ± 3.186
0.5	$E[W]$	.679	.731	.735	.735	.735	.702 ± .015
	$E[W^2]$	2.976	3.132	3.159	3.159	3.168	2.921 ± .136
	$E[W^3]$	-	19.730	19.966	20.011	20.066	17.853 ± 1.890
	$E[D^2]$	34.437	36.900	36.881	36.881	36.883	36.930 ± .210
	$E[D^3]$	-	355.793	355.641	355.649	355.741	354.939 ± 3.469
0.8	$E[W]$	5.628	5.976	5.976	5.968	5.986	5.456 ± .185
	$E[W^2]$	87.875	98.693	98.746	98.842	98.698	82.382 ± 6.506
	$E[W^3]$	-	2435.964	2438.822	2558.740	2433.656	1855.567 ± 300.403
	$E[D^2]$	34.743	36.385	36.384	36.401	35.364	36.429 ± .226
	$E[D^3]$	-	350.382	350.499	351.462	349.585	349.230 ± 3.996

Table 14
$E[W]$ estimates for the network of 8 queues in tandem in Example 7 (E[A] = 1.0, $c_a^2 = 6.0$)

Node	ρ	c_s^2	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
1	.2	6.00	.300	.301	.301	.301	.301	.303 ± .002
2	.6	0.67	3.000	2.562	2.602	2.666	2.670	2.677 ± .019
3	.4	0.41	.598	.324	.228	.245	.258	.259 ± .001
4	.8	0.42	6.261	5.925	6.200	6.254	6.338	7.770 ± .118
5	.4	6.00	1.004	1.030	1.022	1.064	.951	1.000 ± .010
6	.6	6.00	3.709	3.898	3.986	3.949	3.914	4.374 ± .046
7	.2	0.32	.098	.058	.144*	.171*	-	.087 ± .000
8	.8	6.00	15.142	16.349	19.351*	18.463*	-	17.678 ± .410
Avg. abs. % error			25.75	13.57	15.21	17.89	5.86	

Table 15
$E[W^2]$ estimates for the network of 8 queues in tandem in Example 7

Node	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation
1	.979	.895	.895	.895	.895	.911 ± .016
2	22.078	15.478	15.913	16.693	16.729	16.602 ± .251
3	1.140	.384	.227	.252	.272	.274 ± .002
4	84.708	76.067	79.762	73.827	80.118	144.381 ± 5.743
5	6.277	7.059	7.005	7.228	6.503	7.101 ± .157
6	50.990	58.165	59.871	58.817	58.504	69.925 ± 1.722
7	.059	.020	.074*	.097*	-	.038 ± .000
8	596.913	685.244	743.234*	672.438*	-	757.862 ± 38.629
Avg. abs. % error	64.13	21.29	22.53	30.43	12.09	

Table 16
$E[W]$ estimates for the network of 8 queues in tandem in Example 8 (E[A] = 1.0, $c_a^2 = 0.572$)

Node	ρ	c_s^2	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	Simulation
1	.4	6.00	.852	.825	.825	.825	.834 ± .007
2	.8	0.42	2.979	2.870	2.861	2.908	2.903 ± .022
3	.2	0.32	.025	.011	.008	.008	.006 ± .000
4	.6	6.00	3.036	2.994	3.008	2.985	2.951 ± .034
5	.2	6.00	.216	.229	.253	.217	.260 ± .002
6	.8	6.00	14.058	14.401	14.748	14.500	13.015 ± .236
7	.4	0.41	.700	.496	.463	.442	.574 ± .002
8	.6	0.67	2.160	1.909	2.210	1.733	2.168 ± .012
Avg. abs. % error			47.90	16.89	9.38	13.34	

Table 17
$E[W^2]$ estimates for the network of 8 queues in tandem in Example 8

Node	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	Simulation
1	6.787	5.448	5.448	5.448	5.573 ± .098
2	20.505	19.479	19.366	19.689	19.521 ± .374
3	.008	.002	.002	.002	.001 ± .000
4	42.380	40.088	40.199	39.931	38.899 ± .970
5	.558	.651	.705	.624	.771 ± .013
6	521.326	549.374	559.370	587.937	441.774 ± 23.293
7	1.536	.757	.678	.628	1.014 ± .009
8	11.722	19.038	13.939	7.585	12.260 ± .153
Avg. abs. % error	104.66	28.26	23.55	29.26	

Table 18
$E[W]$ estimates for a network of 9 queues in series in Example 9 (E[A] = 1.0, c_a^2 = 8.0)

Node	ρ	c_s^2	QNA	SBD	MSA $\eta = 3$	MSA $\eta = 4$	Simulation
1	0.6	1.0	4.05	4.05	3.44	3.44	3.28 ± .12
2	0.6	1.0	2.92	1.82	2.08	2.05	2.32 ± .10
3	0.6	1.0	2.19	1.49	1.70	1.71	1.91 ± .07
4	0.6	1.0	1.73	1.19	1.50	1.52	1.72 ± .07
5	0.6	1.0	1.43	1.10	1.38	1.39	1.60 ± .06
6	0.6	1.0	1.24	1.06	1.29	1.30	1.48 ± .06
7	0.6	1.0	1.12	1.03	1.22	1.23	1.42 ± .05
8	0.6	1.0	1.04	1.01	1.17	1.18	1.41 ± .07
9	0.9	1.0	8.90	36.45	13.08	13.03	30.12 ± 5.07
Avg. abs. % error			23.22	26.05	16.98	16.70	

Table 19
$E[W]$ estimates for Queue 2 in Example 10 (E[A] = 1.0, c_a^2 = 1.083)

ρ Queue 1	ρ Queue 2	c_s^2	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation $E[W]$
	.2	.05	0.027	0.027	.024	.020	.018	0.008 ± .000
.2	.5	.05	0.273	0.266	.248	.239	.237	0.252 ± .008
	.8	.05	1.747	1.719	1.692	1.692	1.694	1.769 ± .090
	.2	.05	0.022	0.007	.004	.002	.001	0.000 ± .000
.5	.5	.05	0.219	0.133	.105	.093	.080	0.076 ± .002
	.8	.05	1.400	1.187	1.145	1.175	1.828	1.520 ± .085
	.2	.05	0.012	0.002	.000	.000	-	0.000 ± .000
.8	.5	.05	0.118	0.054	.019	.018	-	0.006 ± .000
	.8	.05	0.755	0.551	.351	.346	-	0.489 ± .036

Table 20
$E[W]$ estimates for Queue 2 in Example 11 (E[A] = 1.0, c_a^2 = 0.05)

ρ Queue 1	ρ Queue 2	c_s^2	QNA	MSA $\eta = 3$	MSA $\eta = 4$	MSA $\eta = 5$	MSA $\eta = 6$	Simulation $E[W]$
	.2	.5	0.014	0.002	.001	.000	.000	0.001 ± .000
.2	.5	.5	0.142	0.290	.026	.067	.068	0.062 ± .002
	.8	.5	0.909	1.678	1.785	1.600	1.560	1.715 ± .020
	.2	.5	0.017	0.036	.032	.030	.030	0.006 ± .000
.5	.5	.5	0.166	0.273	.265	.201	.200	0.106 ± .002
	.8	.5	1.060	1.661	1.639	1.378	1.323	0.819 ± .025
	.2	.5	0.021	0.012	-	-	-	0.010 ± .000
.8	.5	.5	0.210	0.231	-	-	-	0.157 ± .003
	.8	.5	1.341	1.553	-	-	-	1.062 ± .036