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Theory and Methodology

Approximations for the departure process of the G/G/1 queue with Markov-modulated arrivals

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Abstract

The interarrival time random variables at the nodes of queueing networks are typically correlated. The analysis, by ignoring the correlations, may not be very accurate in many instances, especially when the correlations are high. The G/G/1 queue with Markov-modulated arrivals is one of the few models that can be used to study queues with non-renewal arrival processes. The waiting time of the Markov-modulated G/G/1 queue has been studied widely in the literature. In this paper, we study its departure process. We derive the MacLaurin series for the moments and the lag-1 autocorrelation of the departure process. The coefficients of the MacLaurin series are calculated through simple recursive equations, which are then used to approximate the moments and the lag-1 autocorrelation based on Padé approximation. Several numerical examples are provided which show that our method gives very good estimates. Our results on the departure process can be potentially used to develop better approximation methods for general queueing networks. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Queueing; Markov-modulated process; G/G/1 queue; MacLaurin series; Autocorrelations

1. Introduction

The performance evaluation of many complex manufacturing and communication systems is achieved by modeling them as queues or in general as queueing networks. It is well known that the arrival and service processes in a wide variety of cases are correlated. Examples of these systems include many communication systems and transport mechanisms developed recently; such as the routers that are deployed in Internet backbone networks and asynchronous transfer mode (ATM) based broadband

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integrated services digital networks (B-ISDN) where voice, data and video are sent which induce large correlations. Empirical and in some cases, theoretical evidence has been established about the significant impact of the correlations on the performance measures. Livny et al. (1993) studied single server queues with correlated arrival and service distributions which were generated by the transform expand sample (TES) method (see Melamed, 1991). Their simulation results of this system suggest that the mean waiting time varies drastically with increased correlations. In the analysis of queueing networks, the arrival process to any node is usually approximated as a renewal process. However, in many situations, the accuracy of the performance measures deteriorate whenever the correlations are significant (e.g., see Girish, 1996; Girish and Hu, 1996).

Many papers have addressed the issue of correlated queues. Fendick et al. (1989) investigated correlated single server queues under heavy traffic conditions. Simple approximations for the lag-1 correlation and the first two moments of the departure process in a tandem queueing network have been studied by Whitt (1984). The waiting time distribution and queue size for a queue with gamma distributed arrival process with a special Markovian structure and i.i.d. exponentially distributed service times were studied by Tin (1985). One of the few models that have been developed to study queues with non-renewal arrival processes is the G/G/1 queue with Markov-modulated arrival process. For a review on recent work on Markov-modulated queues, refer to Neuts (1992, 1995), Prabhu and Zhu (1989), Szekli et al. (1993, 1994), and the references therein. Despite the tremendous progress achieved in the study of the Markov-modulated G/G/1 queue, little work has been done to study its departure process. If we consider the queue to be part of a queueing network, then the analysis of the network will be possible only if its departure process is characterized which essentially becomes the arrival process to the downstream node(s). Our derivation of the lag-1 autocorrelation and the moments of the departure process can potentially be used in the development of more accurate and robust approximations for queueing networks.

Analysis of queueing systems based on the MacLaurin series method has been shown to be effective and useful. Gong and Hu (1992) and Hu (1996) showed that the MacLaurin series of the moments of the waiting, system and the interdeparture time for the GI/G/1 queue can be calculated easily using simple recursive formulas. Sufficient improvements in the accuracy of the performance measures can be obtained for queueing networks using this approach (see Girish and Hu, 1996, 2000). Recently Zhu and Li (1993) derived the MacLaurin series of the moments of the waiting and the system time for the Markov-modulated G/G/1 queue. The performance measures were approximated using polynomial approximation. In this paper, we derive the MacLaurin series for the moments of the departure process and the lag-1 autocorrelation. We then use these MacLaurin series to obtain the moments and the lag-1 autocorrelation based on Padé approximation, which gives much better results than those based on polynomial approximation.

In order to test the applicability and accuracy of the proposed method, a number of numerical examples are studied in this paper. The performance measures of interest are the lag-1 autocorrelation and the second and third moments of the departure process as well as the first three moments of the waiting time. These measures are approximated by rational functions based on their MacLaurin series and are compared with that of simulation estimates. In all cases, the estimates at light and medium traffic are very accurate; but the accuracy of these estimates decreases as the traffic intensity gets closer to one. We also observe that the departure process moments can be estimated with significantly higher accuracy than the waiting time moments.

The rest of this paper is organized as follows. In Section 2, the MacLaurin series of the moments of the interdeparture time is derived and in Section 3, the MacLaurin series of the lag-1 correlation of the departure process is derived. In Section 4, we show how to approximate the performance measures based on their MacLaurin series using Padé approximation. Some numerical results are presented in Section 5 and some discussions and conclusion are given in Section 6.

2. The moments of the departure process

We consider the G/G/1 queue with Markov-modulated arrival process and i.i.d. service distribution which is independent of the arrival process. The underlying Markov chain for the arrival process is denoted by $\{J_n;\ n \geqslant 0\}$ which we assume to be irreducible, aperiodic and recurrent on a state space E. The transition probability matrix is denoted by P with elements $(p_{ij}),\ i,j \in E$ and the stationary probability of being in state $i \in E$ is given by π_i. Moreover, this Markov chain is assumed to be stationary which implies that $\text{Prob}\{J_0 = i\} = \pi_i$, for $i \in E$. The interarrival time between customers n and $n+1$ is denoted by A_n and the service time of customer n is denoted by S_n and A and S are the generic interarrival time and service time. The conditional distributions, $F_{ij}(x)$ for A_n are given by

$$F_{ij}(x) = P\{A_n \leqslant x \,|\, J_n = i,\ J_{n+1} = j\}.$$

In this paper, for convenience we work with the quantity, $E[A_1A_2]$ (and similarly for the interdeparture time in Section 3), which along with the first two moments can be used to determine the lag-1 covariance and autocorrelation coefficient given as follows:

$$\text{Lag-1 Covariance} = E[A_1A_2] - (E[A])^2,$$

$$\text{Lag-1 Autocorrelation Coefficient} = \frac{E[A_1A_2] - (E[A])^2}{E[A^2] - (E[A])^2}.$$

Note that we loosely use the term autocorrelation in this paper and the reader is cautioned to keep in mind the correct definition. Then the moments of A_n and the autocorrelation, $E[A_1A_2]$ can be calculated from

$$E[A_n^k] = \sum_{i\in E}\sum_{j\in E} k!\pi_i p_{ij}\gamma_{ijk}, \quad E[A_1A_2] = \sum_{i\in E}\sum_{j\in E}\sum_{q\in E} \pi_i p_{ij} p_{jq}\gamma_{ij1}\gamma_{jq1},$$

where

$$\gamma_{ijk} \equiv E_{ij}\left[\frac{A_n^k}{k!}\right] = E\left[\frac{A_n^k}{k!}\,|J_n = i,\ J_{n+1} = j\right] = \int_0^\infty \frac{x^k}{k!}\,\mathrm{d}F_{ij}(x). \tag{1}$$

Let $f_{ij}(x)$ denote the conditional interarrival density function. We assume that $f_{ij}(x)$ can be expanded as a MacLaurin series over $x \in [0,\infty)$ for all $i,\ j \in E$. An example is the class of phase type distributions. We now parameterize the queue by introducing a scale parameter θ in the service time, which makes the service time of customer n to be θS_n, where θ is independent of S_n. Note that $\{S_n,\ n = 1,2,\ldots\}$ are independent and identically distributed. With the introduction of θ, the moments of the waiting time and the moments and autocorrelations of the departure process all become functions of θ. The MacLaurin series for the moments of the waiting and system time have been derived by Zhu and Li (1993). We concentrate on the moments of the generic stationary interdeparture time and the lag-1 autocorrelation. Let D_n denote the interdeparture time between customers n and $n+1$. Let W_n and T_n be the waiting and system time, respectively of the nth customer. Let A, D, W and T represent the generical random variables interarrival time, interdeparture time, waiting time and the system time, respectively. Starting from Lindley's equation and following the approach adopted in Hu (1996) and Zhu and Li (1993) we have:

$$W_n = (T_{n-1} - A_{n-1})^+, \tag{2}$$

$$T_n = W_n + \theta S_n, \tag{3}$$

$$D_n = (A_n - T_n)^+ + \theta S_{n+1}, \tag{4}$$

where $(y)^+ = \max(y, 0)$. Moreover, we introduce the following notation for simplicity:

$$\beta_k \equiv \frac{E[S^k]}{k!}, \quad \alpha_{ijk} \equiv f_{ij}^{(k)}(0), \quad J(n_1, n_2, \ldots, n_m; i_1, i_2, \ldots, i_m) \equiv 1_{\{J_{n_1}=i_1; J_{n_2}=i_2; \ldots; J_{n_m}=i_m\}},$$
$$X_n^l(n_1, n_2, \ldots, n_m; i_1, i_2, \ldots, i_m) \equiv E\left[\frac{X_n^l}{l!} J(n_1, n_2, \ldots, n_m; i_1, i_2, \ldots, i_m)\right]$$

for $m \geqslant 1$, where $f_{ij}^{(k)}(0)$ is the kth derivative of $f_{ij}(x)$ at $x = 0$ and the random variable X stands for A, D, W or T. Then for any $n \geqslant 1$, $k \geqslant 1$, $i \in E$, $j \in E$, we have

$$\begin{aligned} D_n^k(n, n+1; i, j) &= \frac{1}{k!} E\left[((A_n - T_n)^+ + \theta S_{n+1})^k J(n, n+1; i, j)\right] \\ &= \sum_{l=0}^{k} \theta^{k-l} \beta_{k-l} E\left[\frac{((A_n - T_n)^+)^l}{l!} J(n, n+1; i, j)\right]. \end{aligned} \tag{5}$$

Note that for $l \geqslant 1$

$$((A_n - T_n)^+)^l = (-1)^l \left[(T_n - A_n)^l - W_{n+1}^l\right]. \tag{6}$$

Substituting Eq. (6) in (5), we get

$$\begin{aligned} D_n^k(n, n+1; i, j) &= \pi_i p_{ij} \beta_k \theta^k + \sum_{l=1}^{k} \theta^{k-l} \beta_{k-l} \frac{(-1)^l}{l!} E\left[((T_n - A_n)^l - W_{n+1}^l) J(n, n+1; i, j)\right] \\ &= \pi_i p_{ij} \sum_{l=0}^{k} \beta_l \gamma_{ij(k-l)} \theta^l + \sum_{l=1}^{k} (-1)^l \beta_{k-l} \theta^{k-l} \left[\sum_{h=1}^{l} (-1)^{l-h} \gamma_{ij(l-h)} T_n^h(n, n+1; i, j) \right. \\ &\qquad \left. - W_{n+1}^l(n, n+1; i, j)\right]. \end{aligned} \tag{7}$$

This queueing system being stable and ergodic is tantamount to the following:

$$(W_n, T_n, D_n, J_n, J_{n+1}) \xrightarrow{d} (W, T, D, \tilde{J}, J)$$

($\xrightarrow{d}$ means converging in distribution). If we express the moments of D, W and T as

$$E\left[\frac{D^k}{k!} 1_{\{\tilde{J}=i; J=j\}}\right] = \sum_{m=0}^{\infty} d_{ijkm} \theta^m, \tag{8}$$

$$E\left[\frac{W^k}{k!} 1_{\{J=j\}}\right] = \sum_{m=0}^{\infty} w_{jkm} \theta^m, \tag{9}$$

$$E\left[\frac{T^k}{k!} 1_{\{J=j\}}\right] = \sum_{m=0}^{\infty} t_{jkm} \theta^m, \tag{10}$$

then based on Eqs. (7)–(10), we have

$$d_{ijkm} = \pi_i p_{ij} \sum_{l=\max(1,k-m)}^{k} (-1)^l \beta_{k-l} \left(\sum_{h=1}^{l} (-1)^{l-h} \gamma_{ij(l-h)} t_{jh(m-k+l)} - w_{jh(m-k+l)} \right) + \begin{cases} \pi_i p_{ij} \beta_m \gamma_{ij(k-m)}, & m \leqslant k, \\ 0, & m > k. \end{cases} \tag{11}$$

Note that the above expression involves the terms t_{jkm} and w_{jkm} which can be calculated iteratively from the following equations (for their derivations, refer to Zhu and Li, 1993):

$$t_{jkm} = \begin{cases} \beta_k, & m = k, \\ \sum_{n=1}^{k} \beta_{k-n} w_{jn(m-k+n)}, & m > k, \\ 0, & m < k, \end{cases} \tag{12}$$

$$w_{jkm} = \begin{cases} \sum_{q \in E} \sum_{n=0}^{m-k-1} p_{qj} \alpha_{qjn} t_{q(k+1+n)m}, & m > k, \\ 0, & m \leqslant k. \end{cases} \tag{13}$$

We would like to point out that the common factor $\pi_i p_{ij}$ in Eq. (11) comes about from the fact that Eq. (7) has $J(n, n+1; i, j)$ on the right-hand side, whereas Eqs. (9) and (10) depend only on $1_{\{J=j\}}$.

3. The lag-1 autocorrelation of the departure process

In Section 2, we derived the MacLaurin series for the moments of the interdeparture process. In this section, we derive the MacLaurin series for the lag-1 autocorrelation of the departure process. The same idea can be extended to derive the higher lag correlations. We start with the following well-known relationship:

$$D_n = T_{n+1} - T_n + A_n \quad \text{for } n \geqslant 1. \tag{14}$$

From Eq. (14) we have

$$E[D_1 D_2 - A_1 A_2] = E\left[T_1 T_2 + T_2 T_3 - T_1 T_3 - T_2^2 + A_1 T_3 - A_1 T_2 + A_2 T_2 - A_2 T_1\right]. \tag{15}$$

Now, we express the expectations on the right-hand side as functions of θ for $h \geqslant 1,\ i \in E,\ j \in E,\ q \in E$ as follows:

$$E[T_1 T_3 J(2,3;i,j) \mid J_1 = q] = E[T_1 W_3 J(2,3;i,j) \mid J_1 = q] + \beta_1 E[T_1 J(2,3;i,j) \mid J_1 = q]\theta,$$

where

$$\begin{aligned} E[T_1 W_3 J(2,3;i,j) \mid J_1 = q] &= E\left[T_1 (T_2 - A_2)^+ J(2,3;i,j) \mid J_1 = q\right] \\ &= E\left[\int_0^{T_2} T_1 (T_2 - x) f_{ij}(x)\, \mathrm{d}x J(2,3;i,j) \mid J_1 = q\right] \\ &= E\left[\int_0^{T_2} T_1 (T_2 - x) \sum_{m=0}^{\infty} \frac{f_{ij}^{(m)}(0)}{m!} x^m\, \mathrm{d}x J(2,3;i,j) \mid J_1 = q\right] \\ &= \sum_{m=0}^{\infty} \alpha_{ijm} E\left[\frac{T_1 T_2^{m+2}}{(m+2)!} J(2,3;i,j) \mid J_1 = q\right], \end{aligned}$$

$$E\left[\frac{T_1 W_2^h}{h!} J(2,3;i,j) \,|\, J_1 = q\right] = E\left[\frac{T_1((T_1 - A_1)^+)^h}{h!} J(2,3;i,j) \,|\, J_1 = q\right]$$
$$= \sum_{m=0}^{\infty} (m+h+2) p_{ij} \alpha_{qim} E\left[\frac{T_1^{m+h+2}}{(m+h+2)!} J(2;i) \,|\, J_1 = q\right],$$

$$E\left[\frac{T_1 T_2^k}{k!} J(2,3;i,j) \,|\, J_1 = q\right] = p_{ij} \beta_k E[T_1 J(2;i) \,|\, J_1 = q] \theta^k + \sum_{h=1}^{k} \beta_{k-h} E\left[\frac{T_1 W_2^h}{h!} J(2,3;i,j) \,|\, J_1 = q\right] \theta^{k-h}$$

and in a very similar way, we can show that:

$$E[A_1 T_3 J(2,3;i,j) \,|\, J_1 = q] = E[A_1 W_3 J(2,3;i,j) \,|\, J_1 = q] + p_{qi} p_{ij} \beta_1 \gamma_{qi1} \theta, \tag{16}$$

$$E[A_1 W_3 J(2,3;i,j) \,|\, J_1 = q] = \sum_{m=0}^{\infty} \alpha_{ijm} E\left[\frac{A_1 T_2^{m+2}}{(m+2)!} J(2,3;i,j) \,|\, J_1 = q\right], \tag{17}$$

$$E\left[\frac{A_1 W_2^h}{h!} J(2,3;i,j) \,|\, J_1 = q\right] = \sum_{m=0}^{\infty} (m+1) p_{ij} \alpha_{qim} E\left[\frac{T_1^{m+h+2}}{(m+h+2)!} J(2;i) \,|\, J_1 = q\right], \tag{18}$$

$$E\left[\frac{A_1 T_2^k}{k!} J(2,3;i,j) \,|\, J_1 = q\right] = p_{qi} p_{ij} \beta_k \gamma_{qi1} \theta^k + \sum_{h=1}^{k} \beta_{k-h} E\left[\frac{A_1 W_2^h}{h!} J(2,3;i,j) \,|\, J_1 = q\right] \theta^{k-h}. \tag{19}$$

It is easy to verify that:

$$E[A_2 T_2 J(2,3;i,j) \,|\, J_1 = q] = \gamma_{ij1} \left(p_{qi} p_{ij} \beta_1 \theta + \sum_{m=0}^{\infty} p_{ij} \alpha_{qim} E\left[\frac{T_1^{m+2}}{(m+2)!} J(2;i) \,|\, J_1 = q\right] \right), \tag{20}$$

$$E[T_2 T_3 J(2,3;i,j) \,|\, J_1 = q] = p_{qi} \sum_{s \in E} E[T_1 T_2 J(2,3;j,s) \,|\, J_1 = i], \tag{21}$$

$$E[A_2 T_1 J(2,3;i,j) \,|\, J_1 = q] = p_{ij} \gamma_{ij1} E[T_1 J(2;i) \,|\, J_1 = q]. \tag{22}$$

Note that in our derivations, we assumed that the expectations and integrals can be freely exchanged. We have not investigated the conditions under which this is true in this paper. But, this issue has been studied by Hu (1996) for the case of the GI/GI/1 queue in which the arrival process is a renewal process. It is likely that the results in Hu (1996) can be extended to the Markov-modulated G/G/1 queue. The numerical studies that we conducted show that these formulae are correct. Nevertheless, this serves as a possible future research problem. Since the expectations in Eqs. (15)–(22) are all functions of θ, let

$$E\left[\frac{T_1T_2^k}{k!}J(2,3;i,j)\,|\,J_1=q\right]=\sum_{m=0}^{\infty}t_{qijkm}^{(12)}\theta^m,\quad E\left[\frac{A_1T_2^k}{k!}J(2,3;i,j)\,|\,J_1=q\right]$$
$$=\sum_{m=0}^{\infty}a_{qijkm}^{(12)}\theta^m,\quad E\left[\frac{T_1T_3^k}{k!}J(2,3;i,j)\,|\,J_1=q\right]$$
$$=\sum_{m=0}^{\infty}t_{qijkm}^{(13)}\theta^m,\quad E\left[\frac{A_rT_n^k}{k!}J(2,3;i,j)\,|\,J_1=q\right]$$
$$=\sum_{m=0}^{\infty}a_{qijkm}^{(rn)}\theta^m,\quad E[(D_1D_2-A_1A_2)J(2,3;i,j)\,|\,J_1=q]=\sum_{m=0}^{\infty}d_{qijm}^{(12)}\theta^m.$$

If the coefficients of the powers of θ are compared, then for $m\geqslant 0$,

$$d_{qijm}^{(12)}=t_{qij1m}^{(12)}+p_{qi}\sum_{s\in E}t_{ijs1m}^{(12)}-t_{qij1m}^{(13)}-2p_{qi}t_{ij2m}+a_{qij1m}^{(13)}-a_{qij1m}^{(12)}+a_{qij1m}^{(22)}-p_{ij}\gamma_{ij1}t_{qi1m},\tag{23}$$

where

$$t_{qijkm}^{(12)}=\begin{cases}p_{ij}\beta_k t_{qi1(m-k)}, & \\ +p_{ij}\sum_{h=1}^{k}\beta_{k-h}\sum_{r=0}^{m-k-2}(r+h+2)\alpha_{qir}t_{qi(r+h+2)(m-k+h)}, & m>k+1,\\ p_{ij}\beta_k t_{qi11}, & m=k+1,\\ 0, & m<k+1,\end{cases}$$

$$a_{qijkm}^{(12)}=\begin{cases}p_{ij}\sum_{h=1}^{k}\beta_{k-h}\sum_{r=0}^{m-k-1}(r+1)\alpha_{qir}t_{qi(r+h+2)(m-k+h)}, & m>k+1,\\ 0, & m=k+1,\\ p_{qi}p_{ij}\beta_k\gamma_{qi1}, & m=k,\\ 0, & m<k,\end{cases}$$

$$t_{qij1m}^{(13)}=\begin{cases}p_{ij}\beta_1 t_{qi1(m-1)}+\sum_{r=0}^{m-3}\alpha_{ijr}t_{qij(r+2)m}^{(12)}, & m>2,\\ p_{ij}\beta_1 t_{qi11}, & m=2,\\ 0, & m<2,\end{cases}$$

$$a_{qij1m}^{(13)}=\begin{cases}\sum_{r=0}^{m-2}\alpha_{ijr}a_{qij(r+2)m}^{(12)}, & m>1,\\ p_{qi}p_{ij}\beta_1\gamma_{qi1}, & m=1,\\ 0, & m<1,\end{cases}$$

$$a_{qij1m}^{(22)}=\begin{cases}\sum_{r=0}^{m-2}p_{ij}\alpha_{qir}\gamma_{ij1}t_{qi(r+2)m}, & m>1,\\ p_{qi}p_{ij}\beta_1\gamma_{ij1}, & m=1,\\ 0, & m<1.\end{cases}$$

Moreover, the other coefficients, t_{ijkm} and w_{ijkm} can be calculated iteratively via the following:

$$t_{ijkm}=\begin{cases}p_{ij}\beta_k, & m=k,\\ \sum_{n=1}^{k}\beta_{k-n}w_{ijn(m-k+n)}, & m>k,\\ 0, & m<k,\end{cases}\tag{24}$$

$$w_{ijkm} = \begin{cases} \sum_{q\in E} \sum_{n=0}^{m-k-1} p_{ij}\alpha_{qin}t_{qi(k+1+n)m}, & m > k, \\ 0, & m \leqslant k. \end{cases} \tag{25}$$

Note that $E[(D_1D_2 - A_1A_2)J(2,3;i,j)\,|\,J_1 = q]$ can be approximated using polynomial or rational functions based on its MacLaurin series. Our numerical studies indicate that in most cases polynomial approximations converge only for a very limited range of θ and the convergence rates are also very poor in general. On the other hand, Padé approximation is robust and has excellent convergence properties. In Section 4, we will present the Padé approximation method.

4. Padé approximation

The simplest way to approximate the moments and the autocorrelations based on their MacLaurin series is to use polynomial functions. But, it is well known that the convergence properties of polynomial approximations are poor, especially for queues in heavy traffic. Therefore, we use Padé approximation. A detailed account of Padé approximation based on MacLaurin series can be found in Petrushev and Popov (1987) and Baker (1975). Recently, Gong et al. (1995) proposed the use of Padé approximation to estimate performance measures for a class of discrete event stochastic systems, including the mean system time of the GI/G/1 queue. Padé approximation is also used in Girish and Hu (1996) for estimating the performance measures in tandem queueing networks. In what follows, we summarize the Padé approximation procedure based on MacLaurin series.

Let Y be the variable to be approximated and let the MacLaurin series coefficients of $E[Y^k]/k!$ be denoted by y_{kl}. Suppose we use the following $[L/M]$ Padé approximant to approximate $E[Y^k]/k!$:

$$\frac{E[Y^k]}{k!} \approx \frac{\sum_{h=0}^{L} q_{kh}\theta^h}{\sum_{h=0}^{M} r_{kh}\theta^h},$$

where q_{kh} and r_{kh} are coefficients to be determined (without loss of generality, we can select $r_{k0} = 1$). To obtain the coefficients q_{kh} and r_{kh}, the first $L + M + 1$ MacLaurin coefficients $y_{k0}, y_{k1}, \ldots, y_{k(L+M)}$ are needed. The coefficients q_{kh} and r_{kh} can be determined by solving for r_{kh} for $h = 1, 2, \ldots, M$ from the following set of linear equations:

$$\sum_{h=0}^{M-1} y_{k(L-M+l+h)} r_{k(M-h)} = -y_{k(L+l)} \quad \text{for } \; l = 1, \ldots, M. \tag{26}$$

Now set $q_{k0} = y_{k0}$ and then evaluate

$$q_{kl} = y_{kl} + \sum_{h=1}^{\min(l,M)} r_{kh}y_{k(l-h)} \quad \text{for } \; l = 1, \ldots, L. \tag{27}$$

It can be shown that the computational complexity of estimating the first k moments of the interdeparture time by a $[L/M]$ Padé approximant is

$$\mathrm{O}[k((L + M + 1)^2 g^2 + M^3 + (L + 1)^2)], \tag{28}$$

where g is the total number of states in the underlying Markov chain.

5. Numerical results

In this section, some examples of the Markov-modulated G/G/1 queue are presented. We calculate the lag-1 autocorrelation, second and third non-central moments of the interdeparture time and the first three moments of the waiting time. The MacLaurin series of the waiting times are evaluated using Eqs. (12) and (13) and that for the departure process are evaluated through Eqs. (11) and (23). The moments are then approximated by rational functions based on Padé approximation which is described in Section 4. In the MacLaurin series approximation (MSA), we set $[L/M] = [10/10]$ and the simulation estimates are obtained for 40 replications each with 50 000 customers with warm-up time of 10 000 customers. The standard deviations for the simulation estimates are also shown in the tables. Of course, the accuracy of our estimates can be improved if we increase L and M. The tables containing the numerical results also include $E[A_1A_2]$, $E[A^2]$, $E[A^3]$ in order to help the reader compare the corresponding departure process parameters with these. Example 8 studies the effect of changing the number of coefficients of the Padé approximant.

Example 1. The Markov chain has two states and the transition probability matrix is given by

$$P = \begin{pmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{pmatrix}.$$

The conditional arrival distributions are exponential with rates $= (0.51, 0.50, 0.52, 0.45)$. The service distribution is three-staged Erlang with rates, $\mu = (7.5, 3.0, 1.875)$ corresponding to traffic intensities, $\rho = (0.2, 0.5, 0.8)$, respectively. Numerical results are given in Table 1. The lag-1 autocorrelation and the second and third moments of the interdeparture time evaluated using MSA are extremely accurate in all cases. The first three moments of the waiting time are very close to the simulation estimates when $\rho = 0.2$; the first moment is also close to the simulation estimates when $\rho = 0.5$ and 0.8; but $E[W^2]$ and $E[W^3]$ are highly inaccurate under medium and heavy traffic.

Table 1
Numerical results for Example 1

ρ	Performance measure	MSA	Simulation	Arrival measure	Value
0.20	$E[D_1D_2]$	4.057	4.079±0.004	$E[A_1A_2]$	4.023
	$E[D^2]$	7.953	7.920±0.009	$E[A^2]$	8.065
	$E[D^3]$	47.986	47.363±0.125	$E[A^3]$	48.792
	$E[W]$	0.068	0.067±0.000		
	$E[W^2]$	0.039	0.038±0.000		
	$E[W^3]$	0.032	0.030±0.000		
0.50	$E[D_1D_2]$	4.124	4.147±0.005	$E[A_1A_2]$	4.023
	$E[D^2]$	7.360	7.352±0.008	$E[A^2]$	8.065
	$E[D^3]$	42.655	42.130±0.112	$E[A^3]$	48.792
	$E[W]$	0.676	0.683±0.001		
	$E[W^2]$	1.649	1.718±0.009		
	$E[W^3]$	5.908	6.620±0.053		
0.80	$E[D_1D_2]$	4.078	4.119±0.005	$E[A_1A_2]$	4.023
	$E[D^2]$	6.218	6.301±0.008	$E[A^2]$	8.065
	$E[D^3]$	30.717	30.386±0.089	$E[A^3]$	48.792
	$E[W]$	4.292	4.449±0.017		
	$E[W^2]$	43.859	51.396±0.532		
	$E[W^3]$	670.805	954.012±17.886		

Example 2. The arrival process in this example is the same as in Example 1. But, the service distributions are hyperexponential with three branches. The parameters are

$$(P_1, P_2, P_3) = (0.3, 0.2, 0.5), \quad (\mu_1, \mu_2, \mu_3) = \begin{cases} (1.5, 2.0, 5.0), & \text{which corresponds to } \rho = 0.2, \\ (0.6, 0.8, 2.0), & \text{which corresponds to } \rho = 0.5, \\ (0.375, 0.5, 1.25), & \text{which corresponds to } \rho = 0.8. \end{cases}$$

Numerical results are given in Table 2. All the performance measures are very accurate in light and medium traffic conditions; but they deviate from the simulation estimates when $\rho = 0.8$, notably the higher moments of the waiting time.

Example 3. The Markov chain has two states and the transition probability matrix is given by

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}.$$

The conditional arrival distributions are all two-branched hyperexponential with parameters

States		Prob.		Stages		Rates	
J_n	J_{n+1}	P_1	P_2	n_1	n_2	λ_1	λ_2
1	1	0.3	0.7	1	1	0.30	0.75
1	2	0.8	0.2	1	1	0.50	0.347
2	1	0.9	0.1	1	1	0.49	0.60
2	2	0.4	0.6	1	1	0.45	0.55

The service distributions are two-branched mixed Erlang with parameters

Table 2
Numerical results for Example 2

ρ	Performance measure	MSA	Simulation	Arrival measure	Value
0.20	$E[D_1D_2]$	3.999	4.035±0.004	$E[A_1A_2]$	4.023
	$E[D^2]$	8.141	8.184±0.010	$E[A^2]$	8.065
	$E[D^3]$	49.452	49.674±0.133	$E[A^3]$	48.792
	$E[W]$	0.129	0.129±0.000		
	$E[W^2]$	0.183	0.187±0.001		
	$E[W^3]$	0.403	0.446±0.008		
0.50	$E[D_1D_2]$	3.960	4.029±0.004	$E[A_1A_2]$	4.023
	$E[D^2]$	8.536	8.625±0.008	$E[A^2]$	8.065
	$E[D^3]$	55.065	55.354±0.109	$E[A^3]$	48.792
	$E[W]$	1.288	1.273±0.003		
	$E[W^2]$	6.988	6.907±0.035		
	$E[W^3]$	57.916	57.111±0.467		
0.80	$E[D_1D_2]$	3.939	4.046±0.004	$E[A_1A_2]$	4.023
	$E[D^2]$	9.193	9.439±0.006	$E[A^2]$	8.065
	$E[D^3]$	61.850	70.254±0.119	$E[A^3]$	48.792
	$E[W]$	8.169	7.769±0.040		
	$E[W^2]$	169.201	148.158±1.890		
	$E[W^3]$	5349.617	4101.200±91.335		

Table 3
Numerical results for Example 3

ρ	Performance measure	MSA	Simulation	Arrival measure	Value
0.20	$E[D_1D_2]$	4.021	4.093±0.003	$E[A_1A_2]$	3.999
	$E[D^2]$	8.308	8.364±0.006	$E[A^2]$	8.424
	$E[D^3]$	55.456	56.382±0.076	$E[A^3]$	56.351
	$E[W]$	0.083	0.083±0.000		
	$E[W^2]$	0.056	0.059±0.000		
	$E[W^3]$	0.056	0.063±0.000		
0.50	$E[D_1D_2]$	4.016	4.161±0.003	$E[A_1A_2]$	3.999
	$E[D^2]$	7.653	7.851±0.006	$E[A^2]$	8.424
	$E[D^3]$	49.370	51.270±0.076	$E[A^3]$	56.351
	$E[W]$	0.849	0.822±0.001		
	$E[W^2]$	2.455	2.585±0.008		
	$E[W^3]$	10.498	12.803±0.075		
0.80	$E[D_1D_2]$	3.876	4.132±0.003	$E[A_1A_2]$	3.999
	$E[D^2]$	6.296	6.854±0.005	$E[A^2]$	8.424
	$E[D^3]$	36.621	38.309±0.064	$E[A^3]$	56.351
	$E[W]$	5.565	6.068±0.015		
	$E[W^2]$	67.998	100.003±0.638		
	$E[W^3]$	1240.491	2612.878±27.442		

$$(P_1, P_2) = (0.7, 0.3), \quad (n_1, n_2) = (2, 3), \quad (\mu_1, \mu_2) = \begin{cases} (4.80, 8.25), & \text{which corresponds to } \rho = 0.2, \\ (1.92, 3.30), & \text{which corresponds to } \rho = 0.5, \\ (1.20, 2.07), & \text{which corresponds to } \rho = 0.8. \end{cases}$$

Numerical results are given in Table 3. Similar to Example 2, all the performance measures are accurate under low and medium traffic conditions. The moments of the waiting time are prone to very large errors in heavy traffic; whereas the errors on the departure process moments are of lesser magnitude.

Example 4. The Markov chain has two states and the transition probability matrix is the same as the one used in Example 1. The conditional arrival distributions are all two-branched mixed Erlang with parameters

States		Prob.		Stages		Rates	
J_n	J_{n+1}	P_1	P_2	n_1	n_2	λ_1	λ_2
1	1	0.8	0.2	3	1	2.84	1.20
1	2	0.9	0.1	3	2	2.80	2.70
2	1	0.55	0.45	1	4	3.00	2.30
2	2	0.75	0.25	2	3	2.82	1.65

The service distributions are two-branched mixed Erlang with parameters

$$(P_1, P_2) = (0.6, 0.4), \quad (n_1, n_2) = (3, 4), \quad (\mu_1, \mu_2) = \begin{cases} (14.0, 22.4), & \text{which corresponds to } \rho = 0.2, \\ (5.60, 8.96), & \text{which corresponds to } \rho = 0.5, \\ (3.50, 5.60), & \text{which corresponds to } \rho = 0.8. \end{cases}$$

Numerical results are given in Table 4. The moments of the departure process are extremely accurate in all cases. The lag-1 autocorrelation estimates are also accurate. The moments of the waiting time estimates are accurate only in light traffic. The errors increase in medium traffic and the estimates are very poor in heavy

Table 4
Numerical results for Example 4

ρ	Performance measure	MSA	Simulation	Arrival measure	Value
0.20	$E[D_1D_2]$	1.006	1.009±0.000	$E[A_1A_2]$	0.999
	$E[D^2]$	1.637	1.649±0.000	$E[A^2]$	1.639
	$E[D^3]$	3.634	3.733±0.001	$E[A^3]$	3.633
	$E[W]$	0.017	0.021±0.000		
	$E[W^2]$	0.004	0.005±0.000		
	$E[W^3]$	0.001	0.002±0.000		
0.50	$E[D_1D_2]$	1.024	1.017±0.000	$E[A_1A_2]$	0.999
	$E[D^2]$	1.594	1.596±0.000	$E[A^2]$	1.639
	$E[D^3]$	3.495	3.569±0.001	$E[A^3]$	3.633
	$E[W]$	0.203	0.225±0.000		
	$E[W^2]$	0.192	0.222±0.000		
	$E[W^3]$	0.557	0.325±0.001		
0.80	$E[D_1D_2]$	1.035	1.016±0.000	$E[A_1A_2]$	0.999
	$E[D^2]$	1.457	1.460±0.000	$E[A^2]$	1.639
	$E[D^3]$	2.983	2.925±0.001	$E[A^3]$	3.633
	$E[W]$	1.355	1.702±0.002		
	$E[W^2]$	29.994	7.899±0.018		
	$E[W^3]$	–	55.624±0.191		

traffic. In fact, for the third moments of the waiting time the estimate is negative when the traffic intensity is 0.8, so it is not included in the table.

Example 5. The Markov chain has two states and the transition probability matrix is given by

$$P = \begin{pmatrix} 0.70 & 0.30 \\ 0.60 & 0.40 \end{pmatrix}.$$

The conditional arrival distributions are exponential, mixed Erlang, Erlang and hyperexponential, respectively, whose parameters are

States		Prob.		Stages		Rates	
J_n	J_{n+1}	P_1	P_2	n_1	n_2	λ_1	λ_2
1	1	1.0		1		0.700	
1	2	0.6	0.4	2	4	2.815	1.500
2	1	1.0		2		1.280	
2	2	0.8	0.2	1	1	0.800	0.300

The service distributions are hyperexponential with two branches. The parameters are

$$(P_1, P_2) = (0.2, 0.8), \quad (\mu_1, \mu_2) = \begin{cases} (0.966, 8.599), & \text{which corresponds to } \rho = 0.2, \\ (0.386, 3.440), & \text{which corresponds to } \rho = 0.5, \\ (0.241, 2.150), & \text{which corresponds to } \rho = 0.8. \end{cases}$$

Numerical results are given in Table 5. For this example, the estimates for the moments of the departure process are very accurate in all cases. The lag-1 autocorrelation estimates are also accurate. The moments of the waiting time are accurate only in light traffic. But their errors are large in medium and especially in

Table 5
Numerical results for Example 5

ρ	Performance measure	MSA	Simulation	Arrival measure	Value
0.20	$E[D_1D_2]$	2.196	2.171±0.001	$E[A_1A_2]$	2.253
	$E[D^2]$	4.645	4.614±0.002	$E[A^2]$	4.365
	$E[D^3]$	22.898	22.050±0.040	$E[A^3]$	20.601
	$E[W]$	0.186	0.178±0.000		
	$E[W^2]$	0.438	0.422±0.001		
	$E[W^3]$	1.578	1.486±0.003		
0.50	$E[D_1D_2]$	2.130	2.123±0.001	$E[A_1A_2]$	2.253
	$E[D^2]$	5.948	6.138±0.004	$E[A^2]$	4.365
	$E[D^3]$	43.633	42.405±0.037	$E[A^3]$	20.601
	$E[W]$	1.977	1.865±0.005		
	$E[W^2]$	19.245	15.670±0.055		
	$E[W^3]$	–	186.709±0.735		
0.80	$E[D_1D_2]$	2.081	2.168±0.001	$E[A_1A_2]$	2.253
	$E[D^2]$	8.584	8.945±0.009	$E[A^2]$	4.365
	$E[D^3]$	87.003	100.781±0.142	$E[A^3]$	20.601
	$E[W]$	11.930	14.167±0.103		
	$E[W^2]$	56.326	545.496±6.952		
	$E[W^3]$	–	31471.729±509.489		

heavy traffic conditions. The estimates for the third moment of the waiting time, which are not included in the table, are negative in medium and heavy traffic.

Example 6. The Markov chain has two states and the transition probability matrix is given by

$$P = \begin{pmatrix} 0.05 & 0.95 \\ 0.45 & 0.55 \end{pmatrix}.$$

The conditional arrival distributions are exponential with rates $= (1.664, 1.872, 0.5193, 1.56)$. The service distribution is three-staged Erlang with rates, $\mu = (15.0, 6.0, 3.75)$ corresponding to traffic intensities, $\rho = (0.2, 0.5, 0.8)$, respectively. Numerical results are given in Table 6. The estimates are accurate in light traffic. In medium traffic, the estimates become less accurate, and in heavy traffic, the errors of the estimates are very large. For this example, we note that the conditional traffic intensities in heavy traffic are given by

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{11} \end{pmatrix} = \begin{pmatrix} E[S]/E_{11}[A] & E[S]/E_{12}[A] \\ E[S]/E_{21}[A] & E[S]/E_{22}[A] \end{pmatrix} = \begin{pmatrix} 1.33 & 1.50 \\ 0.42 & 1.25 \end{pmatrix}.$$

So three of the conditional traffic intensities are greater than one, though the overall traffic intensity is less than one. As also demonstrated by the next example, when some of the conditional traffic intensities are greater than one, the estimates obtained based on our method often have large errors. This has been observed in other numerical examples we did which are not reported in this paper. In fact, we further observe that when some of the conditional traffic intensities are close to one, then our estimates become less accurate. For this example, the conditional traffic intensity $\rho_{12} = 0.94$ in medium traffic.

Diffusion limits and interpolation approximations based on these heavy traffic limits and the light traffic derivatives have been developed for the queueing system considered in this paper in Chen et al. (1997). The numerical studies there indicate that heavy traffic and interpolation approximations indeed produce more accurate estimates than the light traffic approximation of this paper when the queue is in heavy traffic.

Table 6
Numerical results for Example 6

ρ	Performance measure	MSA	Simulation	Arrival measure	Value
0.20	$E[D_1D_2]$	0.823	0.844±0.001	$E[A_1A_2]$	0.812
	$E[D^2]$	2.726	2.674±0.004	$E[A^2]$	2.759
	$E[D^3]$	13.877	13.155±0.059	$E[A^3]$	13.981
	$E[W]$	0.061	0.046±0.000		
	$E[W^2]$	0.019	0.014±0.000		
	$E[W^3]$	0.008	0.006±0.000		
0.50	$E[D_1D_2]$	0.806	0.923±0.001	$E[A_1A_2]$	0.812
	$E[D^2]$	2.653	2.373±0.004	$E[A^2]$	2.759
	$E[D^3]$	12.898	11.092±0.053	$E[A^3]$	13.981
	$E[W]$	0.587	0.459±0.001		
	$E[W^2]$	0.831	0.664±0.003		
	$E[W^3]$	1.746	1.445±0.011		
0.80	$E[D_1D_2]$	0.396	0.987±0.001	$E[A_1A_2]$	0.812
	$E[D^2]$	3.290	1.820±0.004	$E[A^2]$	2.759
	$E[D^3]$	–	6.575±0.048	$E[A^3]$	13.981
	$E[W]$	3.594	3.017±0.009		
	$E[W^2]$	23.679	21.291±0.156		
	$E[W^3]$	–	229.557±3.084		

Example 7. The Markov chain has two states and the transition probability matrix is given by

$$P = \begin{pmatrix} 0.05 & 0.95 \\ 0.85 & 0.15 \end{pmatrix}.$$

The conditional arrival distributions are three-staged Erlang with rates $= (2.25, 0.99, 2.70, 2.145)$. The service distribution is also three-staged Erlang with rates, $\mu = (7.5, 3.0, 1.875)$ corresponding to traffic intensities, $\rho = (0.2, 0.5, 0.8)$, respectively. Numerical results are given in Table 7. For this example, the estimates for the moments and the lag-1 autocorrelation of the departure process deviate from simulation values to a certain extent, while the estimates for the moments of the waiting time are very poor in medium and heavy traffic (most of them are negative). We note that in medium traffic the conditional traffic intensity $\rho_{21} = 0.9$ and in heavy traffic the conditional traffic intensities are

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{11} \end{pmatrix} = \begin{pmatrix} 1.20 & 0.53 \\ 1.44 & 1.14 \end{pmatrix}.$$

Similar to the previous example, the MSA does not work well in heavy traffic in this case. Using heavy traffic approximations or interpolation approximations (Chen et al., 1997) enhance the accuracy significantly.

Example 8. We study the impact of the coefficients of the Pade approximant on the accuracy of the performance measures in this example. This example considers the system in Example 1. Numerical results are given in Table 8. In light traffic, all performance measures quickly converge starting from the [2/2] approximant itself. In medium traffic, a slightly higher order of the approximant is needed for reasonable accuracy and in heavy traffic the higher orders of the approximant are needed to arrive at an acceptable level of accuracy. Also note that the third moment of the interdeparture time needs higher order of the approximant than the second moment for similar accuracy.

Table 7
Numerical results for Example 7

ρ	Performance measure	MSA	Simulation	Arrival measure	Value
0.20	$E[D_1D_2]$	3.262	3.297±0.010	$E[A_1A_2]$	3.258
	$E[D^2]$	6.566	6.578±0.020	$E[A^2]$	6.491
	$E[D^3]$	30.204	30.310±0.134	$E[A^3]$	29.701
	$E[W]$	0.015	0.017±0.000		
	$E[W^2]$	0.006	0.007±0.000		
	$E[W^3]$	0.004	0.004±0.000		
0.50	$E[D_1D_2]$	3.542	3.545±0.011	$E[A_1A_2]$	3.258
	$E[D^2]$	6.750	6.412±0.021	$E[A^2]$	6.491
	$E[D^3]$	30.995	29.000±0.153	$E[A^3]$	29.701
	$E[W]$	0.279	0.294±0.003		
	$E[W^2]$	–	0.436±0.007		
	$E[W^3]$	–	0.910±0.022		
0.80	$E[D_1D_2]$	4.054	3.874±0.010	$E[A_1A_2]$	3.258
	$E[D^2]$	6.581	5.919±0.022	$E[A^2]$	6.491
	$E[D^3]$	27.353	24.235±0.183	$E[A^3]$	29.701
	$E[W]$	1.826	2.254±0.034		
	$E[W^2]$	–	13.256±0.375		
	$E[W^3]$	–	113.429±4.888		

Table 8
Numerical results for Example 8

ρ	$[L/M]$	$E[D_1D_2]$	$E[D^2]$	$E[D^3]$
0.2	[2/2]	4.063	7.954	47.967
	[4/2]	4.059	7.954	47.990
	[6/2]	4.057	7.954	47.999
	[4/4]	4.057	7.954	47.999
	[6/4]	4.057	7.954	47.999
	[10/10]	4.057	7.953	47.986
	Simulation	4.079 ± 0.004	7.920 ± 0.009	47.363 ± 0.125
0.5	[2/2]	4.303	7.346	39.629
	[4/2]	4.318	7.076	36.259
	[6/2]	4.157	7.371	42.744
	[4/4]	4.292	7.355	43.032
	[6/4]	4.126	7.318	44.411
	[10/10]	4.124	7.360	42.655
	Simulation	4.147 ± 0.005	7.352 ± 0.008	42.130 ± 0.112
0.8	[2/2]	3.593	6.000	35.810
	[4/2]	4.621	6.305	37.452
	[6/2]	4.222	6.283	30.470
	[4/4]	5.981	6.204	32.454
	[6/4]	3.846	6.336	35.005
	[10/10]	4.078	6.218	30.717
	Simulation	4.119 ± 0.005	6.301 ± 0.008	30.386 ± 0.089

It may be observed that the performance estimates for the hyperexponential conditional arrival distributions are in general more accurate than those with Erlang or mixed Erlang. This can be explained by looking at the GI/G/1 queue in which the interarrival time distribution is Erlang with n stages. It can be shown that the first $k + n - 1$ light traffic derivatives or MacLaurin series coefficients of the kth moment of the steady state waiting time are zero (see Girish and Hu, 1997). Therefore, if a low Padé approximant is used to estimate the waiting time moments, then it can lead to large errors. On the other hand, since $n = 1$ for hyperexponential distributions, then even lower order approximants lead to more accurate estimates. For mixed Erlang distributions, n is the minimum among the stages of the Erlang branches.

6. Discussions and conclusion

The departure process of the G/G/1 queue with Markov-modulated arrival process was studied in this paper. Simple recursive equations to calculate the MacLaurin series of the departure process moments and the lag-1 autocorrelation were derived. We then proposed the use of Padé approximation to estimate the performance measures based on their MacLaurin series. The proposed method can be used to estimate any order of the moments of the departure process. It can also be used to estimate any lag of the autocorrelations, though in this paper we focused on the lag-1 autocorrelation. Our numerical experiments indicate that our estimates are very accurate in light and medium traffic, and their errors increase in heavy traffic. Furthermore, if some of the conditional traffic intensities are very close to or greater than one, then our method may not yield very good estimates even when the overall traffic intensity is less than one. This is an interesting issue which needs to be further investigated.

Our numerical results also show that the estimates for the moments and lag-1 autocorrelation of the departure process is much more accurate than those for the moments of the waiting time, especially for heavy traffic cases. This is mainly because the moments of the waiting time go to infinity as the traffic intensity approaches to one, while the moments and lag-1 autocorrelation of the departure process are bounded. To obtain more accurate estimates for the moments of the waiting time, we can use the moments of the departure process to estimate the moments of the waiting time, a method which is proposed in Girish and Hu (1996, 1997). Their numerical results show that very accurate estimates for the moments of the waiting time can be obtained based on this method in heavy traffic cases.

One possible future research direction is to develop better approximation techniques for general queueing networks based on the results in this paper that take correlations of departure and arrival processes into consideration.

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