

(s,S) Inventory Systems with Random Lead Times: Harris Recurrence and Its Implications in Sensitivity Analysis

Michael C. Fu
College of Business and Management
University of Maryland

Jian-Qiang Hu*
Manufacturing Engineering Dept.
Boston University

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Abstract

Most of the previous work on (s, S) inventory systems assume that lead times for orders are such that orders never cross in time, i.e., the arrival of orders follows the same sequence as the placement of the orders. In this paper, we consider more general mechanisms for random lead times. Since the introduction of a general random lead time mechanism makes the system essentially intractable for most performance measures of interest, simulation is a natural candidate for estimating performance and/or optimizing the system. Two important issues in simulation are the stability and ergodicity of the system. Therefore, we first study some theoretical implications of the mechanism by providing conditions for which the system is stable and Harris ergodic, with the accompanying wide-sense regenerative properties. We then consider the problem of gradient estimation during simulation. Using the technique of perturbation analysis, we derive sample path-based gradient estimates for the finite-horizon average cost per period with respect to the parameters s and S , and give a sample path proof of unbiasedness. We then show how stability and ergodicity can be used to simplify the estimators in the limiting infinite horizon case and to establish strong consistency of the resulting estimators.

1 Introduction

There exists an extensive body of literature on inventory systems operating under an (s, S) ordering policy (see, e.g., references in Sahin 1990 and Zheng and Federgruen 1991). In order to achieve some degree of tractability, almost all analytical models have assumed that lead times for orders do not cross in time, i.e., orders arrive in the order in which they were placed. However, it is clear that this is not always the case in practice, especially when multiple suppliers are involved. Thus, the allowance of a general random lead time mechanism makes the system essentially intractable for most performance measures of interest, which suggests simulation as a possible alternative for analysis or optimization. In either case – performance analysis or optimization – it would probably be beneficial to begin with the available analytical means to study the system of interest even under the more restrictive assumptions prior to undertaking a simulation study. For example, in the context of optimization, one might use the technique of Zheng and Federgruen (1992) to get approximate estimates of the optimal values, and then use simulation to “fine-tune” the optimization. A recent review of techniques for this purpose is contained in Fu (1994). One approach highlighted there is

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the use of gradient-based techniques, of which a critical component is the estimation of the gradient in the simulation environment.

This paper is concerned with two topics for (s, S) systems with general random lead times: the theoretical issues of stability and ergodicity of the system, and their application to the problem of gradient estimation in a simulation environment. For the former, we provide conditions for which the inventory level process is stable and Harris ergodic, with the accompanying wide-sense regenerative properties. The (wide-sense) regeneration points are given explicitly and the conditions are quite intuitive. For the latter, we consider the standard periodic review (s, S) inventory system with full backlogging. We derive estimators for gradients of average cost per period with respect to the parameters s and S using the technique of perturbation analysis. In particular, we use the technique of smoothed perturbation analysis (Gong and Ho 1987). We show how stability and ergodicity can be used to simplify and to establish strong consistency of the estimators. Other approaches to gradient estimation, which are not investigated here, include the likelihood ratio method and frequency domain experimentation (see Fu 1994 for an overview and references), as well as other forms of perturbation analysis.

A related work which also uses Harris recurrence is Browne and Zipkin (1991). However, the focus in their work is on a continuous stochastic demand process which serves as the natural generalization of the deterministic demand case used in the EOQ model and its variations. Our model for demand is simply a discrete-time renewal process, and our focus is not on the demand but on the stochastic lead time mechanism. Their results lead to a computationally useful insensitivity property, whereas our attempts are to characterize the effect of general stochastic lead time mechanisms on performance analysis and gradient estimation in sample path analysis or discrete event simulation. Moreover, they consider continuous review systems operating under a reorder point/order quantity (r, q) policy, whereas we are considering periodic review (s, S) systems.

Previous work on gradient estimation using perturbation analysis for inventory systems includes Fu (1993), the first such application of perturbation analysis to inventory systems, and Bashyam and Fu (1993), all considering various versions of (s, S) inventory systems. More recently, Glasserman and Tayur (1992a,b) have addressed the gradient estimation problem and investigated stability issues for multi-echelon production-inventory systems with limited production capacity operating under a base-stock ordering policy.

The remainder of the paper is organized as follows. In Section 2, we review the non-crossing assumption for lead times and its consequences, discuss more general random lead times based on a queueing mechanism, and establish conditions for Harris recurrence. In Section 3, we move on to the gradient estimation problem. We first derive the finite horizon estimators and establish their unbiasedness. These estimators would usually require additional simulation above and beyond that needed just to estimate the average cost per period. However, when considering the long-run limit, we are able to simplify the estimators based on the stability

and ergodicity properties of the system such that the final form contains only quantities that can easily be estimated from a single sample path (or simulation), thus resulting in estimators which are both efficient and easily implementable in a gradient-based optimization algorithm. Lastly, the stability and ergodicity properties are used to establish the consistency of the estimators.

2 Stability and Ergodicity Issues

2.1 A Review of Random Lead Times

We consider the standard periodic review (s, S) inventory system where excess demand is backordered. Previous results on (s, S) inventory systems has focused almost exclusively on those systems in which lead times of orders do not overlap (in most cases the lead times are constant, e.g., Zheng and Federgruen 1991). We define

$$\begin{aligned}
 W_n &= \text{inventory level (on hand minus backorders) at the beginning of period } n, \\
 Y_n &= \text{inventory position (inventory level plus outstanding orders) at the beginning of period } n, \\
 D_n &= \text{demand in period } n, \text{ i.i.d. with c.d.f. } F, \\
 D(j, k) &= \text{total demand for periods } j, j + 1, \dots, k, \\
 L_m &= \text{lead time of the } m\text{th order placed,} \\
 \eta_m &= \text{period the } m\text{th order was placed,} \\
 q &= S - s,
 \end{aligned}$$

and assume that at the end of a period demand is satisfied *before* the order placement decision is made, i.e., the inventory position follows the recursive equation

$$Y_{n+1} = \begin{cases} Y_n - D_n & \text{if } Y_n - D_n \geq s \\ S & \text{if } Y_n - D_n < s \end{cases} . \quad (1)$$

For convenience, we also define

$$V_n = Y_n - D_n,$$

which is the inventory position in period n after demand satisfaction but before order placement. From Equation (1), it is this quantity on which ordering decisions are made. Also let L and D represent generic lead times and demands, respectively. Note that L is a discrete random variable. By definition of η_k , we have

$$Y_{\eta_k+1} = S.$$

For convenience in the analysis that follows, we will assume the initial conditions $Y_1 = W_1 = S$ with no orders outstanding. Arrival of orders are assumed to be taken at the beginning of the period, so as far as inventory level W_n is concerned, the sequence of events in a single period occurs as follows: at the beginning

of the period, arrival of orders are taken and the inventory is measured; at the end of the period, demand for the period is subtracted and an order may be placed.

The advantage of non-crossing lead times is in the resulting analysis, which leads to (see, e.g., Zipkin 1986)

$$W_{n+L} = Y_n - D(n, n + L - 1) \quad (2)$$

in finite time, and

$$W = Y - D(L) \quad (3)$$

in the limiting case, assuming there exist limiting random variables such that

$$W_n \Rightarrow W, \quad Y_n \Rightarrow Y \text{ which implies } V_n \Rightarrow V, \quad D(n, n + L - 1) \Rightarrow D(L), \quad (4)$$

(\Rightarrow denoting weak convergence), with $D(L)$ representing demand over lead time L . Whereas the convergence of the inventory position process is independent of the nature of the lead times, convergence of the inventory level process must obviously involve conditions on both the demand process and the lead time process, and finding such conditions is the focus of this section. As we shall see, the inventory level process does not in general possess i.i.d. regenerative structure, but does retain a wide-sense regenerative structure.

Equations (2) and (3) decompose the inventory level into the difference of inventory position and demand over lead time, which are *independent*. This independence facilitates the calculation of the inventory level distribution and accompanying performance measures by reducing the calculation to convolutions of quantities which can be computed with relative ease. Non-crossing allows this decomposition by guaranteeing that all orders outstanding in period n will have arrived by period $n + L$ and any new orders placed in the interim will not have arrived yet. The work by Browne and Zipkin (1991) for continuous review (r, q) systems is one of the few that we know of where the assumption of non-crossing is not needed for the decomposition in (2) and (3) to hold.

However, it is clear that the requirement of non-crossing lead times can be restrictive in practice, e.g., it precludes multiple independent suppliers. This paper investigates random lead times governed by very general mechanisms, which we model using a simple queueing system. However, it soon becomes clear that since Equation (2) does not hold, analytic determination of the resulting quantities is a formidable task. By definition, we can express the relationship between inventory level and inventory position by

$$W_n = Y_n - \sum_{i=1}^{N_n} O_i^{(n)}, \quad (5)$$

where N_n represents the number of outstanding orders in period n and $O_i^{(n)}$ represents the corresponding order amounts outstanding in period n . Although it can be shown that the order amounts $\{O_i^{(n)}\}$ are i.i.d. and independent of N_n and Y_n , unfortunately N_n is not independent of Y_n . Intuitively, smaller values of Y_n

would tend to indicate smaller N_n , since that would signal some time since the last order was placed and hence the likelihood that some outstanding orders would have arrived in the meantime.

In this section, we concentrate on the following two issues (following the definitions of Walrand 1988):

- Stability of the process $\{W_n\}$, i.e., whether there exists a random variable W such that $W_n \Rightarrow W$,
- Ergodicity of the process $\{W_n\}$, which would give

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n g(V_i, W_i) = E[g(V, W)], \quad \text{w.p.1}$$

where g is a function such that $E[|g(V, W)|] < \infty$.

One of the needs for establishing these properties comes up in the practical matter of simulation. Due to complexity of inventory systems with general stochastic lead times, it is likely that at some point simulation of the system would be undertaken. The above results imply the convergence of the usual estimates for costs and service levels, which are critical (though often taken for granted) assumptions in the simulation of the system. The conditions needed are based upon the stability of the underlying queueing replenishment model. Furthermore, the results are then used in the gradient estimation derivations discussed in Section 3.

Before proceeding, we note that although we will be considering the standard periodic review case, essentially the same arguments also apply to the standard continuous review case.

2.2 The Queueing Model for Orders

As has been noted before (cf. Zipkin 1986), the underlying replenishment mechanism can often be modeled as a queueing process. Thus, we will model the process as a multi-server queueing system as follows:

- The “interarrival” times of order *requests* are generated by the underlying condition $\sum D_i > q$, where the summation is reset at each order epoch (regeneration point for the inventory position process). If the demand process is renewal, then the resulting arrival process will also be renewal. For the periodic case we consider, we have a discrete arrival process.
- The “service” times are the times spent on processing the order.
- The time in system is the service time plus any waiting time, where “waiting” may not be in the usual sense of the word (e.g., it can be used to enforce the no-crossing requirement); in the inventory system, this quantity corresponds to the order lead time L .
- N_n is just a queue-length (number in system) process.

This model allows the whole spectrum of replenishment possibilities, three of which we label as follows:

M1 The GI/GI/1 queue enforces a FIFO departure process and hence corresponds to non-crossing of orders.

M2 The GI/GI/ ∞ queue generates independent system times (equal to the service times) and hence corresponds to the case of independent order lead times. Note also that the G/D/ ∞ queue would cover the fixed lead time case.

M3 The GI/GI/ m queue allows both crossing of orders and non-independence of order lead times.

In the next subsection, we see that with this model, the stability and ergodicity properties of the inventory process are closely tied with the corresponding properties of the queueing process. In particular, since the arrival process is a function of the demand process, the usual stability condition for queueing systems of the arrival rate not exceeding the maximum system service rate will translate into a requirement on demand sizes with respect to the replenishment process. Finally, we note that the queueing model can be easily extended to a tandem queueing network if for example processing and transportation times need to modeled separately.

2.3 Harris Recurrence and Regeneration

Now we establish regenerative structure for the inventory process $\{W_n\}$. The key steps we need to show in our analysis are the following:

1. To establish Harris recurrence, we need to find a regeneration set, a set of states to which our process returns i.o. (and the respective conditions to ensure the latter).
2. To establish positive recurrence, we need to find conditions for which the times between returns to the regeneration set has finite expectation.

Under the assumption of continuous demands, Harris ergodicity then follows because our process is aperiodic.

First we introduce the following definition in Asmussen (1987, pp.125-129) for a regenerative process with regeneration in a wider sense than the usual definition.

Definition 1. A discrete-time stochastic process $\{X_n\}$ is said to be (pure or delayed) *wide-sense regenerative* if there exists a (pure or delayed) renewal process $\{\tau_k\} = \{\sigma_0 + \sigma_1 + \dots + \sigma_k\}$ such that for each $k \geq 0$, the post- τ_k -process

$$\{\sigma_{k+1}, \sigma_{k+2}, \dots, \{X_{\tau_k}, X_{\tau_k+1}, X_{\tau_k+2}, \dots\}\}$$

is independent of $\{\tau_0, \tau_1, \dots, \tau_k\}$ and its distribution does not depend on k . $\{\tau_n\}$ is called the *embedded renewal process* and each τ_n a *regenerative point*.

Clearly, classical regeneration is a special case of the above definition in which the post- τ_k process is also *independent* of $\{X_0, \dots, X_{\tau_k-1}\}$ as well. Under this more restrictive definition, $\{X_n, n \in [\tau_{k-1}, \tau_k)\}$ for each $k \geq 1$ are i.i.d., so we will call the process $\{X_n\}$ *i.i.d. regenerative*. For example, in our system, $\{Y_n\}$ is an i.i.d. regenerative process with respect to the renewal process $\{\eta_k\}$.

However, the inventory process $\{W_n\}$ is not i.i.d. regenerative, but we shall show that it is wide-sense regenerative. We now introduce the definition of Harris recurrence (see Asmussen 1987, pp.150-151).

Definition 2. Let $\{X_n\}$ be a Markov process on state space \mathbf{E} with Borel sets \mathcal{B} , and define the *first hitting time* for a set of states $R \in \mathcal{B}$ be given by $\tau(R) = \inf\{t \geq 1 : X_n \in R\}$. We call R *recurrent* if

$$P(\tau(R) < \infty | X_0 = y) = 1 \quad \text{for all } y \in \mathbf{E},$$

and $\{X_n\}$ is said to be *Harris recurrent* with regeneration set $R \in \mathcal{B}$ if R is recurrent and for some $r > 0$, $\epsilon \in (0, 1)$, and some probability measure λ on \mathbf{E} ,

$$P(X_r \in B | X_0 = x) \geq \epsilon \lambda(B), x \in R, B \in \mathcal{B}. \quad (6)$$

For our system, it turns out that the following specialization is useful (Asmussen 1987):

Lemma 1. If $P(X_{n+1} = dy | X_n = x_1) = P(X_{n+1} = dy | X_n = x_2)$ for all $x_1, x_2 \in R, y \in \mathbf{E}$, then $\{X_n\}$ is *Harris recurrent* with regeneration set R if and only if R is recurrent.

The specialization is to a regeneration set with identical one-step transition kernels. The proof comes from, for example, taking $r = 1$, $\epsilon = 1/2$, $\lambda(B) = P(X_r \in B | X_0 = x)$.

For our purposes the key property is that if X is Harris recurrent with regeneration set R , then it is wide-sense regenerative, with the (wide-sense) regenerative points defined by

$$\tau_k = \min\{n : X_n \in R, n > \tau_{k-1}\}$$

with $\tau_{-1} = 0$, and the cycle lengths given by $\{\tau_k - \tau_{k-1}\}$, which form a (pure or delayed) renewal process. For convenience, we will assume that $X_0 \in R$, so that we have a pure (wide-sense) regenerative (and corresponding renewal) process. For our inventory system, we will see that the (wide-sense) regenerative points form a subsequence of $\{\eta_k + 1\}$.

Positive recurrence is the case where the expectation of the cycle lengths is finite:

Definition 3. A Harris recurrent chain is positive recurrent if

$$E[\tau_1 - \tau_0] < \infty.$$

For our system, we define the Markov process by taking the vector of the inventory position and the queue-length process with the usual supplementary variables (the ages of all outstanding orders and their respective sizes):

$$X_n = (Y_n; N_n; Q_n; (\alpha_i^{(n)}, O_i^{(n)}, i = Q_n + 1, \dots, N_n)), \quad (7)$$

where Q_n is the number of orders “in queue” in period n , and $\alpha_i^{(n)}$ and $O_i^{(n)}$ are the age and order size of the i th order “in service” in period n . Of course, for the infinite-server case (independent orders), Q_n is

identically zero, and for the single-server case, $Q_n = (N_n - 1)^+$. Also, note that the inventory process $\{W_n\}$ is a function of the process $\{X_n\}$ via (5).

We can now define the regeneration set by the set of states

$$R = \{(S; 1; 0; (0, y)) \quad y \geq q\}, \quad (8)$$

i.e., it is the set of states at which all outstanding orders have come in, and a new order was just placed. Note that the transition probability out of a state in R to any other state is independent of the value of $y \geq q$, so we can apply Lemma 1. In fact, if the state of the process were defined without the order amounts, then the process would be i.i.d. regenerative.

Thus, for Harris recurrence we need to show that $P(\tau(R) < \infty | X_0) = 1$ for all initial states X_0 , and for positive recurrence, we need to show $E[\tau_k - \tau_{k-1}] < \infty$.

In our queueing process, a state in R corresponds to the beginning of a busy period. Thus, in order to reach this state i.o., the queueing system must empty i.o. Note that this condition is independent of order size. However, the presence of order size will introduce dependencies between adjacent regenerative cycles.

For the queueing model of order replenishment, let

$$\begin{aligned} A &= \text{generic interarrival time,} \\ T &= \text{generic service time,} \\ \lambda &= 1/E[A] = \text{arrival rate,} \\ \mu &= 1/E[T] = \text{service rate.} \end{aligned}$$

In the inventory model, only information on T would be available directly, whereas information on A would be derived from the demand information. Harris recurrence for $\{X_n\}$ then translates to the usual stability requirements for the corresponding queueing systems:

- For the G/G/1 queue, $\lambda < \mu$.
- For the G/GI/ ∞ queue, $P\{A > T\} > 0$.
- For the G/GI/ m queue, $\lambda < m\mu$ and $P\{A > T\} > 0$.

However, unlike the busy periods of “standard” multi-server queues, but as is the usual case with Harris recurrence, there is *one-dependence*: dependence between adjacent cycles, with independence for cycles separated by more than one cycle. As stated earlier, the one-dependence comes from the order sizes. We note here that milder conditions can be derived if we don’t require the system to empty i.o., by redefining our regeneration set. This is discussed in more detail at the end of the section following our main result.

So what remains is to relate the interarrival times and demand sizes. We first relate λ to the demand sizes. Defining $M(t)$ as the number of renewals generated by epoch t for the renewal process generated by

demands, we have

$$E[A] = 1 + E[M(q)], \quad (9)$$

i.e., the mean interarrival time is given by the renewal function (plus 1) for the demand renewal process.

Thus, our first condition is

A1. $(1 + \sum_{n=0}^{\infty} F^{(n)}(q)) m\mu > 1$, where m is the number of servers, and $F^{(n)}$ denotes the n -fold convolution of F .

Next, we have

$$A = k \iff \sum_{i=1}^k D_i > q, \quad \sum_{i=1}^{k-1} D_i < q. \quad (10)$$

In particular,

$$\sum_{i=1}^k D_i < q \iff A > k, \quad (11)$$

so we rewrite the condition $P\{A > T\} > 0$ as $P\{A > k, T \leq k\} = P\{A > k\}P\{T \leq k\} > 0$ for some k , since lead times and demands are independent of each other. Thus, the other main condition for the multi-server case is

A2. There exists an integer $k \geq 0$ such that $P(T \leq k) > 0$ and $P(\sum_{i=1}^k D_i < q) > 0$.

Using these two conditions as needed for the three models establishes Harris ergodicity for $\{X_n\}$ under continuous demands, so putting this all together, we have our main result:

Theorem 1. Assume that demands are continuous with density f .

- Under **A1**, $\{X_n\}$ with M1 is Harris ergodic.
- Under **A2**, $\{X_n\}$ with M2 is Harris ergodic.
- Under **A1** and **A2**, $\{X_n\}$ with M3 is Harris ergodic.

Remarks. If we are only interested in Harris recurrence, then the condition **A2** is actually not needed. We have stated our main theorem with this condition, because we require ergodicity for our work in the succeeding section. In terms of the underlying queueing models, without the condition $P\{A > T\} > 0$, it is possible that the system time process not be ergodic, and hence $\{X_n\}$ would not be Harris ergodic. For example, see Whitt (1972) for counterexamples in which no limiting distribution exist.

To demonstrate Harris recurrence when $P\{A > T\} > 0$ is not required, we redefine the regeneration set below. The result is that instead of one-dependence, we get a more complicated multi-period dependence. For the GI/GI/ m queue, if $\lambda < m\mu$ then the process $\{X_n\} = \{(Y_n; N_n; Q_n; (\alpha_i^{(n)}, O_i^{(n)}, i = Q_n + 1, \dots, N_n))\}$ is Harris regenerative with regeneration set

$$R^j = \{S_j; j; 0; (\alpha_i, O_i^{(n)}, i = 1, \dots, j), \alpha_i \geq 0, O_i^{(n)} \geq q\}, \quad j \geq 1,$$

i.e., we generalize our previous case of $j = 1$, one outstanding order, to $j \geq 1$, any number of outstanding orders in the system at the beginning of each regenerative cycle. (See Sigman 1988b; also Asmussen 1987, p.249. For Harris recurrence of the G/G/ ∞ queue, see Sigman 1988a.) Whereas In the case of $j = 1$ – only one outstanding order – that order must have been generated in the previous (adjacent) regenerative cycle, under the new regeneration set, some of the outstanding orders may have been generated in any of the earlier (non-adjacent) regenerative cycles. We now define a new regeneration set

$$R_K^j = \{s \in R^j : O_1^{(n)}, \dots, O_j^{(n)} \text{ generated within the previous } K \text{ regenerative cycles}\}.$$

If K is large enough then the probability that these j outstanding orders are generated within the previous K regenerative cycles as defined by R_j is non-zero. Since R_K is a regenerative set for $\{X_n\}$, let $\tau_0^K, \tau_1^K, \tau_2^K, \dots$ be the corresponding regenerative points, so that $\{X_n\}$ is again regenerative with respect to $\{\tau_{nK}^K, n = 0, 1, \dots\}$, and the j outstanding orders at τ_{nK}^K must have been generated during $[\tau_{(n-1)K}^K, \tau_{nK}^K)$. This in turn implies that $\{X_n\}$ is one-dependent regenerative with respect to $\{\tau_{nK}^K, n = 0, 1, \dots\}$, as is the inventory process $\{W_n\}$, based on (5), although both are K -dependent regenerative with respect to $\{\tau_n^K, n = 0, 1, \dots\}$. In any case, without **A2** we have the regenerative structure, but not necessarily ergodicity.

3 Gradient Estimation

Theorem 1 established Harris ergodicity with well-defined (wide-sense) regenerative points. Since the random lead time model is intractable in general, simulation may be needed for which this result gives us the useful (and usual) regenerative relationship

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n g(V_i, W_i) = E[g(V, W)] = \frac{E \left[\sum_{i=\tau_{k-1}}^{\tau_k-1} g(V_i, W_i) \right]}{E[\tau_k - \tau_{k-1}]},$$

for $E[|g(V, W)|] < \infty$, where g could represent any of the various performance measures of interest of either the cost or service level type. In this section, we derive gradient estimators, where the performance measure of interest is the average cost per period, with one-period costs given by

$$J_i(s, q) = H(W_i^+) + B(W_i^-) + I\{V_i < s\}(K + c(s + q - V_i)),$$

where $H(x) =$ holding costs per period for level $x > 0$ (0 otherwise),

$B(x) =$ shortage (backorder) costs per period for backordered level $x > 0$ (0 otherwise),

$K =$ set-up cost for placing an order,

$c =$ per-unit ordering cost,

where $I\{\cdot\}$ represents the indicator function. We have assumed a fixed set-up cost with linear per-unit cost for ordering, general continuous holding and shortage cost functions. In analytical work, the linear per-unit

ordering cost is usually omitted, since it can be easily calculated independently of the values of s and q . However, we will include it in our discussion for completeness. It is well-known, but not our primary concern, that additional conditions on form of these cost functions are needed to be able to assert that a policy of the (s, S) type is optimal (see, e.g., Sahin 1990).

We further assume:

C0. $H(\cdot)$ and $B(\cdot)$ are non-decreasing, piecewise differentiable, and Lipschitz continuous, i.e.,

$$|H(w_1) - H(w_2)| \leq K_h |w_1 - w_2|,$$

$$|B(w_1) - B(w_2)| \leq K_b |w_1 - w_2|,$$

for all w_1, w_2 , where $K_h > 0$ and $K_b > 0$ are the Lipschitz constants.

C1. $F(\cdot)$ is Lipschitz continuous with Lipschitz constant K_f and density $f(\cdot)$.

C2.

$$E \left[B \left(\sum_{i=1}^n D_i \right) \right] < \infty,$$

All three conditions are relatively mild. For example, taking H and B as any polynomial functions (which are right continuous at 0) would satisfy **C0** and **C2**, as would convex functions which are bounded on any finite range (as long as $E[D] < \infty$). Most of the “common” continuous distributions (normal, uniform, exponential, and the Weibull or gamma distributions which are not infinite at 0) satisfy **C1**.

For the finite horizon case, we define the random variable

$$\bar{J}_n = \frac{1}{n} \sum_{i=1}^n J_i, \tag{12}$$

and for the infinite horizon case, we consider its a.s. limit

$$J = \lim_{n \rightarrow \infty} \bar{J}_n. \tag{13}$$

Our goal then is to derive estimators for both

$$\frac{\partial E[\bar{J}_n]}{\partial \theta} \quad \text{and} \quad \frac{\partial J}{\partial \theta},$$

for $\theta = s$ and $\theta = q$.

For fixed lead times, Fu (1993) derived efficient sample path estimators and proved strong consistency by comparison with analytic results. Since no analytic results exist for general random times, here we use the more general sample path proofs to establish unbiasedness and consistency of our estimators. Our results can also be extended to service level performance measures in a natural way.

3.1 Derivation of the Estimators

Recall that we assume $Y_1 = W_1 = s + q$ (with no orders outstanding). For $\theta = s$, we begin by showing that the following IPA estimator

$$\frac{\partial \bar{J}_n}{\partial s} = \frac{1}{n} \sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) \quad (14)$$

is unbiased for $\partial E[\bar{J}_n]/\partial s$.

The key result is

Lemma 2. For all $\delta > 0$ and all i ,

$$\begin{aligned} I\{V_i(s') < s'\} &= I\{V_i(s) < s\}, \quad \text{where } s' = s + \delta, \\ \text{and hence } V_i(s + \delta) &= V_i(s) + \delta, \\ W_i(s + \delta) &= W_i(s) + \delta. \end{aligned}$$

Intuitively, the shape of sample paths are unaltered by changes in s , if q is held constant; the entire sample path is merely shifted by the size of the change.

Proof. The proof is by induction.

For $i = 1$, we have

$$\begin{aligned} V_1(s) &= s + q - D_1, \\ V_1(s + \delta) &= s + \delta + q - D_1 = V_1(s) + \delta, \\ I\{V_1(s') < s'\} &= I\{s + \delta + q - D_1 < s + \delta\} = I\{s + q - D_1 < s\} = I\{V_1(s) < s\}. \end{aligned}$$

Now, we assume our statement holds for i and establish the result for $i + 1$. First, we write Equation (7) in terms of V_i :

$$V_{i+1} = \begin{cases} V_i - D_{i+1} & \text{if } V_i \geq s \\ S - D_{i+1} & \text{if } V_i < s \end{cases} \quad (15)$$

There are two cases to consider.

- $V_i \geq s$: By the induction hypothesis

$$V_i(s + \delta) = V_i(s) + \delta \quad \text{and} \quad V_i(s') < s', \quad \text{since } \{I\{V_i(s') < s'\} = I\{V_i(s) < s\},$$

$$\text{so } V_{i+1}(s') = V_i(s') - D_{i+1} = V_i(s) + \delta - D_{i+1} = V_{i+1}(s) + \delta$$

$$\text{and } I\{V_{i+1}(s') < s'\} = I\{V_{i+1}(s) + \delta < s + \delta\} = I\{V_{i+1}(s) < s\}.$$

- $V_i < s$: In this case, V_{i+1} merely imitates V_1 above, and the induction proof is complete for V_i .

For W_i , a similar argument holds after the critical observation that the induction hypothesis implies that the outstanding orders are always identical since $I\{V_i(s') < s'\} = I\{V_i(s) < s\}$ for $i = 1, \dots, n$. This completes our proof. \square

Note that this result is *completely* independent of the nature of the order lead times.

Thus, we have our first main result:

Theorem 2. Under **C0**, the IPA estimator for $\partial E[\bar{J}_n]/\partial s$ given by (14) is unbiased.

Proof. Using Lemma 2, we have

$$\lim_{\delta \rightarrow 0} \frac{J_i(s + \delta) - J_i(s)}{\delta} = \begin{cases} H'(W_i) & \text{if } W_i > 0, \\ B'(-W_i) & \text{if } W_i < 0. \end{cases} \quad (16)$$

On the other hand, invoking **C0**, we have

$$\begin{aligned} |J_i(s + \delta) - J_i(s)| &= |H((W_i(s) + \delta)^+) - H(W_i^+(s)) + B((W_i(s) + \delta)^-) - B(W_i^-(s))| \\ &\leq K_h |(W_i(s) + \delta)^+ - W_i^+(s)| + K_b |(W_i(s) + \delta)^- - W_i^-(s)| \\ &\leq (K_h + K_b)\delta, \end{aligned}$$

and hence

$$\left| \frac{\bar{J}_n(s + \delta) - \bar{J}_n(s)}{\delta} \right| \leq K_h + K_b$$

Applying the dominated convergence theorem, we have

$$\begin{aligned} \frac{\partial E[\bar{J}_n]}{\partial s} &= \lim_{\delta \rightarrow 0} \frac{E[(\bar{J}_n(s + \delta) - \bar{J}_n(s))]}{\delta} = E \left[\lim_{\delta \rightarrow 0} \frac{(\bar{J}_n(s + \delta) - \bar{J}_n(s))}{\delta} \right] \\ &= E \left[\frac{1}{n} \sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) \right] = E \left[\frac{\partial \bar{J}_n}{\partial s} \right]. \end{aligned}$$

This completes our proof. \square

For $\theta = q$, Lemma 2 does not hold, and the IPA estimator is not unbiased by itself. As in Fu (1993), we use smoothed perturbation analysis (SPA) to derive an unbiased estimator. To derive the SPA estimator, we introduce the following additional notation:

$$\begin{aligned} \eta(k) &= \max\{m : \eta_m \leq k\}, \\ y_k &= \{D_1, D_2, \dots, D_{k-1}, D_{k+1}, D_{k+2}, \dots, D_n, L_1, L_2, \dots, L_{\eta(k)}\} \\ \mathcal{A}_k &= \{V_i(q) \geq s \text{ or } V_i(q) < s - \delta, i = 1, \dots, k\} \\ \mathcal{B}_k &= \mathcal{A}_{k-1} - \mathcal{A}_k = \{V_i(q) \geq s \text{ or } V_i(q) < s - \delta, i = 1, \dots, k-1\} \cup \{s - \delta \leq V_k(q) < s\}, \end{aligned}$$

$k = 1, \dots, n$, where $\delta > 0$. The notation $\eta(k)$ gives the number of total orders placed (received or outstanding) as of period k , i.e., the last order placed up to (and including) period k was the $\eta(k)$ th order placed. The set y_k is the so-called characterization (Gong and Ho 1987), the set of conditioning quantities for our estimator.

Thus, \mathcal{A}_k and \mathcal{B}_k are both functions of δ , though we omit explicit display of the argument. The event \mathcal{B}_k indicates that a perturbation in the value of q from q to $q + \delta$, with s held fixed, first causes a change in the ordering pattern in period k , i.e., at q an order is placed in period k , whereas at $q + \delta$ no order is placed in period k . The event \mathcal{A}_n then represents the case where the perturbation does not cause a change in the ordering pattern over the entire sample path, i.e., the periods in which orders are placed remain the same at $q + \delta$ as for q .

Since $\mathcal{B}_1, \dots, \mathcal{B}_n, \mathcal{A}_n$ partition our sample space, we can write

$$E[\bar{J}_n(q + \delta)] - E[\bar{J}_n(q)] = E[(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{A}_n)] + \sum_{i=1}^n E[(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{B}_i)]. \quad (17)$$

The key result is the natural extension of Lemma 2 for this case:

Lemma 3.

(a) If $I(\mathcal{A}_n) = 1$, then

$$\begin{aligned} V_i(q + \delta) &= V_i(q) + \delta \text{ for } i = 1, \dots, n + 1, \\ W_i(q + \delta) &= W_i(q) + \delta \text{ for } i = 1, \dots, n + 1, \\ I\{V_i(q + \delta) < s\} &= I\{V_i(q) < s\} \text{ for } i = 1, \dots, n. \end{aligned}$$

(b) If $I(\mathcal{B}_k) = 1$, then

$$\begin{aligned} V_i(q + \delta) &= V_i(q) + \delta \text{ for } i = 1, \dots, k, \\ W_i(q + \delta) &= W_i(q) + \delta \text{ for } i = 1, \dots, k, \\ I\{V_i(q + \delta) < s\} &= I\{V_i(q) < s\} \text{ for } i = 1, \dots, k - 1, \\ V_k(q + \delta) &\geq s, \text{ whereas } V_k(q) < s, \\ V_{k+1}(q + \delta) &= V_k(q + \delta) - D_{k+1}, \\ V_{k+1}(q) &= s + q - D_{k+1}. \end{aligned}$$

Proof. Since the proof is nearly identical to the proof of Lemma 2, we omit the details here. Basically, $I(\mathcal{A}_n) = 1$ corresponds to the situation in Lemma 2, whereas $I(\mathcal{B}_k) = 1$ indicates that the situation in Lemma 2 holds up to period k , with the difference indicated occurring in period $k + 1$. \square

Using techniques similar to (and actually simpler than, due to the structure of the problem) those used in Fu and Hu (1992), we will show that the SPA estimator

$$\left(\frac{\partial \bar{J}_n}{\partial q} \right)_{\text{SPA}} = \frac{1}{n} \left[\sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) + \sum_{i \in L^*(n)} \frac{f(z_i)}{1 - F(z_i)} \sum_{k=1}^n (J_k^{i2} - J_k^{i1}) \right] \quad (18)$$

is unbiased for $\partial E[\bar{J}_n]/\partial q$, where

- $z_i = Y_i - s$;
- $L^*(n) = \{i | i \leq n, V_i < s\}$, the set of potential ordering changes;
- NP , called the nominal path, is the original sample path for the Markov process $\{X_i\}$ defined by (7) and is generated by $y_i \cup \{D_i\}$; DNP_i , called the i th degenerated nominal sample path, is the sample path generated by y_i with $D_i = z_i^+$; and PP_i , called the i th perturbed path, is the sample path generated by y_i with $D_i = z_i^-$. Thus, an order of size q is placed for period i in the sample path DNP_i whereas no order is placed in the sample path PP_i . We add the superscript $i1$ to all quantities for the sample path DNP_i and the superscript $i2$ for the sample path PP_i , so J_k^{i1} and J_k^{i2} represent the period k costs in DNP_i and PP_i , respectively.

Thus, every time the simulation encounters the condition $V_i < s$, the estimator requires the estimation of costs on two sample paths where the i th demand is set to z_i^+ and z_i^- , respectively. Otherwise, the same set of demands is used. In the next section, we simplify the estimator to make it more efficient, but first we prove unbiasedness.

Lemma 4. Under **C0**,

$$\lim_{\delta \rightarrow 0} \frac{E[(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{A}_n)]}{\delta} = E \left[\frac{1}{n} \sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) \right].$$

Proof. Based on Lemma 3 (a) and **C0**, we have

$$\begin{aligned} & |(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{A}_n)| \\ &= \frac{1}{n} \left| \sum_{i=1}^n (J_i(q + \delta) - J_i(q))I(\mathcal{A}_n) \right| \\ &= \frac{1}{n} \left| \sum_{i=1}^n (H((W_i(q) + \delta)^+) - H(W_i^+(q)) + B((W_i(q) + \delta)^-) - B(W_i^-(q)))I(\mathcal{A}_n) \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n (K_h |(W_i(q) + \delta)^+ - W_i^+(q)| + K_b |(W_i(q) + \delta)^- - W_i^-(q)|) \\ &\leq (K_h + K_b)\delta. \end{aligned}$$

This and the dominated convergence theorem lead to

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{E[(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{A}_n)]}{\delta} &= E \left[\lim_{\delta \rightarrow 0} \frac{(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{A}_n)}{\delta} \right] \\ &= E \left[\frac{1}{n} \sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) \right]. \end{aligned}$$

This completes our proof. □

Now we consider $E[(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{B}_i)]$, $i = 1, \dots, n$. First we rewrite it as

$$E[E[(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{B}_i)|y_i]].$$

We note that $s - \delta \leq V_i(q) < s$ is equivalent to $z_i < D_i \leq z_i + \delta$ and also that z_i only depends on D_1, \dots, D_{i-1} , which are independent of D_i . Based on this, Lemma 3 (b), and **C1**, we have

$$\begin{aligned} |E[(\bar{J}_n(q + \delta)I(\mathcal{B}_i)|y_i)]| &= \frac{1}{n} \left| \sum_{i=1}^n J_i(q + \delta)I(\mathcal{B}_i)P(z_i < D_i \leq z_i + \delta) \right| \\ &= \frac{1}{n} \left| \sum_{i=1}^n J_i(q + \delta)I(\mathcal{B}_i)(F(z_i + \delta) - F(z_i)) \right| \leq K_f \delta \bar{J}_n(q + \delta). \end{aligned}$$

From **C2**, we have

$$E \left[\sup_q \bar{J}_n(q) \right] \leq K + H(s + q) + E \left[B \left(\sum_{i=1}^n D_i \right) \right] < \infty,$$

so once again invoking the dominated convergence theorem, we get

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{E[\bar{J}_n(q + \delta)I(\mathcal{B}_i)]}{\delta} &= E \left[\lim_{\delta \rightarrow 0} \frac{E[\bar{J}_n(q + \delta)I(\mathcal{B}_i)|y_i]}{\delta} \right] \\ &= E \left[\frac{1}{n} \lim_{\delta \rightarrow 0} \frac{\sum_{k=1}^n J_k(q + \delta)I(\mathcal{B}_i)(F(z_i + \delta) - F(z_i))}{\delta} \right] \\ &= E \left[\frac{1}{n} f(z_i) \sum_{k=1}^n J_k^{i2} \right] = E \left[\frac{1}{n} \frac{f(z_i)}{1 - F(z_i)} I\{D_i > z_i\} \sum_{k=1}^n J_k^{i2} \right] \\ &= E \left[\frac{1}{n} \frac{f(z_i)}{1 - F(z_i)} I\{V_i < s\} \sum_{k=1}^n J_k^{i2} \right] \end{aligned}$$

Similarly, we can show

$$\lim_{\delta \rightarrow 0} \frac{E[\bar{J}_n(q)I(\mathcal{B}_i)]}{\delta} = E \left[\frac{1}{n} \frac{f(z_i)}{1 - F(z_i)} I\{V_i < s\} \sum_{k=1}^n J_k^{i1} \right].$$

To summarize, we present

Lemma 5. If **C1** and **C2** hold, then

$$\lim_{\delta \rightarrow 0} \frac{E[(\bar{J}_n(q + \delta) - \bar{J}_n(q))I(\mathcal{B}_i)]}{\delta} = E \left[\frac{1}{n} \frac{f(z_i)}{1 - F(z_i)} I\{V_i < s\} \sum_{k=1}^n (J_k^{i2} - J_k^{i1}) \right].$$

Combining Lemmas 4 and 5 with (17), we have

Theorem 3. Under **C0-C2**, the SPA estimator for $\partial E[\bar{J}_n]/\partial q$ given by (18) is unbiased.

3.2 Simplification of the SPA Estimator

Utilizing the ergodicity properties, we now simplify the SPA estimator given by (18), which is not computationally attractive, since it requires the construction of two sample paths to estimate $\sum_{k=1}^n (J_k^{i2} - J_k^{i1})$ every time $V_i < s$. We first construct a sample path $\tilde{P}P_i$ based on PP_i as follows:

$$\tilde{D}_k = D_k, \quad k = 1, \dots, i, \quad \tilde{D}_i = z_i^-, \quad \tilde{D}_{i+1} \sim F, \quad \tilde{D}_k = D_{k-1}, \quad k = i + 2, \dots$$

$$\tilde{L}_k = L_k, \quad k = 1, \dots$$

and initial conditions $\tilde{Y}_1 = \tilde{W}_1 = s + q$ with no orders outstanding. Thus, other than the ‘‘inserted’’ demand \tilde{D}_{i+1} , the two sample paths $\tilde{P}P_i$ and PP_i are coupled by the same demand and lead time processes. Since $\tilde{D}_{i+1} \sim F$, the two sample paths $\tilde{P}P_i$ and PP_i are equal in probability law, so $E[J_k^{i3}] = E[J_k^{i2}]$, where we use the superscript $i3$ to indicate all quantities for the sample path $\tilde{P}P_i$, and hence

$$\left(\frac{\partial \bar{J}_n}{\partial q} \right)_{\text{SPA}} = \frac{1}{n} \left[\sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) + \sum_{i \in L^*(n)} \frac{f(z_i)}{1 - F(z_i)} \sum_{k=1}^n (J_k^{i3} - J_k^{i1}) \right] \quad (19)$$

is also unbiased for $\partial E[\bar{J}_n]/\partial q$.

Recalling that if an order is placed in period k , then $k + L_{\eta(k)} + 1$ is the period in which the order arrives (because orders are placed at the end of a period and arrive at the beginning of a period), we define

$$\tilde{m}(i) = \max_{k: V_k < s, k \leq i} \{k + L_{\eta(k)} + 1\},$$

which represents the period in which the last of all orders outstanding in period i arrives. (Recall that orders can cross.) Note that DNP_i and $\tilde{P}P_i$ are identical before period i , and that the segment of DNP_i after period $\tilde{m}(i) - 1$ is identical to the segment of $\tilde{P}P_i$ after period $\tilde{m}(i)$, so we have

$$\sum_{k=1}^n (J_k^{i3} - J_k^{i1}) = \sum_{k=i}^{m(i)-1} (J_k^{i3} - J_k^{i1}) + J_{m(i)}^{i3} - J_n^{i1},$$

where $m(i) = \min(\tilde{m}(i), n)$. We further note that

$$\begin{aligned} V_{i+1}^{i3} &= s - \tilde{D}_{i+1}, \\ V_{k+1}^{i3} &= V_k^{i1}, \text{ for } k = i + 1, i + 2, \dots, \\ W_i^{i3} &= W_i^{i1}, \quad W_{i+1}^{i3} = W_{i+1}^{i1}, \end{aligned}$$

which leads to

$$\begin{aligned} & \sum_{k=1}^{m(i)-1} (J_k^{i3} - J_k^{i1}) + J_{m(i)}^{i3} - J_n^{i1} \\ = & c\tilde{D}_{i+1} + \sum_{k=i+1}^{m(i)-1} (H((W_k^{i3})^+) + B((W_k^{i3})^-) - H((W_k^{i1})^+) - B((W_k^{i1})^-)) \\ & + \sum_{k=i+1}^{m(i)-1} (I\{V_{k+1}^{i3} < s\}(K + c(S - V_{k+1}^{i3})) - I\{V_k^{i1} < s\}(K + c(S - V_k^{i1}))) \\ & + H((W_{m(i)}^{i3})^+) + B((W_{m(i)}^{i3})^-) - J_n^{i1} \\ = & c\tilde{D}_{i+1} + \sum_{k=i+1}^{m(i)-1} (H((W_k^{i3})^+) + B((W_k^{i3})^-) - H((W_k^{i1})^+) - B((W_k^{i1})^-)) \\ & + H((W_{m(i)}^{i3})^+) + B((W_{m(i)}^{i3})^-) - J_n^{i1}. \end{aligned}$$

Now we simplify the estimator further. First, suppose that there are orders that are placed after period i that arrive before period $m(i)$ in sample path DNP_i . Let $\Gamma_i = \{\gamma_1, \gamma_2, \dots, \gamma_{p(i)}\}$ be the set of periods in

which such orders arrive, where $p(i)$ denotes the number of such periods, $\gamma_{p(i)} < \tilde{m}(i)$ (if no such orders exist, then $p(i) = 0$ and Γ_i is empty), and let $O_i(1), O_i(2), \dots, O_i(p(i))$ denote the corresponding order amounts. Note that this includes the case of multiple orders arriving in the same period. Then, we then have

$$\begin{aligned} W_{k+1}^{i1} - W_{k+1}^{i3} &= D_k - \tilde{D}_{i+1}, \text{ for } k = i+1, \dots, m(i), k \notin \Gamma_i, \\ W_{k+1}^{i1} - W_{k+1}^{i3} &= D_k - \tilde{D}_{i+1} + O_i(j), \text{ for } k = \gamma_j, j = 1, \dots, p(i). \end{aligned} \quad (20)$$

Next we note that (z_i, W_k^{i1}) and (z_i, W_k^{i3}) , $k = i+1, \dots, m(i), k \notin \Gamma_i$ are equal in distribution, so we have

$$\begin{aligned} & E \left[\frac{f(z_i)}{1-F(z_i)} \sum_{k=i+1}^{m(i)-1} (H((W_k^{i3})^+) + B((W_k^{i3})^-) - H((W_k^{i1})^+) - B((W_k^{i1})^-)) \right] \\ &= E \left[\frac{f(z_i)}{1-F(z_i)} \sum_{k \in \Gamma_i} (H((W_k^{i3})^+) + B((W_k^{i3})^-) - H((W_k^{i1})^+) - B((W_k^{i1})^-)) \right], \quad \forall i \in L^*. \end{aligned}$$

Therefore, (19) is equal in expectation to

$$\begin{aligned} & \frac{1}{n} \left[\sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) + cE[D] \sum_{i \in L^*(n)} \frac{f(z_i)}{1-F(z_i)} \right. \\ & + \sum_{i \in L^*(n)} \frac{f(z_i)}{1-F(z_i)} \sum_{k \in \Gamma_i} (H((W_k^{i3})^+) + B((W_k^{i3})^-) - H((W_k^{i1})^+) - B((W_k^{i1})^-)) \\ & \left. + \sum_{i \in L^*(n)} \frac{f(z_i)}{1-F(z_i)} (H((W_{m(i)}^{i3})^+) + B((W_{m(i)}^{i3})^-)) - \sum_{i \in L^*(n)} \frac{f(z_i)}{1-F(z_i)} J_n^{i1} \right]. \end{aligned} \quad (21)$$

Let us consider the last term in (21). Recalling that $\{X_i\}$ is Harris ergodic, let us denote the regenerative points by $\tau_1, \tau_2, \dots, \tau_{r(n)}$, where $r(n)$ is the number of regenerative points up to period n , so $\tau_{r(n)}$ denotes the period of the most recent regenerative point of the process up to period n . Then, by definition, there are no outstanding orders in period $\tau_{r(n)}$ in both NP and DNP_i . For $i \leq \tau_{r(n)}$, the two sample paths NP and DNP_i become exactly the same after period $\tau_{r(n)}$, and hence, $J_n^{i1} = J_n$. In addition, V_n and W_n are independent of V_i and W_i for $i < \tau_{r(n)-1}$, the regenerative point just prior to $\tau_{r(n)}$, since regenerative cycles of a Harris ergodic process are one-dependent (see Asmussen 1987, p. 151). Therefore, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} E \left[\sum_{i \in L^*(n)} \frac{f(z_i)}{1-F(z_i)} J_n^{i1} \right] = \lim_{n \rightarrow \infty} \frac{1}{n} E \left[\sum_{i \in L^*(n), i < \tau_{r(n)-1} } \frac{f(z_i)}{1-F(z_i)} J_n^{i1} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} E \left[\sum_{i \in L^*(n), i < \tau_{r(n)-1} } \frac{f(z_i)}{1-F(z_i)} \right] E[J_n] = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i \in L^*(n), i < \tau_{r(n)-1} } \frac{f(z_i)}{1-F(z_i)} \right) J \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i \in L^*(n)} \frac{f(z_i)}{1-F(z_i)} \right) J = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i \in L^*(n)} \frac{f(z_i)}{1-F(z_i)} \right) \bar{J}_n, \end{aligned}$$

where all but the second step are justified by ergodicity, the second step following from the discussion above.

Combining this result with $\lim_{n \rightarrow \infty} m(i) = \tilde{m}(i)$, and the fact that (21) is unbiased for $\partial \bar{J}_n / \partial \theta$, we conclude that

$$\begin{aligned}
& \frac{1}{n} \left[\sum_{i=1}^n (I\{W_i > 0\}H'(W_i) - I\{W_i < 0\}B'(-W_i)) + cE[D] \sum_{i \in L^*(n)} \frac{f(z_i)}{1 - F(z_i)} \right. \\
& + \sum_{i \in L^*(n)} \frac{f(z_i)}{1 - F(z_i)} \sum_{k \in \Gamma_i} (H((W_k^{i3})^+) + B((W_k^{i3})^-) - H((W_k^{i1})^+) - B((W_k^{i1})^-)) \\
& \left. + \sum_{i \in L^*(n)} \frac{f(z_i)}{1 - F(z_i)} (H((W_{\tilde{m}(i)}^{i3})^+) + B((W_{\tilde{m}(i)}^{i3})^-)) - \bar{J}_n \sum_{i \in L^*(n)} \frac{f(z_i)}{1 - F(z_i)} \right] \quad (22)
\end{aligned}$$

is an *asymptotically* unbiased estimator for $\lim_{n \rightarrow \infty} \partial \bar{J}_n / \partial q$. Harris ergodicity and the results in Hu and Strickland (1990) and Glasserman, Hu, and Strickland (1992) can be used to prove that (22) is in fact a strongly consistent estimator for $\partial J / \partial q$, i.e., it converges a.s. to $\partial J / \partial q$, so finally we have

Theorem 4. Assume the system is Harris ergodic. Then under **C0-C2**, the IPA estimator for $\partial J / \partial s$ given by (14) and the SPA estimator for $\partial J / \partial q$ given by (22) are strongly consistent.

The key features distinguishing this final estimator for $\partial J / \partial q$ from the previous ones are that the terms W_k^{i3} ($k \in \Gamma_i, k = \tilde{m}(i)$) can be obtained from NP , the original sample path, via (20), so the estimator given by (22) is efficient in the sense of not requiring any additional simulation. Finally, we remark that (22) simplifies to the SPA estimator given in Fu (1993) when L is a deterministic constant, in which case $\Gamma_i = \emptyset$.

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