

Optimal Amplifier Placement for Optical Networks

Ye Tao J.Q. Hu

Department of Manufacturing Engineering
 Boston University
 15 Saint Mary's Street
 Brookline, MA 02446

Abstract—Amplifiers are one of the most important components in an optical WDM network, and they are used to restore the strength of optical signals which are attenuated by optical fibers as they propagate through the network. However, amplifiers also introduce the spontaneous emission noise which degrades the optical signal-to-noise ratio (OSNR). In this paper, we study the amplifier placement problem with the objective of minimizing the total amplifier cost subject to the constraint on the cumulative spontaneous emission noise. We assume that there are several different types of amplifiers. Therefore, we need to decide not only where to place amplifiers, but also what type of amplifier to use at each location. We consider two different cases: 1) the discrete case (brown-field design) in which the possible locations where amplifiers can be placed are given, and 2) the continuous case (green-field design) in which amplifiers can be placed anywhere in the network with no restriction. For the discrete case, we propose two methods: one based on dynamic programming and the other based on heuristics. For the continuous case, we first formulate the problem as a mixed integer optimization problem, and then propose a numerical method to solve it. Some numerical examples are also provided.

Keywords: optical WDM network, optical amplifier, spontaneous emission noise, optical signal-to-noise ratio, dynamic programming, convex optimization.

I. INTRODUCTION

In an optical network, optical signals from transmitters attenuate gradually as they propagate through optical fibers. Therefore, the strength of these optical signals needs to be restored before they become too weak to be detected by receivers. Before the advent of optical amplifiers, this can only be done by using optoelectronic regenerators to regenerate the signals. A regenerator converts an optical signal to an electrical signal for processing and then converts it back for transmission. This approach requires very expensive high-speed electronic circuitry, and furthermore, at each regenerating point, every wavelength needs to have its own regenerator due to its unique wavelength and power range. Therefore, it is very expensive to use this approach for optical WDM (wavelength division multiplexed) networks. On the contrary, an optical amplifier can restore the strength of an optical signal without any optical-electrical-optical conversion. Furthermore, it can amplify several WDM signals simultaneously. Hence, optical amplifiers have become a very important component of today's WDM networks.

There are several different types of amplifiers available, such as Erbium-doped fiber amplifiers (EDFAs), Praseodymium-doped fiber amplifiers (PDFAs), and semiconductor optical

amplifiers (SOAs). EDFAs are the most commonly used amplifiers in practice. EDFAs are made of short optical fibers doped with the rare earth element "erbium," whose amplification process is caused by the stimulated emission of those excited erbium ions from higher energy level to lower energy level. Recently, Raman EDFAs, which enable optical signals propagate over longer distance, have been made available commercially.

Typically, the cost of a nationwide optical network is dominated by optical transponders and optical amplifiers. However, with a few exceptions ([4], [6], [7], [2]), most works on the design of optical networks have focused on optical transponders, by studying the issues related to traffic grooming and routing and wavelength assignment. The first work to study the amplifier placement problem is [4], where it was considered how to place amplifiers in a star/tree type network so that optical signals are equalized at each receiver. The approach used in [4] is a link-by-link based. This work was improved in [6], where a global based optimization approach was taken to minimize the total number of amplifiers required in the network, and was extended in [7], where unequally powered wavelengths were allowed so that the total number of amplifiers required can be further reduced. In [2], a simulated annealing based method was proposed to solve the problem.

Though an optical amplifier can restore the strength of optical signals, it also introduces the spontaneous emission noise which degrades the optical signal-to-noise ratio (OSNR). This is a major problem associated with amplifiers. In general, the spontaneous emission noise of an amplifier (with saturated gain) grows exponentially with respect to the length of a fiber span. Therefore, for a long transmission system we need to place enough amplifiers in the system in order to limit the cumulative spontaneous emission noise (as well as for adequate amplification gain). On the other hand, we would like to use as fewer number of amplifiers as possible to minimize the cost of amplifiers.

In all the previous works mentioned above, the issue of spontaneous emission noise is completely ignored. Also, it was assumed that only one type of amplifier is available, while in reality there are often several different types of amplifiers available and some may be more suitable for longer spans and some for shorter spans. In addition, it was assumed that amplifiers can be placed anywhere in the network, which is rarely the case in most applications where amplifiers can be only placed in some pre-specified locations.

In this paper, we study the amplifier placement problem for a linear transport system. Our objective is to minimize

the total amplifier cost subject to the constraint on the cumulative spontaneous emission noise. We assume that there are several different types of amplifiers available. Hence, we need to decide not only where to place amplifiers (amplifying locations), and also which type of amplifier to use at each location. We consider two different cases: the discrete case and the continuous case. In the discrete case (sometimes referred as brown-field design), the possible locations where amplifiers can be placed are given. This is the case seen in most applications since there are usually many (physical and economic) restrictions on where amplifiers can be placed. In the continuous case (sometimes referred as green-field design), it is assumed that amplifiers can be placed anywhere in the network without any restriction. Though this is not very realistic in most applications, it may provide some guidelines on choosing amplifier locations. The main constraint we consider is the cumulative spontaneous emission noise of the system and we have to make sure that it does not exceed certain limit. Our work in this paper is significantly different from the previous works, and to the best of knowledge it is the first time that the issue of spontaneous emission noise is taken into consideration.

The rest of this paper is organized as follows. The amplifier placement problem is presented in Section 2, along with the formulations for both continuous and discrete cases. In Section 3, we consider the discrete case and develop a method based on the technique of dynamic programming. To improve computation efficiency, we also propose a heuristic method and provide some numerical results to illustrate it. In Section 4, we consider the continuous case. When there is only one type of amplifier available, we prove that amplifiers should be placed in equal distance in the system. In general, we show that the problem can be formulated as a mixed integer optimization problem. We then propose a numerical method, which is based on convex optimization techniques, to solve the problem, and provide some numerical results to validate the numerical method. We also discuss how the continuous case can be converted to the discrete case by using the method of discretization. Finally, a conclusion and some discussions are provided in Section 5.

II. PROBLEM FORMULATION

In this section, we present the mathematical formulation for the amplifier placement problem. In this paper, we only consider a linear transport system, in which we need to determine amplification locations (i.e., where amplifiers are placed), and also what type of amplifier should be placed at each location (since we assume that there are several types of amplifiers available). Our objective is to minimize the total cost of amplifiers subject to the constraint that the cumulative spontaneous emission noise of the linear system does not exceed certain limit. We consider two different cases: the discrete case and the continuous case. In the discrete case, the possible locations (a finite number) where amplifiers can be placed are given, while in the continuous case, amplifiers can be placed anywhere along the linear system.

We first introduce the following notation:

- S : the number of types of amplifiers available;
- c_s : the cost of each type s amplifier ($s = 1, \dots, S$);
- L : the total length of the linear system;
- E : the certain limit of the total cumulative spontaneous emission noise for the linear system;
- $f_s(l)$: the spontaneous emission noise produced by type s amplifier for a span with length l (a span is a fiber segment between two adjacent amplifiers).
- N : the number of amplifiers to be placed in the system;
- L_n : the length of the n th fiber span ($n = 1, \dots, N$).

Note that in practice the spontaneous emission noise of an amplifier can be either measured or estimated. For example, the spontaneous emission noise of an amplifier with saturated gain G can be estimated as $p(G - 1)$, where p is the spontaneous emission factor associated with the amplifier (e.g., see [8]). For a fiber span with length l , the power loss associated with it is $e^{\alpha l}$, where α is the fiber attenuation factor. So the saturated gain G for the amplifier is $G = e^{\alpha l}$ and its spontaneous emission noise is given by $p(e^{\alpha l} - 1)$. However, in reality, the performance of an amplifier often becomes poor for a very short span (which leads to a very small saturated gain for the amplifier). One way to solve this problem is to pad a short span by increasing the VOA (variable optical attenuator) on the output of the previous amplifier (the one launching into the span). With this padding, the power loss associated with a span with length l becomes $\max(k, e^{\alpha l})$, where k is the minimum power loss for a span, hence, in this case the spontaneous emission noise of an amplifier with saturated gain is given by $p(\max(k, e^{\alpha l}) - 1) \approx p \max(k, e^{\alpha l})$ (since in general $k \gg 1$). We should emphasize that most results in this paper are independent of whatever formula used for estimating the spontaneous emission noise of an amplifier.

In what follows, we first consider the discrete case, and then the continuous case.

A. The Discrete Case

In the discrete case, we assume that there are only a finite number of points (locations) available where amplifiers can be placed.

- M : the number of possible amplification points (locations) available;
- l_m : the length of the fiber span between points $m - 1$ and m , $m = 1, 2, \dots, M$ (point 0 is the starting point of the system and point M is the end point of the system);
- $x_{ms} = \begin{cases} 1 & \text{if a type } s \text{ amplifier is placed at point } m; \\ 0 & \text{otherwise;} \end{cases}$
- m_n : the index of the point where the n th amplifier is placed, $n = 1, \dots, N$ (without loss of generality, we assume that no amplifier is needed at point 0, i.e., $m_1 \geq 1$, and the last amplifier is placed at point M , i.e., $m_N = M$).

Based on the above definitions, it is clear that

$$N = \sum_{m=1}^M \sum_{s=1}^S x_{ms}$$

$$m_{n+1} = \min\{m : \sum_{s=1}^S x_{ms} = 1, m > m_n\} \quad \text{with } m_0 = 0$$

$$L_n = \sum_{m=1+m_n}^{m_{n+1}} l_m$$

The problem of optimal amplifier placement can then be formulated as the following integer optimization problem:

$$(\text{OP}_1) \quad \min \quad \sum_{m=1}^M \sum_{s=1}^S c_s x_{ms}$$

$$\text{s.t.} \quad \sum_{s=1}^S x_{ms} \leq 1 \quad m = 1, 2, \dots, M \quad (1)$$

$$\sum_{n=1}^N \sum_{s=1}^S x_{m_n s} f_s(L_n) \leq E \quad (2)$$

x_{ms} are binary variables.

The above integer optimization problem is denoted as (OP_1) . Note that $\sum_{m=1}^M \sum_{s=1}^S c_s x_{ms}$ is the total amplifier cost and the two constraints are

- (1): At each point, at most one amplifier can be placed;
- (2): The cumulative spontaneous emission noise of the system (which is equal to the summation of the spontaneous emission noises produced by all the amplifiers) should not exceed E .

In Section 3, we will use the technique of dynamic programming to solve the amplifier placement problem (OP_1) .

B. The Continuous Case

We now consider the continuous case in which amplifiers can be placed anywhere in the linear system. In this case, we use slight different decision variables from those for the discrete case:

$$x_{ns} = \begin{cases} 1 & \text{if the } n\text{th amplifier placed on the line is type } s \\ 0 & \text{otherwise} \end{cases}$$

The amplifier placement problem for the continuous case can be formulated as:

$$(\text{OP}_2) \quad \min \quad \sum_{n=1}^N \sum_{s=1}^S c_s x_{ns}$$

$$\text{s.t.} \quad \sum_{s=1}^S x_{ns} = 1 \quad n = 1, 2, \dots, N \quad (3)$$

$$\sum_{n=1}^N L_n = L \quad (4)$$

$$\sum_{n=1}^N \sum_{s=1}^S x_{ns} f_s(L_n) \leq E \quad (5)$$

x_{ns} are binary variables.

We denote the above problem as (OP_2) . Note that in (OP_2) L_n 's are also decision variables, while in (OP_1) they are completely determined by binary decision variables x_{ms} . However, it is interesting to note that for the continuous case the order in which different types of amplifiers are placed in the system is not crucial since fiber spans are not fixed any more (they need to be optimized as well), i.e., we can change the order in which different types of amplifiers are placed in the system as long as we also keep the lengths of their associated spans fixed, then the total cumulative spontaneous emission noise remains the same. Therefore, we can modify (OP_2) as follows. First we need to introduce some additional notation.

$N(s)$: the number of type- s amplifiers to be placed in the system ($s = 1, 2, \dots, S$);

$L_n(s)$: the length of the fiber span associated with the n th type- s amplifier ($n = 1, \dots, N(s)$, $s = 1, 2, \dots, S$);

$L(s)$: the total length of spans associated with type- s amplifiers ($s = 1, 2, \dots, S$).

Then, (OP_2) is equivalent to

$$(\text{OP}_3) \quad \min \quad \sum_{s=1}^S c_s N(s)$$

$$\text{s.t.} \quad \sum_{n=1}^{N(s)} L_n(s) = L(s) \quad s = 1, 2, \dots, S \quad (6)$$

$$\sum_{s=1}^S L(s) = L \quad (7)$$

$$\sum_{s=1}^S \sum_{n=1}^{N(s)} f_s(L_n(s)) \leq E \quad (8)$$

$N(s)$ are integer variables.

In general, (OP_3) is much easier to deal with than (OP_2) , so in the rest of this paper, we will exclusively use (OP_3) for the continuous case. In Section 4, we will show how to convert (OP_3) into a mixed convex optimization problem and then propose a numerical method to solve it.

III. SOLVING THE DISCRETE CASE

In this section, we solve the amplifier placement problem for the discrete case by using the technique of dynamic programming. For ease of exposition, we focus exclusively on two special cases: 1) one amplifier type (i.e., $S = 1$), and 2) two amplifier types (i.e., $S = 2$). The extension to general S is quite straightforward. For $S \geq 2$, we will also propose a heuristic method based on $S = 1$. We will provide numerical results to show that this heuristic method provides approximation solutions while reduces computation efforts, in some cases, quite significantly.

A. For $S = 1$

We now try to solve (OP_1) for $S = 1$. First, let us consider an alternative amplifier placement problem: for a fixed N , find

an optimal way to place N amplifiers in the system so that the cumulative spontaneous emission noise is minimized. It is quite clear that once we solve this problem, we can obtain the minimum cumulative spontaneous emission noise for each N , then (OP₁) is reduced to finding the smallest N for which the minimum cumulative spontaneous emission noise is no larger than E . Therefore, in what follows we focus on the problem of finding optimal placement of N amplifiers with the objective of minimizing the cumulative spontaneous emission noise.

Let $c_1(i, j)$ ($i < j$, $i = 0, 1, \dots, M-1$, and $j = 1, 2, \dots, M$) be the spontaneous emission noise for the span between two nodes i and j (i.e., one amplifier is placed at node i except for $i = 0$, another amplifier placed at node j , and there is no amplifier placed between nodes i and j). Also, define $g(k, j)$ as the minimum cumulative spontaneous emission noise between nodes 0 and j with exact k amplifiers placed between nodes 0 and j (assuming that no amplifier is placed at node 0 and the last amplifier is always placed at node j). It is not difficult to see that

$$c_1(i, j) = f_1\left(\sum_{m=i}^{j-1} l_{m+1}\right) \quad (9)$$

$$g(k, j) = \min_{k-1 \leq i \leq j-1} \{g(k-1, i) + c_1(i, j)\} \quad (10)$$

We note that (10) is a typical recursive equation seen in dynamic programming. More specifically, $g(k, j)$ ($j = 1, 2, \dots, M; k = 1, 2, \dots, j$) can be calculated via (10) as follows. First, we have

$$\begin{aligned} g(0, j) &= c_1(0, j), \quad j = 1, 2, \dots, M \\ g(k, k) &= \sum_{i=1}^k c_1(i-1, i), \quad k = 1, 2, \dots, M. \end{aligned}$$

Then, we can obtain $g(k+1, j)$ ($j = k+1, \dots, M$) recursively by using (10) and $g(k, j)$ ($j = k, \dots, M$).

Note that $g(N, M)$ is the minimum cumulative spontaneous emission noise (for the entire linear system) given that N amplifiers are placed in the system, which is exactly what we need. So we just need to find the smallest N such that $g(N, M) \leq E$. Finally, we point out that the complexity of the above recursive algorithm is $O(M^2)$.

B. For $S = 2$

We now consider the case $S = 2$. Similar to $S = 1$, we first consider the optimal amplifier placement problem in which we want to minimize the cumulative spontaneous emission noise with a fixed number (N_1) of type-1 amplifiers and a fixed number (N_2) of type-2 amplifiers. Then the original problem (OP₁) for $S = 2$ is equivalent to finding a pair of (N_1, N_2) with the smallest value $c_1 N_1 + c_2 N_2$ among those whose minimum cumulative spontaneous emission noise is no larger than E . The recursive algorithm developed for $S = 1$ can be easily extended for $S = 2$. Since now we can place either type-1 or type-2 amplifier at each point, let $c_s(i, j)$ be the spontaneous emission noise for the span between adjacent nodes i and j if a type- s amplifier is placed at node j , and another amplifier, either type-1 or type-2, is placed at node i

(except for $i = 0$), $s = 1, 2$. Furthermore, let $g(k_1, k_2, j)$ be the minimum cumulative spontaneous emission noise between nodes 0 and j given that k_1 type-1 amplifiers and k_2 type-2 amplifiers are placed between nodes 0 and j (again assuming that no amplifier is placed at node 0 and the last amplifier is placed at node j). Then we have the following recursive equation:

$$\begin{aligned} &g(k_1, k_2, j) \\ &= \min_{k_1+k_2-1 \leq i \leq j-1} \left[\begin{aligned} &g(k_1-1, k_2, i) + c_1(i, j), \\ &g(k_1, k_2-1, i) + c_2(i, j) \end{aligned} \right] \quad (11) \end{aligned}$$

The boundary values of $g(k_1, k_2, j)$ can be obtained as follows:

- $g(0, 0, j) = 0$ for all j ;
- $g(k_1, 0, j)$ is the minimum cumulative spontaneous emission noise between nodes 0 and j with only type-1 amplifiers being used, hence it can be obtained by using the algorithm developed for $S = 1$. $g(0, k_2, j)$ can be obtained in the same way.

Based on the above boundary values, $g(k_1, k_2, j)$ can be calculated recursively based on (11). It is not difficult to see that the computational complexity of calculating all $g(k_1, k_2, j)$ is $O(M^3)$.

The above procedure for $S = 2$ can be easily extended to general S . It is also clear that for general S , the computational complexity is $O(M^{S+1})$. In the next subsection, we develop a heuristic method that will reduce computational efforts required.

C. A Heuristic Method

In this section, we propose a heuristic method for solving the amplifier placement problem (OP₁). The basic idea of our method is to solve the problem in two steps. In the first step, we obtain N and L_n ($n = 1, 2, \dots, N$), i.e., we determine the total number of amplifiers to be used in the system and the locations where they are to be placed. In the second step, we decide which type of amplifier to be placed at each location.

Without loss of generality, we assume that $c_1 \geq c_2 \geq \dots \geq c_S$, i.e., type-1 amplifier is the most expensive, type-2 amplifier is the second most expensive amplifier, and so on. Clearly, it is reasonable to expect that the more expensive an amplifier is, the better its performance is, i.e., the less spontaneous emission noise it produces. Hence, we assume

$$\text{(A1)} \quad f_1(l) \leq f_2(l) \leq \dots \leq f_S(l) \text{ for } l \geq 0.$$

We note in case $f_s(l) = p_s(e^{al} - 1)$ ($s = 1, 2, \dots, S$), where p_s is the spontaneous emission factor associated with type- s amplifier, $f_s(l) \leq f_t(l)$ implies $p_s \leq p_t$ ($1 \leq s < t \leq S$), which leads to $f_s(l_2) - f_s(l_1) \leq f_t(l_2) - f_t(l_1)$ for $l_2 \geq l_1 \geq 0$. Therefore, we further assume that

$$\text{(A2)} \quad f_s(l_2) - f_s(l_1) \leq f_t(l_2) - f_t(l_1) \text{ for } 1 \leq s < t \leq S \text{ and } l_2 \geq l_1 \geq 0.$$

We now present the following useful result:

Proposition 1: Under (A2), better amplifiers should be assigned to longer spans in order to minimize the total cumulative spontaneous emission noise, i.e., if type- s amplifier is

assigned to the i th span with length L_i and type- t amplifier is assigned to the j th span with length L_j ($s \neq t$) and $L_i \geq L_j$, then $s < t$, i.e., type- s amplifier is better than type- t amplifier.

Proof: We prove it by using contradiction. Suppose that $s > t$. Then based on (A2), we have

$$f_t(L_i) + f_s(L_j) \leq f_s(L_i) + f_t(L_j),$$

which implies that if we switch the two amplifiers for the i th span and the j th span, then the total cumulative spontaneous emission noise is reduced. Therefore, we should assign type- s amplifier to the j th span and type- t amplifier to the i th span. This completes our proof. \blacksquare

We are now ready to present our heuristic method. As mentioned earlier, the first step in our heuristic method is to obtain N and L_n ($n = 1, 2, \dots, N$). To do so, we consider a modified version of (OP₁) in which we replace the original objective function by N , i.e., we want to minimize the number of amplifier locations in the system. It is obvious that since type-1 amplifier has the best performance, we should only use type-1 amplifiers in order to minimize N . But if we only use type-1 amplifiers, then the problem becomes the amplifier placement problem with $S = 1$. Therefore, we can simply use the recursive algorithm in Section 3.1 to obtain N and L_n ($n = 1, 2, \dots, N$).

We should point out that in practice both equipment vendors and network operators have certain requirements on the length of each span, in some cases they are rather restrictive. Therefore, it may not be very difficult to obtain N and L_n based on those requirements, or at least those requirements can lead to a very limited number of options for N and L_n . Under such circumstances, the second step of the problem we will solve next is interesting by itself.

In the second step, we essentially need to consider a special case of the amplifier placement problem (OP₁) in which N and L_n ($n = 1, 2, \dots, N$) are given (they have been determined in the first step), i.e., we only need to decide which type of amplifier to place at each amplifier location. Let L_n^* be the n th longest span among L_1, L_2, \dots, L_N , i.e., $L_1^* \geq L_2^* \geq \dots \geq L_N^*$. Then we can easily derive the following results based on Proposition 1.

Proposition 2: Under (A2), the optimal solution for (OP₁) with fixed N and L_n ($n = 1, 2, \dots, N$) is such that type- s amplifiers are used for spans $L_{i_{s-1}}^*, \dots, L_{i_s}^*$ ($s = 1, 2, \dots, S$), where $1 = i_0 \leq i_1 \leq i_2 \leq \dots \leq i_S = N + 1$.

In particular, let us consider the case $S = 2$. For $S = 2$, Proposition 2 says that for the optimal solution there exists i_1 such that type-1 amplifiers are used for spans $L_1^*, \dots, L_{i_1-1}^*$ and type-2 amplifiers are used for spans $L_{i_1}^*, \dots, L_N^*$. It is clear that i_1 can be obtained as:

$$i_1 = \min \left\{ i : \sum_{j=1}^{i-1} f_1(L_j^*) + \sum_{j=i}^N f_2(L_j^*) \leq E \right\}$$

In other words, if we start with type-1 amplifiers for all spans, then to find i_1 we just need to replace type-1 amplifiers with type-2 amplifiers, starting with the shortest span L_N^* , then the second shortest span L_{N-1}^* , and so on, until the

total cumulative spontaneous emission noise exceeds E . (Each time a type-1 amplifier is replaced by a type-2 amplifier, the total cumulative spontaneous emission noise increases.) This procedure can be easily extended even when (A2) does not hold, in which case instead of starting with the shortest span, we should start with span L_i which has the smallest increment in spontaneous emission noise $f_2(L_i) - f_1(L_i)$ when type-1 amplifier is replaced by type-2 amplifier.

For general S , in order to obtain i_1, i_2, \dots, i_{S-1} we need to solve the following optimization problem:

$$\begin{aligned} \text{(OP}_4\text{)} \quad & \min \quad \sum_{s=1}^S c_s(i_s - i_{s-1}) \\ & \text{s.t.} \quad \sum_{s=1}^S \sum_{j=i_{s-1}}^{i_s-1} f_s(L_j^*) \leq E \\ & \quad 1 = i_0 \leq i_1 \leq \dots \leq i_{S-1} \leq i_S = N + 1 \end{aligned}$$

We note that the computational complexity for solving (OP₄) is $O(M^{S-1})$ in the worst case (if we use the exhaustive search method) and the recursive algorithm for solving (OP₁) with $S = 1$ is $O(M^2)$. Therefore, the computational complexity of our heuristic algorithm is $O(M^2)$ for $S = 2$ and $O(M^{S-1})$ for $S > 2$. Also recall that the computational complexity of the recursive algorithm for solving (OP₁) is $O(M^{S+1})$. So, our heuristic method is more efficient, and in many cases savings in computational efforts are quite significant (e.g., when $S = 2, 3$). This will become much clearer when we present our numerical results later. Furthermore, we should point out that more efficient methods may be developed to solve (OP₄). For example, similar to $S = 2$, we can start with type-1 amplifier for all spans. Then we use type- S amplifiers to replace as many type-1 amplifiers as possible, starting with the longest span, then the second longest span, and so on. Next we use type- $(S - 1)$ amplifiers to replace the remaining type-1 amplifiers, following by type- $(S - 2)$ amplifiers, and finally type-2 amplifiers. The computational complexity of this procedure is only $O(M)$.

Before closing this section, we provide some numerical results to validate our heuristic method. All our numerical results are obtained by using *Matlab 5.3* on a PC with 1.6GHz CPU and 256M memory.

Numerical Example 1 ($S = 2$). In this example, we use $f_s(l) = p_s \max(e^{\alpha l}, k_s)$ ($s = 1, 2$), with $p_1 = 2.51$, $p_2 = 7.08$, $k_1 = 125.89$, and $k_2 = 63.1$. We use five different sets of values for α, c_1, c_2, E, L and M , which are listed in Table 1. For each set of α, c_1, c_2, E, L and M , we randomly generate 100 different data instances for l_1, l_2, \dots, l_M . The numerical results for each set of α, c_1, c_2, E, L and M are displayed in Table 2, which includes

- **Total Instances:** the number of randomly generated data sets of l_1, l_2, \dots, l_M ;
- **Average Error:** the average error for the heuristic method, where the error for each randomly generated instance of l_1, l_2, \dots, l_M is defined as

$$\text{error} = \frac{C_a - C_o}{C_o}$$

where C_a is the total amplifier cost associated with the solution produced by the heuristic method and C_o is the minimum amplifier cost produced by the recursive algorithm;

- *Average Run Time*: the average run time (in seconds) for an instance of l_1, l_2, \dots, l_M based on both the heuristic method and the recursive algorithm.

Set	α	c_1	c_2	E	L	M
1	0.02	2.0	1.0	7944	1000	14
2	0.02	2.0	1.0	7944	2000	30
3	0.02	1.5	1.0	17783	3000	42
4	0.025	1.5	1.0	7944	1000	18
5	0.025	2.0	1.0	7944	2000	38

TABLE I
VALUES FOR α, c_1, c_2, E, L AND M IN EXAMPLE 1

Set	Total	Average	Avg. Run Time (s)	
	Instances	Error	Heuristic	Recursive
1	100	2.89%	0.08	2.18
2	100	20.38%	1.25	107.13
3	100	7.73%	5.70	580.40
4	100	4.71%	0.16	7.66
5	100	18.30%	4.32	306.73

TABLE II
NUMERICAL RESULTS FOR EXAMPLE 1

From the numerical results in Table 2 we can see that the errors of our heuristic method range from 3% to 20% and that the computational efforts are reduced significantly in all the cases.

Numerical Example 2 ($S = 3$). In this example, we still use $f_s(l) = p_s \max(e^{\alpha l}, k_s)$ ($s = 1, 2, 3$), with $p_1 = 2.51$, $p_2 = 7.08$, $p_3 = 5.00$, $k_1 = 125.89$, $k_2 = 63.1$, and $k_3 = 94.65$. We also use five different sets of values for $\alpha, c_1, c_2, c_3, E, L$ and M , which are listed in Table 3. The numerical results are presented in Table 4.

Set	α	c_1	c_2	c_3	E	L	M
1	0.02	2.0	1.5	1.0	7944	1000	14
2	0.02	2.0	1.5	1.0	7944	2000	30
3	0.02	1.5	1.2	1.0	17783	3000	42
4	0.025	1.5	1.2	1.0	7944	1000	18
5	0.025	2.0	1.5	1.0	7944	2000	38

TABLE III
VALUES FOR $\alpha, c_1, c_2, c_3, E, L$ AND M IN EXAMPLE 2

Since the run time is much longer for $S = 3$ than for $S = 2$, so we run 100 instances for three data sets and only 50 instances for the other two data sets. The numerical results are listed in the Table IV. Again, the results are very similar to those in Example 1.

IV. SOLVING THE CONTINUOUS CASE

We first consider $S = 1$, then $S \geq 2$. Throughout this section, we assume that $f_s(l)$ and $xf_s(1/x)$ ($s = 1, \dots, S$)

Set	Total	Average	Avg. Run Time (s)	
	Instances	Error	Heuristic	Recursive
1	100	1.92%	0.11	3.36
2	100	16.77%	5.53	537.74
3	50	6.58%	68.58	8665.53
4	100	3.01%	0.37	20.99
5	50	14.85%	19.88	1695.01

TABLE IV
NUMERICAL RESULTS FOR EXAMPLE 2

are convex functions with respect to l and x , respectively. (This is certainly the case, for example, if $f_s(l) = p_s(e^{\alpha l} - 1)$, $p_s(\max(e^{\alpha l}, k_s) - 1)$, or $p_s \max(e^{\alpha l}, k)$). For $S = 1$, we prove that amplifiers should be placed in equal distance in the system. For $S \geq 2$, we show that the problem can be converted into a mixed integer optimization problem, and then propose a numerical method based on the techniques of convex optimization. We also discuss the possibility of converting the continuous case to the discrete case.

A. For $S = 1$

Same as we did for the discrete case in Section 3.1, for $S = 1$, we can convert (OP₃) to an equivalent problem of finding the smallest N for which the minimum cumulative spontaneous emission noise does not exceed E . Therefore, for a fixed N , we need to find the minimum of $\sum_{n=1}^N f_1(L_n)$ subject to $\sum_{n=1}^N L_n = L$. Note that since $S = 1$ we have $L_n(1) = L_n$.

If $f(\cdot)$ is a convex function, then we have the following result:

Proposition 3: If $f_1(\cdot)$ is a convex function and $\sum_{n=1}^N L_n = L$ ($L_n > 0$), then $\sum_{n=1}^N f_1(L_n)$ is minimized at $L_1 = \dots = L_N = L/N$.

Proof: Since $f_1(\cdot)$ is convex, we have

$$\begin{aligned} f_1\left(\frac{L_1}{N} + \dots + \frac{L_N}{N}\right) &\leq \frac{1}{N}f_1(L_1) + \dots + \frac{1}{N}f_1(L_N) \\ &= \frac{1}{N} \sum_{n=1}^N f_1(L_n), \end{aligned}$$

which results in

$$\sum_{n=1}^N f_1(L_n) \geq N f_1\left(\frac{L_1 + \dots + L_N}{N}\right) = N f_1(L/N).$$

Hence $\sum_{n=1}^N f_1(L_n)$ is minimized at $L_1 = \dots = L_N = L/N$, and the minimum is $N f_1(L/N)$. ■

Based on Proposition 3, we just need to find the smallest N satisfying $N f_1(L/N) \leq E$. To find such an N , we can enumerate N from 1 to M at the worst, which has computational complexity of $O(M)$. Also, it is clear that amplifiers should be placed in equal distance in the optimal solution.

B. For $S \geq 2$

Based on our discussions for $S = 1$ in the previous subsection, it is clear that if $f_s(\cdot)$ is a convex function, then we should have $L_1(s) = L_2(s) = \dots = L_{N(s)}(s) = L(s)/N(s)$ for the optimal solution. Hence, assuming $f_s(\cdot)$ is convex for $s = 1, 2, \dots, S$, (OP₃) can be simplified as:

$$\begin{aligned}
 \text{(OP}_5\text{)} \quad & \min \quad \sum_{s=1}^S c_s N(s) \\
 & \text{s.t.} \quad \sum_{s=1}^S L(s) = L \\
 & \quad \sum_{s=1}^S N(s) f_s \left(\frac{L(s)}{N(s)} \right) \leq E \\
 & \quad N(s) \text{ are integer variables.}
 \end{aligned}$$

In general, one cannot expect to obtain analytical solutions for (OP₅). In what follows, we propose a numerical method to solve (OP₅).

First we relax the integer constraint on $N(s)$ in (OP₅), and consider the following optimization problem:

$$\begin{aligned}
 \text{(OP}_6\text{)} \quad & \min \quad \sum_{s=1}^S c_s x_s \\
 & \text{s.t.} \quad \sum_{s=1}^S L(s) = L \\
 & \quad \sum_{s=1}^S x_s f_s \left(\frac{L(s)}{x_s} \right) \leq E \\
 & \quad L(s), x_s \geq 0.
 \end{aligned}$$

We note that if $x_s f_s(1/x_s)$ ($s = 1, \dots, S$) is convex with respect to x_s , then (OP₆) is a convex optimization problem, which can be solved based on standard techniques for convex optimization problems (e.g., see [1], [5]).

Once we obtain the optimal solution for (OP₆), we can set $N(s)$ to either $\lceil x_s \rceil$ or $\lfloor x_s \rfloor$ (where $\lceil x_s \rceil$ is the smallest integer in $[x_s, \infty)$ and $\lfloor x_s \rfloor$ is the largest integer in $(-\infty, x_s]$). So we have a total of 2^S possible solutions (note that S is usually not very large). To verify whether a set of $\{N(s), s = 1, \dots, S\}$ is feasible or not, we only need to solve following convex optimization problem (which is quite easy to solve)

$$\begin{aligned}
 \text{(OP}_7\text{)} \quad & \min \quad \sum_{s=1}^S N(s) f_s \left(\frac{L(s)}{N(s)} \right) \\
 & \text{s.t.} \quad \sum_{s=1}^S L(s) = L \\
 & \quad L(s) \geq 0.
 \end{aligned}$$

If the optimal value of (OP₇) is no larger than E , then the corresponding $\{N(s), s = 1, \dots, S\}$ is a feasible solution, otherwise it is not. Finally, we can choose either one feasible solution (e.g., the first one we find) among these 2^S solutions or the best solution among all the feasible solutions.

We now provide some numerical results to validate the above numerical method. We used *NPSOL* to solve both

convex optimization problems (OP₆) and (OP₇). *NPSOL* is a software developed at Stanford University for solving non-linear optimization problems ([3]).

Numerical Example 3 ($S = 2$). Same as the previous two examples, we again use $f_s(l) = p_s \max(e^{\alpha l}, k_s)$ ($s = 1, 2$), with $p_1 = 2.51$, $p_2 = 7.08$, $k_1 = 125.89$, and $k_2 = 63.1$. We use six different sets of values for α , c_1 , c_2 , E , and L , which are listed in Table 5. The numerical results are given in Table 6. For each data set, the solution marked with * is in fact the optimal solution for the amplifier placement problem (OP₅), which we can obtain by enumerating all possible integer values for $(N(1), N(2))$. It is clear from Table 6 that in all the cases the best feasible solution leads to the optimal solution of (OP₅). Also, in most cases, any feasible solution is very close to the optimal solution.

Set	α	c_1	c_2	E	L
1	0.02	1.5	1.0	7944	1000
2	0.02	2.0	1.0	7944	2000
3	0.02	2.0	1.0	17783	4000
4	0.025	2.0	1.0	7944	2000
5	0.025	1.5	1.0	17783	3000
6	0.025	2.0	1.0	17783	4000

TABLE V
VALUES FOR α , c_1 , c_2 , E , AND L IN EXAMPLE 3

Set	(x_1, x_2) from (OP ₆)	$(N(1), N(2))$	$(L(1), L(2))$ from (OP ₇)	$c_1 N(1) +$ $c_2 N(2)$
1	(0.1, 9.8)	(0, 10)*	(0.0, 1000.0)*	10.0*
		(1, 10)	(111.4, 888.6)	11.5
		(0, 9)	infeasible	—
		(1, 9)	infeasible	—
2	(13.7, 5.4)	(14, 5)*	(1556.9, 431.1)*	33.0*
		(14, 6)	(1494.8, 505.2)	34.0
		(13, 5)	infeasible	—
		(13, 5)	infeasible	—
3	(12.1, 30.2)	(12, 30)*	(1336.4, 2663.6)*	54.0*
		(12, 31)	(1311.6, 2688.4)	55.0
		(13, 30)	(1414.1, 2585.9)	56.0
		(13, 31)	(1388.6, 2611.4)	57.0
4	no feasible solution			
5	(18.6, 17.5)	(19, 17)*	(1745.6, 1254.4)*	45.5*
		(19, 18)	(1707.8, 1292.3)	46.5
		(18, 17)	infeasible	—
		(18, 18)	infeasible	—
6	(36.6, 12.8)	(36, 13)*	(3184.3, 815.7)*	85.0*
		(37, 13)	(3134.0, 866.0)	87.0
		(36, 12)	infeasible	—
		(37, 12)	infeasible	—

TABLE VI
NUMERICAL RESULTS FOR EXAMPLE 3

C. Discretization Method

The numerical procedure we proposed in the previous subsection is quite efficient for solving the amplifier placement problem for the continuous case. However, it is worth pointing out that we can also solve the continuous case by converting it into the discrete case: we can discretize L and divide it into

M equal segments. Clearly, this method is very simple and straightforward. The only question is how to choose M . In general, the larger M is, the more computational efforts are required, but the better the results are.

Based on some preliminary numerical experiments we did, it seems to be advisable to set M to a value which is $5 \sim 8$ times the number of amplifiers obtained from the discretized problem. Since we do not know the number of amplifiers for the discretized problem in advance, we can use the iterative method based on feedback as follows: we first set M to some initial value and solve the discretized problem. If M is less than 5 times of the number of amplifiers, then we increase M (e.g., double M) and solve the discretized problem again. Our numerical results indicate that the errors of the results from the discretized problem are usually within $2 \sim 5\%$.

V. CONCLUSION

We studied the amplifier placement problem with multiple types of amplifiers. Our objective is to minimize the total amplifier cost subject to the constraint on the cumulative spontaneous emission noise. We considered both discrete and continuous cases. For the discrete case, we proposed two methods: one based on the technique of dynamic programming and the other based on heuristics. When the number of different amplifier types is small (less than 3), the dynamic programming method is quite effective and can produce optimal solutions in seconds or minutes. However, when the number of different amplifier types is large, the dynamic programming method may require long computation time, in which case the heuristic method can be used to reduce the computation time though it only produces approximate solutions. For the continuous case, we formulated the problem as a mixed integer optimization problem and proposed a numerical method to solve it. The numerical method decomposes the original optimization problem into two convex optimization problems and is based on the techniques of convex optimization.

In this paper, we only considered the linear transport system. One future research direction is to study the amplifier placement problem for optical mesh networks with many linear transport sub-systems (lightpaths). For a mesh network, the amplifier placement problem becomes much more complicated since its linear transport sub-systems may overlap with each other. Therefore, one amplifier can be used in more than one linear transport sub-system in the network and each sub-system has its own cumulative spontaneous emission noise limit. We believe many results obtained in this paper can be potentially used in the study of the amplifier placement problem for mesh networks.

REFERENCES

- [1] S. Boyd and L. Vandenberghe, *Convex Optimization*, manuscript, 2002. (<http://www.stanford.edu/~boyd/cvxbook.html>)
- [2] A. Fumagalli, G. Balestra, and L. Valcarenghi, "Optimal Amplifier Placement in Multi-Wavelength Optical Networks Based on Simulated Annealing," *SPIE International Symposium on Voice, Video, and Data Communications - All-Optical Networking: Architecture, Control, and Management Issues*, Boston, MA, SPIE vol. 3531, pp. 268-279, November 1998.
- [3] P. E. Gill, W. Murray, M. A. Saunders, and M. H. Wright, *User's Guide for NPSOL 5.0: A Fortran Package for Nonlinear Programming*, 1998. (<http://www.sbsi-sol-optimize.com/manuals/NPSOL%20Manual.pdf>)
- [4] C.-S. Li, F.F.-K. Tong, C.J. Georgiou, and M. Chen, "Gain equalization in metropolitan and wide area optical networks using optical amplifiers," in *Proceedings, IEEE INFOCOM '94*, pp. 130-137, Toronto, Ont. Canada, June, 1994.
- [5] C. Papadimitriou and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity*, Prentice-Hall, 1982
- [6] B. Ramamurthy, J. Iness, B. Mukherjee, "Optimizing Amplifier Placements in a Multi-Wavelength Optical LAN/MAN: The Equally Powered-Wavelengths Case", *Journal of Lightwave Technology*, Vol. 16, No. 9, pp. 1560-1569, September, 1998
- [7] B. Ramamurthy, J. Iness, B. Mukherjee, "Optimizing Amplifier Placements in a Multi-Wavelength Optical LAN/MAN: The Unequally-Powered-Wavelengths Case", *IEEE/ACM Transactions on Networking*, Vol. 6, No. 6, pp. 755-767, December, 1998.
- [8] R. Ramaswami and K. Sivarajan, *Optical Networks: A Practical Perspective*, Morgan Kaufmann Publishers, 1998.