On a Simple, Static Model of Migration
Part I: General Theory

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Abstract

Economic agents are characterized by their types in a static model of migration. Individuals cannot change their types in one generation, but are free to migrate to communities most beneficial to their interests. Given a large population of fixed type-composition, can we find a migrationally stable partition of this population? (So that no subpopulation will have further incentives to emigrate and form a new community.) In this paper a general theorem is proved to address this question.

1 Introduction

In many situations economic agents can be classified into types, which determine the agents' preferences and how they are to be preferred by other agents, and the only practical strategic possibility open to them is to form communities most advantageous to their members. In a specific theory of migration, types may denote professions, linguistic affiliations, religions, income groups, capital endowments, or any other characteristics that cannot be strategically altered in a relevant scale of time (such as one generation).

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This is complementary to the evolutionary view of social and economic behaviour, which focuses upon the changes in agents' actions and characteristics from generation to generation; whether these changes arise from learning or from demographic selection.

A fundamental problem in a static migration model is to determine when a given population can be partitioned into communities in a way that is migrationally stable, which is to say there will be no incentive for further emigration and cross-migration to take place. In this paper we give general conditions under which this stability may be expected.

We shall formulate the migration problem as a coalition game involving a continuum of players. Migrational stability is achieved when the resulting welfare allocation lies in the core of the game. Amongst coalition games with a continuum of players our model is quite special; in particular we assume that the population falls into finitely many types. We prefer to use the language of "migration" in this paper, not only as specification for this kind of coalition game, but also to put emphasis on the *core-partition*, rather than the core-allocation.

The existence theorem for "migration-proof equilibrium" turns out to be sufficiently general to provide a unified approach to diverse economic problems, involving finite as well as continuum populations. It extends an old theorem of H. Scarf [6] on the core of multi-person games without side payments. Replication phenomena and the existence of approximate core, which have been observed and applied in the theory of coalition games, can also be recovered, simplified and strengthened by means of the model. (For reviews, see [4, 10].) It also generalizes the existence aspect of the well-developed theory of bilateral matchings to a multilateral setting. (See [5] for an introduction to the theory of bilateral matching.) Migration phenomena in socio-economics, such as club formations and the locational aspects of public goods, have also received increasing attention from theoretical researchers in recent times. Some of their results are related to our model. We should also mention that a precursor of our model has been introduced and studied by A.Shaked in [7], where a special case of our Main Theorem is proved[1].

In Part II of the paper we set out to explore the implications of our migration model for finite populations.

We shall also examine briefly the limitations of the existence theorem.

[1]I am indebted to Professor Shinji Yamashige for pointing out this reference to me at the Canadian Economic Theory Conference (McGill University, May 27, 1995), where I presented an earlier version of this paper.

In particular, from this static model one can already see that interactions between communities (for example, by trade) may be a source of migrational instability. This underlines the importance of understanding migration as a dynamic phenomenon.

2 A Simple, Static Model of Migration

Suppose there are n types of agents, indexed by $i = 1, \ldots, n$. Let $\mathcal{X}$ be the standard $(n-1)$-simplex[2], defined by

$$\mathcal{X} = \{x \in \mathbb{R}^n : \textstyle\sum_i x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i.\}$$

The agents' preferences are specified by the *payoff correspondence*

$$W \subset \mathcal{X} \times \mathbb{R}^n.$$

A point $(x, u) \in W$ is called a *community profile*, or simply a *community*, by an abuse of language to be justified by the ensuing discussion. The projection mapping

$$\pi : W \to \mathcal{X} \quad \text{given by } \pi(x, u) = x$$

specifies the *profile* $x = (x_1, \ldots, x_n)$ of *relative populations* of the various types of agents in the community, whilst the component $u = (u_1, \ldots, u_n)$ prescribes a *feasible per-capita payoff* u_i to an agent of each type i in that community. Notice that for some type(s) it may happen that $x_i = 0$, so that no agent of type i is actually present in that community. Nevertheless, u_i is still defined. This redundancy is convenient, and imposes no essential restriction on the model.

We make the following assumptions about W.

Hypothesis 1. The mapping $\pi : W \to \mathcal{X}$ is surjective, so that there is at least one feasible community for every profile of relative populations.

Hypothesis 2. W is a closed and bounded subset of $\mathcal{X} \times \mathbb{R}^n$.

A subset $\mathcal{M} \subseteq W$ is called a *formation* (of communities). Intuitively, the formation $\mathcal{M}$ is not migration-proof if there is incentive for some agents present in $\mathcal{M}$ to form a new community P amongst themselves. According

[2]Note that $n-1$ is the dimension of the simplex.

to our model, an individual agent can be induced to migrate only if its payoff in the defecting community P is *strictly* higher than its status quo in $\mathcal{M}$. This assumption may be justified on various grounds. For instance, the benefits derived from a community may be recurrent, but there is an (unspecified) one-time cost incurred by an act of emigration. This cost can only be recovered if the promised payoff in the new community is strictly, however slightly, higher.

The following definitions formalize these notions.

Definition. *We say that a community* $P = (x, u) \in W$ *is a* possible defection from $\mathcal{M}$ *if, for every* $i \in \{1, \dots, n\}$,

Either $x_i = 0$;

Or $x_i > 0$, *and there exists* $(y, v) \in \mathcal{M}$ *such that* $y_i > 0$ *and* $u_i > v_i$.

Definition. *A formation* $\mathcal{M}$ *is said to be* migration-proof *if there are no possible defections from* $\mathcal{M}$.

Several economic assumptions are implicit in our formulation:

1. (Scale Invariance) The feasible payoffs for an individual agent in a community depend only on the profile of relative populations, but not on the *size* of that community. This would seem to contradict the fact that the welfare of individuals may be affected by the total resources available to the community. However, under some circumstances one can include each factor of production (such as a specific amount of capital) as a type in the model. Then, in the picturesque language of the classical economist, they will migrate towards the highest profits.

Under this assumption, the migrational stability or instability of a collection of communities will not be affected if all the populations are multiplied by the same factor.

2. (Absence of Externalities) We have assumed that the feasible payoffs for an individual agent depend only on its own community and its type, and not on the configuration of other communities. While this assumption may be a good approximation in many situations, it is a serious restriction imposed by the model. Indeed, we shall see that payoff-relevant *interactions* amongst communities may undermine the existence of migration-proof formations.

3. (Infinite Divisibility) In our model, every point in the continuum $\mathcal{X}$ is assumed to be realizable as the relative population profile of a community; and a community defecting from a formation is allowed to have arbitrary size. Clearly that cannot be the case in a world consisting of a finite number of indivisible individuals; but, due to the scale-invariance of the model,

one may regard this assumption as an approximation when the number of individuals becomes extremely large.

In the second place, this assumption is a technical expedient that would allow topological methods to be used in their most natural form. In fact, *significant conclusions about the migrational stability of discrete populations can be drawn from the analysis of the continuum model.*

4. (Equal Treatment within Communities) Since the payoff correspondence prescribes a single per-capita payoff vector to each community, we are assuming that all individuals of the same type in the same community are treated equally. However, this assumption is not necessary for the existence of migration-proof formations, as we shall show in a later section.

Let us now address a fundamental problem raised by the model. In addition to the payoff correspondence W, we are also given a *total population*, described by a vector

$$\Xi = (N_1, \dots, N_n),$$

where N_i is the total number (or mass) of individuals of type i. Let $N = \sum N_i$. The *profile of the total population* is a point $\xi \in \mathcal{X}$, given by

$$\xi = \Xi/N = (N_1/N, \dots, N_n/N).$$

We would like to know *whether the individuals in Ξ can be partitioned into communities in such a way that the resulting formation $\mathcal{M}$ is migration-proof.*

Suppose that is the case. Then

$$\Xi = N\xi = \sum_{P \in \mathcal{M}} N(P)\pi(P),$$

and

$$N = \sum_{P \in \mathcal{M}} N(P),$$

where $N(P)$ is the total population of the community P, and $\pi(P) \in \mathcal{X}$ is its profile. Hence

$$\xi \in \text{the convex hull of } \pi(\mathcal{M}).$$

Motivated by this observation, we make the following definition.

Definition. *Given a population profile $\xi \in \mathcal{X}$ and a formation $\mathcal{M} \subseteq W$, we say that ξ* can be distributed into $\mathcal{M}$ *if and only if ξ is in the convex hull of $\pi(\mathcal{M})$.*

Thus the question above is answered by the following theorem.

Main Theorem. *Suppose the payoff correspondence W satisfies Hypotheses 1 and 2. Then, given any total population Ξ with profile $\xi \in \mathcal{X}$, there exists a migration-proof formation $\mathcal{M} \subseteq W$ such that ξ can be distributed into $\mathcal{M}$.*

Furthermore, the number of communities in $\mathcal{M}$ may be taken to be no greater than n, the number of types in the model.

3 Proof of the Main Theorem

We now give a proof of the Main Theorem. The argument is geometric. First we interpret the migration-proof formations as the *vanishing loci* for certain *critical payoff levels* (Lemma 2). This enables us to reduce the solution of the fundamental problem to a well-known topological theorem.

Brouwer's Fixed-Point Theorem. *If $f : \mathcal{X} \to \mathcal{X}$ is a continuous mapping from the $(n-1)$-simplex to itself, then f has a fixed-point. In other words, there is a point $x \in \mathcal{X}$ such that $f(x) = x$.*
See e.g. [3, 9] for a proof. (Aside from this proposition our proof of the Main Theorem will be self-contained.)

NOTATION. The set $\mathbb{R}^n_+$ is defined to be $\{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}$, and the set $\mathbb{R}^n_-$ is defined to be $\{x \in \mathbb{R}^n : -x \in \mathbb{R}^n_+\}$.
Given a subset $I \subseteq \{1, \ldots, n\}$, the *I-face* of the standard $(n-1)$-simplex $\mathcal{X} \subset \mathbb{R}^n_+$ is

$$\mathcal{X}^I = \{x \in \mathcal{X} : x_i = 0 \ \text{ for all } \ i \notin I\}.$$

The *boundary* of the simplex $\mathcal{X}$ is

$$\dot{\mathcal{X}} = \bigcup_{I \neq \{1,\ldots,n\}} \mathcal{X}^I.$$

For an arbitrary subset $\mathcal{A}$ of the simplex $\mathcal{X}$ (or of $\mathbb{R}^n$), its convex hull will be denoted by $Cvx(\mathcal{A})$.
A *correspondence* G from X to Y is a subset $G \subseteq X \times Y$ which projects surjectively onto X. (This is consistent with our previous usage when defining the payoff correspondence.) One may regard G as a point-to-set mapping. Given a point $x \in X$, by definition $G(x) = \{y \in Y : (x, y) \in G\}$. Similarly, for a subset $A \subseteq X$, by definition $G(A) = \{y \in Y : (x, y) \in$

G and $x \in A\} = \bigcup_{x \in A} G(x)$. Thus a correspondence G is said to be surjective if $G(X) = Y$; and it reduces to an ordinary mapping $G : X \to Y$ if $G(x)$ is always a singleton.
A mapping or correspondence $G \subseteq \mathcal{X} \times \mathcal{Y}$ between $(n-1)$-simplices is said to *preserve faces* if $G(\mathcal{X}^I) \subseteq \mathcal{Y}^I$ for all I.

Consider the *payoff space* $H = \mathbb{R}^n$. For each $\lambda \in H$, let

$$W_\lambda = \{P = (x, u) \in W : \text{For all } i, \text{ either } x_i = 0; \text{ or } x_i > 0 \text{ and } u_i \geq \lambda_i\}.$$

Also define

$$E = \{\lambda \in H : W_\lambda \neq \emptyset\}.$$

Thus E consists of those payoff vectors that are offered (or surpassed) by some feasible community, whilst W_λ is the set of feasible communities offering the payoff vector λ (or better). The following lemma records two straightforward consequences of these definitions. (See Figures 1 and 2.)

Lemma 1. *As a subset of H,*
(a) E is closed;
(b) E is comprehensive, that is to say: for all λ, λ', if $\lambda \in E$ and $\lambda' \leq \lambda$ then $\lambda' \in E$.

Let $L = 1 + \sup\{|u_i| : (x, u) \in W\}$. By the boundedness of W, $L < \infty$. The topological frontier of E is defined to be

$$F = E \setminus \text{Interior}(E).$$

Note that F is a closed set.

Lemma 2.
(a) If $\lambda \in F$, then W_λ is migration-proof.
(b) If a formation $\mathcal{M} \subseteq W$ is migration-proof, then $\mathcal{M} \subseteq W_\lambda$ for some $\lambda \in F$.

PROOF. (a) Suppose $\lambda \in F$, but W_λ were not migration-proof. Then there is a possible defection from W_λ, say $P = (x, u) \in W$. Consider the payoff vector $\lambda' \in H$ with components

$$\lambda'_i = \begin{cases} u_i & \text{if } x_i > 0, \\ L & \text{if } x_i = 0. \end{cases}$$

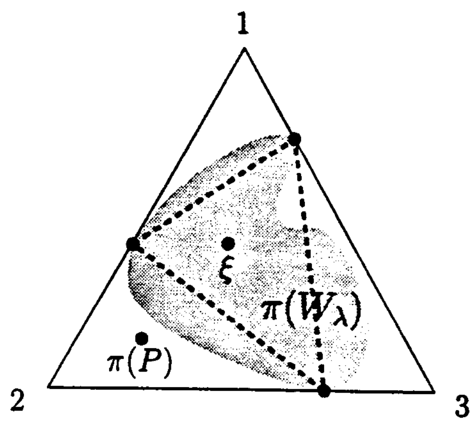


Figure 1: Population space $\mathcal{X}$

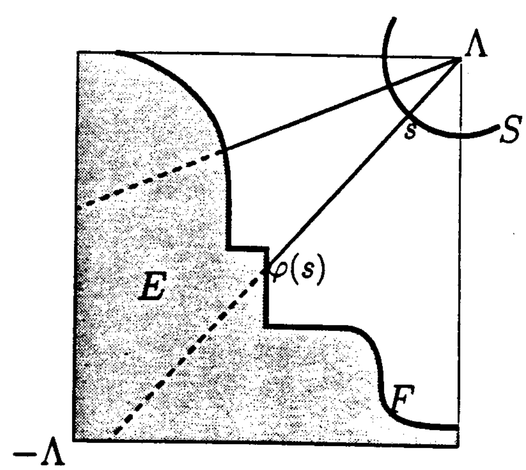


Figure 2: Payoff space $\mathbb{R}^n$

Then $\lambda' \in E$ and $\lambda'_i > \lambda_i$ for all i. Since E is a comprehensive set, this implies that $\lambda \in \text{Interior}(E)$ and we arrive at a contradiction.
(b) For any nonempty formation $\mathcal{M}$, let us define a payoff vector $\lambda(\mathcal{M})$ with components

$$\lambda_i(\mathcal{M}) = \begin{cases} \inf\{v_i : (y,v) \in \mathcal{M}, y_i > 0\} & \text{if this set is non-empty,} \\ L & \text{otherwise.} \end{cases}$$

Thus $\lambda_i(\mathcal{M})$ is the *minimal payoff level for type-i individuals, wherever they can be found in the formation* $\mathcal{M}$.

Observe that $\mathcal{M} \subseteq W_{\lambda(\mathcal{M})}$ and hence $\lambda(\mathcal{M}) \in E$. Now suppose $\mathcal{M}$ is migration-proof, but $\lambda(\mathcal{M}) \in \text{Interior}(E)$. Then there is a vector $\lambda' \in E$ such that $\lambda'_i > \lambda_i(\mathcal{M})$ for all i. Since $W_{\lambda'}$ is non-empty, we can find a community $P \in W_{\lambda'}$. Then P is a possible defection from $\mathcal{M}$, a contradiction. Hence $\lambda(\mathcal{M}) \in F$ and the lemma is proved.

Let $\Lambda \in H$ be the vector $(L, \ldots, L)$, and consider the hypercube

$$H_L = \{\lambda \in H : |\lambda_i| \leq L \text{ for all } i\}.$$

The I-face of the hypercube is defined to be

$$H_L^I = \{\lambda \in H_L : \lambda_i = +L \text{ for all } i \notin I\}.$$

These faces meet at the vertex Λ. Preliminary to the next lemma let us state a useful fact.

Geometric Estimate for Comprehensive Sets. *For $\lambda, \lambda' \in \mathbb{R}^n$, define $d(\lambda, \lambda') = \max_i |\lambda_i - \lambda'_i|$. Note that d is a metric. Also define $\mu(\lambda) = \min\{|\lambda_i| : \lambda_i < 0\}$ if the latter set is non-empty, and $+\infty$ otherwise.*

Suppose $K \subseteq \mathbb{R}^n$ is a comprehensive set, and $\lambda \in K \cap \mathbb{R}^n_-$. Given any $\delta > 0$,

$$\textit{if } z \in \mathbb{R}^n_- \textit{ and } d(z, \lambda) \leq \mu(\lambda)\frac{\delta}{1+\delta}, \textit{ then } (1+\delta)z \in K.$$

PROOF. We show that under the stated conditions $(1+\delta)z_i \leq \lambda_i$ for all i. Then since K is comprehensive and $\lambda \in K$, we get the desired conclusion $(1+\delta)z \in K$.

Suppose $z_i \leq \lambda_i$, then $(1+\delta)z_i \leq z_i \leq \lambda_i$. On the other hand, suppose $z_i > \lambda_i$. Then $\lambda_i < 0$ since $z \in \mathbb{R}^n_-$, and

$$z_i - \lambda_i \leq d(z, \lambda) \leq \mu(\lambda)\frac{\delta}{1+\delta} \leq -\lambda_i \frac{\delta}{1+\delta} = -\lambda_i \left(1 - \frac{1}{1+\delta}\right).$$

Hence $z_i \leq \lambda_i/(1+\delta)$, or $(1+\delta)z_i \leq \lambda_i$, as required.

Lemma 3. *Let $\hat{F} = F \cap H_L$ and $\hat{F}^I = F \cap H_L^I$. Then $\hat{F}$ is homeomorphic to a standard $(n-1)$-simplex with $\hat{F}^I$ as its faces.*

PROOF. In the space $H = \mathbb{R}^n$, let S be the unit sphere centred at Λ. Define $\hat{S} = S \cap H_L$ and $\hat{S}^I = S \cap H_L^I$. Then $\hat{S}$ is an $(n-1)$-simplex, with $\hat{S}^I$ as its faces. We shall define a mapping $\varphi : \hat{S} \to \hat{F}$, and show that it is a face-preserving homeomorphism. (See Figure 2.)

For each $s \in \hat{S}$, consider the *ray* emanating from Λ in the direction s, that is:

$$\Lambda - r(s - \Lambda) \text{ for } r > 0.$$

For $s \in \hat{S}$, let $r(s) = \inf\{r : \Lambda - r(s-\Lambda) \in E\}$. Let $\lambda = \Lambda - r(s - \Lambda)$. When r becomes sufficiently large at least one coordinate, say λ_i, will be less than $-L$. Then a community consisting solely of a type-i population will be in W_λ, and hence $\Lambda - r(s-\Lambda) \in E$. Therefore $r(s)$ has a well-defined value. In fact, $r(s) \leq$ (the Euclidean distance between Λ and $-\Lambda$) $= 2\sqrt{n}L$. Also, from the definition of L it follows that $r(s) \geq 1$. In particular, $\Lambda \notin E$.

From the geomtric estimate for comprehensive sets, if $\Lambda - r(s-\Lambda) \in E$, $s' \in \hat{S}$ and

$$rd(s, s') = rd(s - \Lambda, s' - \Lambda) \leq \mu(s - \Lambda)\frac{\delta}{1+\delta},$$

then $\Lambda - r(1+\delta)(s' - \Lambda) \in E$. Note that $\mu(s - \Lambda) \geq 1/\sqrt{n}$ for all $s \in \hat{S}$.

This implies that for any $\delta > 0$,

$$(1+\delta)^{-1} \leq r(s')/r(s) \leq (1+\delta)$$

whenever $s, s' \in \hat{S}$ and

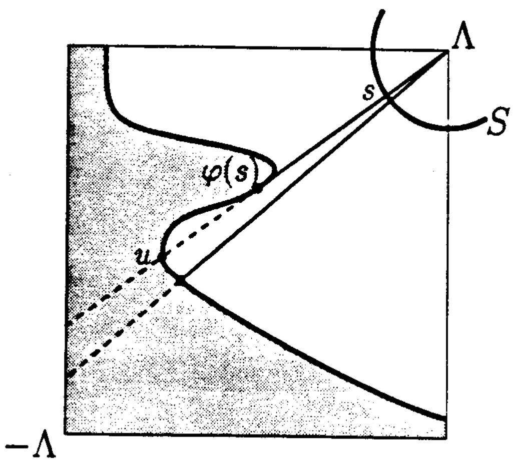
$$d(s, s') \leq \frac{(\sqrt{n})^{-1}\delta}{2\sqrt{n}L(1+\delta)} = \frac{\delta}{2nL(1+\delta)}.$$

It follows that $r(s)$ is a continuous function of s.

We now define the mapping φ by

$$\varphi(s) = \Lambda - r(s)(s - \Lambda) \quad \text{for} \quad s \in \hat{S}.$$

Thus $\varphi(s)$ is the *first entry-point* of the ray into the set E. Hence $\varphi(s) \in \hat{F}$. Since $r(s)$ is continuous, so is φ.



φ is discontinuous at s and u is not in Image(φ)

Figure 3: Noncomprehensive Set

For $s \neq s'$, $\varphi(s)$ and $\varphi(s')$ belong to distinct rays emanating from Λ, and $\varphi(s)$ and $\varphi(s')$ cannot be at the origin Λ. Hence $\varphi(s) \neq \varphi(s')$, showing that φ is injective. Given any point $\lambda \in \hat{F}$, consider the ray joining Λ to λ. Let $s \in \hat{S}$ be the direction of this ray. Since E is a comprehensive set, this ray cannot have an entry-point into E earlier than λ. Hence $\lambda = \varphi(s)$, showing that φ is surjective. If $s \in \hat{S}^I$, then the ray in the direction s remains in H_L^I and so $\varphi(s) \in \hat{F}^I$; whence φ is face-preserving.

Now $\hat{F}$, being a subspace of $\mathbb{R}^n$, is a Hausdorff space; and $\hat{S}$ is compact. Hence the continuous bijective mapping $\varphi : \hat{S} \to \hat{F}$ is a homeomorphism, proving the lemma. (Note that the comprehensiveness of E is crucial for the continuity and surjectivity of φ, see Figure 3.)

Lemma 4. (Carathéodory's Theorem) *Suppose A is a subset of $\mathbb{R}^n$ and $\xi \in Cvx(A)$. Then there is an affine independent subset $B \subseteq A$ such that $\xi \in Cvx(B)$. In particular, the cardinality of B is no greater than $n+1$.*

PROOF. There are many proofs and many applications of this theorem, see e.g. [1]. We sketch an argument. Consider the set

$$Cvx_0(A) = \{\text{all finite convex linear combinations of points in } A\}.$$

Note that $Cvx_0(A)$ is a convex set containing A, hence $Cvx_0(A) = Cvx(A)$. Therefore if $\xi \in Cvx(A)$, there is a finite subset $B \subseteq A$ such that $\xi \in Cvx(B)$. Furthermore, we may assume that B has minimal cardinality amongst such subsets. Let $m = |B|$. We will show that B is affine independent.

Let $\Pi = \{\alpha \in \mathbb{R}^m : \sum \alpha_i = 1\}$, and $\Delta = \{\alpha \in \Pi : \alpha_i \geq 0 \text{ for all } i\}$. The space $\mathbb{R}^m$ can be identified with the set of all functions $\alpha : B \to \mathbb{R}$. We shall do so, and define a linear mapping

$$\ell : \mathbb{R}^m \to \mathbb{R}^n, \quad \ell(\alpha) = \textstyle\sum_{x \in B} \alpha(x)x.$$

Let B' be the set $\{(x, 1) \in \mathbb{R}^{n+1} : x \in B\}$. Then B is affine independent in $\mathbb{R}^m$ if and only if B' is linearly independent in $\mathbb{R}^{m+1}$. These conditions hold if and only if

$$\dim \text{Null-space}(\ell) \leq 1.$$

Note that $\dim \text{Null-space}(\ell) \geq m - n$. Hence if B is affine independent then $m \leq n + 1$.

Consider the set

$$L = \left\{\alpha \in \Pi : \textstyle\sum_{x \in B} \alpha(x)x = \xi\right\}.$$

Since $\xi \in Cvx(B)$, there is a point $\alpha_0 \in \Delta \cap L$. Indeed,

$$L = \Pi \cap (\alpha_0 + \text{Null-space}(\ell)).$$

Now Π is an affine linear space of dimension $m - 1$, and L is an affine linear subspace of Π. The dimension of L is given by

$$\dim L = \dim \Pi + \dim \text{Null-space}(\ell) - \dim \mathbb{R}^m = \dim \text{Null-space}(\ell) - 1.$$

If it were true that $\dim \text{Null-space}(\ell) > 1$, then $\dim L > 0$. In that case L contains a line through the point α_0. Since the simplex Δ is a closed and bounded set, this line must intersect its boundary $\dot{\Delta}$, say in the point α_1. But that means

$$\textstyle\sum_{x \in B} \alpha_1(x)x = \xi, \ \alpha_1 \in \Delta, \text{ and } \alpha_1(x) = 0 \text{ for some } x \in B.$$

This expresses ξ as a convex linear combination of a proper subset of B, contradicting its minimality. Hence $\dim \text{Null-space}(\ell) \leq 1$, B is affine independent, $m \leq n + 1$, and the lemma is proved.

We now define a correspondence $\Gamma \subset \hat{F} \times \mathcal{X}$, as the composite

$$\lambda \longmapsto W_\lambda \longmapsto Cvx(\pi(W_\lambda)).$$

Lemma 5.
(a) Γ *is closed.*
(b) $\Gamma(\hat{F}^I) \subseteq \mathcal{X}^I$ *for all* I.
(c) $\Gamma(\lambda)$ *is a convex subset of* $\mathcal{X}$ *for every* $\lambda \in \hat{F}$.

PROOF.
(a) Suppose $(\lambda_k, x_k) \in \Gamma$ is a convergence sequence with limit point (λ, x). We need to show that $(\lambda, x) \in \Gamma$.

Since $\hat{F}$ is closed, $\lambda \in \hat{F}$. For each k, we have $x_k \in Cvx(\pi(W_{\lambda_k}))$. By Carathéodory's theorem we can find an n-tuple of points $(P_k^1, \ldots, P_k^n)$ in W_{λ_k}, so that $P_k^i = (x_k^i, u_k^i)$ and $x_k = \sum_i \alpha_k^i x_k^i$ is a convex linear combination. (Thus $\sum_i \alpha_k^i = 1, \alpha_k^i \geq 0$.)

By compactness of the simplex, we can extract a convergent subsequence of $(P_k^1, \ldots, P_k^n; \alpha_k^1, \ldots, \alpha_k^n)$. Let the limit be $(P^1, \ldots, P^n; \alpha^1, \ldots, \alpha^n)$, and write $P^i = (x^i, u^i)$. Recall that each x^i is a point in $\mathcal{X}$, denote by $(x^i)_j$ the j-component of x^i. Similarly, denote by $(u^i)_j$ the j-component of u^i. (The j-component corresponds to individuals of type j in the model.)

First observe that

$$\textstyle\sum_i \alpha^i = 1, \quad \alpha^i \geq 0, \quad \text{and} \quad x = \sum_i \alpha^i x^i.$$

Secondly, observe that $P^i \in W_\lambda$ for every i. We can verify it as follows:

$$\begin{aligned} (x^i)_j > 0 &\Rightarrow (x_k^i)_j > 0 && \text{for sufficiently large } k \\ &\Rightarrow (u_k^i)_j \geq (\lambda_k)_j && \text{since } P_k^j \in W_{\lambda_k} \\ &\Rightarrow (u^i)_j \geq (\lambda)_j. \end{aligned}$$

This holds for every type j. Combining these observations, we conclude that $x \in Cvx(\pi(W_\lambda)) = \Gamma(\lambda)$.
(b) Suppose $\lambda \in \hat{F}^I$. Consider any point $P = (x, u) \in W_\lambda$. Then for all $i \notin I$, $\lambda_i = +L > u_i$. Hence x_i must be 0 for $i \notin I$. In other words, $\pi(W_\lambda) \subseteq \mathcal{X}^I$. Since $\mathcal{X}^I$ is convex, $Cvx(\pi(W_\lambda)) \subseteq \mathcal{X}^I$.
(c) Follows directly from the definition of the correspondence Γ.

We can now complete the proof of the Main Theorem. By Lemma 2, as λ vary through $\hat{F}$ the formations W_λ (and their subsets) vary through (all) the migration-proof formations. The desired conclusion is that every point $\xi \in \mathcal{X}$ should be in the convex hull $Cvx(\pi(\mathcal{M}))$ for some migration-proof formation $\mathcal{M}$. This follows directly from the following topological proposition.

Lemma 6. *Let $\Gamma \subset \hat{F} \times \mathcal{X}$ be any correspondence between $(n-1)$-simplices. Suppose it satisfies the conditions (a), (b), and (c) of Lemma 5. Then Γ is surjective.*

PROOF. We identify the $(n-1)$-simplex $\hat{F}$ with the standard $(n-1)$-simplex $\mathcal{X}$, and consider Γ as a correspondence $\Gamma \subset \mathcal{X} \times \mathcal{X}$. Let us first consider the special case when Γ is a continuous, face-preserving *mapping*, and show that it is surjective. It is sufficient to show that Interior$(\mathcal{X}) \subseteq \Gamma(\mathcal{X})$. Indeed, since $\mathcal{X}$ is compact $\Gamma(\mathcal{X})$ is closed, so if Interior$(\mathcal{X}) \subseteq \Gamma(\mathcal{X})$ then $\mathcal{X} = \Gamma(\mathcal{X})$.

Assume, to the contrary, that y is a point in the interior of $\mathcal{X}$, but $y \notin \Gamma(\mathcal{X})$.

The *retraction mapping* $\rho : \mathcal{X} \setminus \{y\} \to \dot{\mathcal{X}}$ is defined as follows. Consider a ray in $\mathcal{X}$ emanating from y and terminating in a point z on the boundary $\dot{\mathcal{X}}$. Then ρ maps every point on the ray (except y) to z. Since every point of $\mathcal{X} \setminus \{y\}$ lies on exactly one such ray, ρ is well-defined; and ρ is continuous. Furthermore,

$$\rho(z) = z \quad \text{for every } z \in \dot{\mathcal{X}}.$$

Also define the *antipodal mapping* $\sigma : \dot{\mathcal{X}} \to \dot{\mathcal{X}}$, as follows. Let $x_0 = (1/n, \ldots, 1/n)$ be the barycenter of $\mathcal{X}$. Consider a straight line in $\mathcal{X}$ through x_0. This line interects $\dot{\mathcal{X}}$ in exactly two points, say z and z'. Then $\sigma(z) = z'$ and $\sigma(z') = z$. Note that σ is well-defined and continuous. Furthermore, for every face $\mathcal{X}^I \subset \dot{\mathcal{X}}$,

$$\sigma(\mathcal{X}^I) \cap \mathcal{X}^I = \emptyset.$$

Let $\iota : \dot{\mathcal{X}} \hookrightarrow \mathcal{X}$ denote the inclusion mapping.

By assumption, $\Gamma : \mathcal{X} \to \mathcal{X} \setminus \{y\}$. Consider the composite mapping f, given by

$$\mathcal{X} \xrightarrow{\Gamma} \mathcal{X} \setminus \{y\} \xrightarrow{\rho} \dot{\mathcal{X}} \xrightarrow{\sigma} \dot{\mathcal{X}} \xrightarrow{\iota} \mathcal{X}.$$

Clearly no interior point can be a fixed-point of f, due to the retraction ρ. On the other hand, suppose $x \in \mathcal{X}^I$ is a boundary point. Then $\rho(\Gamma(x)) \in \mathcal{X}^I$ since Γ is face-preserving; and $\sigma(\rho(\Gamma(x))) \notin \mathcal{X}^I$ since $\sigma(\mathcal{X}^I) \cap \mathcal{X}^I = \emptyset$. Therefore $f(x) \neq x$. Hence the mapping $f : \mathcal{X} \to \mathcal{X}$ is continuous, but it has no fixed-point. This contradicts Brouwer's Fixed-Point Theorem and proves the surjectivity of Γ.

We turn to the general case of the lemma, when Γ is not necessarily a mapping.

Let $\{K^m\}$ be a sequence of (finite) simplicial decompositions of $\mathcal{X}$, with mesh$(K^m) \to 0$. Let $\mathcal{X}^{(m)} \subset \mathcal{X}$ be the set of vertices in the decomposition K^m. (For example, we can take $\{K^m\}$ to be the successive *barycentric* *subdivisions* of $\mathcal{X}$. The *mesh* of a simplicial decomposition is the maximal diameter of all the simplices in the decomposition. For more details on the properties of simplicial decompositions, see e.g. [9].)

We now approximate Γ by a sequence of continuous mappings

$$f_m : \mathcal{X} \to \mathcal{X},$$

contructed as follows. For $x \in \mathcal{X}^{(m)}$, define $f_m(x)$ to be any point in $\Gamma(x)$. Now every point $x \in \mathcal{X}$ belongs to the interior of a unique k-simplex in K^m. (This simplex is called the *carrier* of x.) More precisely, for every m and every $x \in \mathcal{X}$, there is a unique number $k = k(m, x) \in \{0, \dots, n-1\}$ and a unique subset $\kappa_m(x) = \{z_0, \dots, z_k\} \subseteq \mathcal{X}^{(m)}$, such that $\kappa_m(x)$ is a simplex in K^m and $x \in$ Interior $Cvx(\kappa_m(x))$. Thus x can be uniquely expressed as a convex linear combination $x = \alpha_0 z_0 + \dots + \alpha_k z_k$, with strictly positive weights $\alpha_0, \dots, \alpha_k$. We define $f_m(x)$ to be $\alpha_0 f_m(z_0) + \dots + \alpha_k f_m(z_k)$.

Note that f_m is a continuous mapping. Since Γ is face-preserving by hypothesis, so is f_m. Hence, by what has been proved above, f_m is surjective. Given any $y \in \mathcal{X}$, there is a point $x_m \in \mathcal{X}$ so that $f_m(x_m) = y$. By compactness of $\mathcal{X}$, we can extract a subsequence of $\{x_m\}$ converging to a point x. We show that $y \in \Gamma(x)$. This will establish the surjectivity of Γ.

Since $k(m, x_m)$ can take on only finitely many values, we can extract a subsequence of $\{x_m\}$ so that the values $k(m, x_m)$ are all equal for the indices m in the subsequence. Suppose that has been done. Let k be the common value of $k(m, x_m)$, and let

$$\kappa_m(x_m) = \{z_0^m, \dots, z_k^m\},$$

$$x_m = \alpha_0^m z_0^m + \dots + \alpha_k^m z_k^m, \quad \text{with} \quad \alpha_i^m > 0, \quad \textstyle\sum_i \alpha_i^m = 1,$$
$$w_i^m = f_m(z_i^m) \qquad \text{for} \qquad i = 0, \dots, k.$$

Then, since $f_m(x_m) = y$,

$$y = \alpha_0^m w_0^m + \dots + \alpha_k^m w_k^m.$$

Consider the sequence

$$(z_0^m, \dots, z_k^m; w_0^m, \dots, w_k^m; \alpha_0^m, \dots, \alpha_k^m),$$

for $m = 1, 2, \dots$. By compactness of the simplex, we can extract a convergent subsequence. Let the limit be

$$(z_0, \dots, z_k; w_0, \dots, w_k; \alpha_0, \dots, \alpha_k).$$

First note that

$$x = \alpha_0 z_0 + \ldots + \alpha_k z_k, \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1.$$

Recall that $z_0^m, \ldots, z_k^m$ are vertices of a single simplex in the decomposition K^m. Since $\text{mesh}(K^m) \to 0$, we have

$$z_0 = z_1 = \ldots = z_k = x.$$

Secondly, note that $(z_i^m, w_i^m) \in \Gamma$. Since Γ is closed by hypothesis, $(z_i, w_i) \in \Gamma$. Hence

$$w_0, \ldots, w_k \in \Gamma(x).$$

Thirdly, observe that

$$y = \alpha_0 w_0 + \ldots + \alpha_k w_k, \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1.$$

This means that $y \in Cvx(\Gamma(x))$. But $\Gamma(x)$ is convex by hypothesis, so $y \in \Gamma(x)$. The lemma is proved.

Finally, it is clear that a subset of a migration-proof formation is again migration-proof. Thus the second conclusion of the Main Theorem will follow if, for every $\lambda \in \hat{F}$ and $\xi \in Cvx(\pi(W_\lambda))$, we can find a subset $\mathcal{M} \subseteq W_\lambda$ of cardinality n (or less) so that the relation $\xi \in Cvx(\pi(\mathcal{M}))$ already holds. This follows immediately from Carathéodory's Theorem (Lemma 4).

4 Equal Treatment

In order to achieve migrational-proofness, a population of the same type often have to be divided into more than one community. In general, the per-capita payoffs to these individuals may be different in different communities. We now give a sufficient condition that will guarantee the equal treatment of individuals of the same type *across* the communities of a migration-proof formation.

For each point $x \in \mathcal{X}$, let $I(x) = \{i : x_i > 0\}$. Note that $x \in \mathcal{X}^{I(x)}$.

Definition. *A community $P = (x, u) \in W$ is said to be* nonsingular (with respect to the payoff correspondence W) *if the following conditions hold:*
1.(Continuity) For $\epsilon > 0$ and $i \in I(x)$, let us define

$$U_i(P, \epsilon) = \{y \in \mathcal{X}^{I(x)} : \exists v \text{ such that } (y, v) \in W \text{ and } |u_i - v_i| < \epsilon\}.$$

Then $U_i(P,\epsilon)$ contains an open neighborhood of x in $\mathcal{X}^{I(x)}$, for every $\epsilon > 0$ and $i \in I(x)$.
2.(Non-satiation) In every open neighborhood U of x in $\mathcal{X}^{I(x)}$, and for every subset $J \subset I(x)$ with $|J| = |I(x)| - 1$, there is a community $(y, v) \in W$ such that $y \in U$ and $v_j > u_j$ for every $j \in J$.

Theorem. (An Equal Treatment Principle) *Suppose $\mathcal{M} \subset W$ is a migration-proof formation in which every community $P \in \mathcal{M}$ is nonsingular. Consider any type i.*

Then the per-capita payoffs to the individuals of type i are the same in every community $P = (x, u) \in \mathcal{M}$ where they are present (i.e. $x_i > 0$).

PROOF. Assume the contrary. Then there will be two communities in $\mathcal{M}$, say $P = (x, u)$ and $P' = (x', u')$, and a type i, such that $x_i > 0$, $x'_i > 0$, and $u_i > u'_i$.

Let us choose ϵ to satisfy $u_i - u'_i > \epsilon > 0$, and $J = I(x) \setminus \{i\}$. From the nonsingularity of P we see that there will be a community $(y, u) \in U_i(P, \epsilon) \subseteq \mathcal{X}^{I(x)}$ which will attract individuals of type i to defect from P' and individuals of types $j \in J$ to defect from P. But this contradicts the migration-proofness of $\mathcal{M}$, and the theorem is proved.

Despite the intrincate appearance of the definition, the nonsingularity condition can be verified in several broad and natural classes of payoff correspondences, and thus yielding a variety of special equal treatment principles. The following propositions are quite straightforward.

Proposition.[3] *Suppose W is a payoff correspondence which permits* full tranfer of utility *amongst all individuals of all types in a community. That is to say, if $(x, u) \in W$ and $\sum x_i u_i = \sum x_i v_i$ then (x, v) is also in W.*

Then every community is nonsingular. Hence equal-treatment prevails in every migration-proof formation.

Proposition. *Suppose W is the graph of a* differentiable payoff mapping *$w : \mathcal{X} \to \mathbb{R}^n$. Write $w(x) = (w_1(x), \dots, w_n(x))$, and denote by grad$(w_i)$ the gradient vector of the function w_i.*

Then a community $P = (x, u)$ is nonsingular if grad$(w_i) \neq 0$ for all i such that $x_i > 0$. If a migration-proof formation $\mathcal{M}$ consists solely of such communities, then equal-treatment prevails in $\mathcal{M}$.

[3] I am indebted to Professor Aaron Eidlin for drawing my attention to this interesting application of the general equal treatment principle.

We turn to the question of equal treatment *within* communities. Thus far in our analysis this is an assumption in our model. As promised, we will now give a version of the existence theorem which will allow for unequal treatment of individuals of the same type in the same community.

NOTATION. Let $\mathfrak{M}(\mathbb{R})$ denote the space of probability measures on the real line $\mathbb{R}$. For $\mu \in \mathfrak{M}(\mathbb{R})$ and $u \in \mathbb{R}$, let $\mu[u] = \mu(\{x : x < u\})$. We say that u is the *essential infimum* of μ if and only if $\mu[u] = 0$ and $\mu[u + \epsilon] > 0$ for all $\epsilon > 0$; and write $u = ess.\, inf\; \mu$.

The payoff correspondence is now a subset

$$W \subset \mathcal{X} \times \mathfrak{M}(\mathbb{R})^n.$$

We interpret μ as the *payoff schedule* for the community $P = (x, \mu) \in W$, where $\mu_i[u]$ is the relative mass of individuals of type i in P who receive a payoff less or equal to u. As with the original model, an individual will leave an existing formation $\mathcal{M}$ to join a proposed defection P only if there is strictly positive advantage to do so; and the defection is successful if it can recruit all its member from $\mathcal{M}$. Thanks to the assumptions of scale invariance and infinite divisibility, which we maintain, it is easy to see that if one can recruit the individuals who are worst-off (amongst its type) in the defection P then one can certainly recruit the requisite proportions of those who are better-off. These considerations reduce to the following definitions.

Definitions. *A* formation *is a subset $\mathcal{M} \subseteq W$, as before. The community $P = (x, \mu) \in W$ is a* possible defection from $\mathcal{M}$ *if, for every $i \in \{1, \dots, n\}$,*

Either $x_i = 0$;

Or $x_i > 0$, and there exists $(y, \nu) \in \mathcal{M}$ such that $y_i > 0$ and $ess.\, inf\; \mu_i > ess.\, inf\; \nu_i$.

Again, $\mathcal{M}$ is migration-proof *if and only if there are no possible defections from it.*

Let us consider the *reduction map*

$$R : \mathcal{X} \times \mathfrak{M}(\mathbb{R})^n \to \mathcal{X} \times \mathbb{R}^n, \text{ with } R(x, \mu) = (x, ess.\, inf\; \mu).$$

Directly from these defintions we gather the following observation.

Observation. *A formation $\mathcal{M} \subseteq W$ is migration-proof with respect to the payoff correspondence W (with unequal treatment within community) if and only if $R(\mathcal{M}) \subseteq R(W)$ is migration-proof with respect to the* reduced *payoff correspondence $R(W)$ (with equal treatment within community).*

By virtue of this observation we can now recover our Main Theorem for the more general payoff correspondences.

Main Theorem—under Unequal Treatment Within Community. *Suppose the reduced payoff correspondence $R(W)$ satisfies Hypotheses 1 and 2. (That is, $\pi : W \to \mathcal{X}$ is surjective and $R(W)$ is a closed and bounded subset of $\mathcal{X} \times \mathbb{R}^n$.) Then, given any total population Ξ with profile $\xi \in \mathcal{X}$, there exists a migration-proof formation $\mathcal{M} \subseteq W$ such that ξ can be distributed into $\mathcal{M}$.*

Furthermore, the number of communities in $\mathcal{M}$ may be taken to be no greater than n, the number of types in the model.

5 Interaction and Instability

According to our model, when a community defects from an existing formation the welfare of its members is determined purely by the internal structure of this new community, and no consideration need be paid to any potential reconfiguration of the world that it leaves behind.

Furthermore, we may assume that the defecting community is internally stable. Formally, let us call a community P *emigration-proof* if the singleton formation $\{P\}$ is migration-proof according to our definition. Then the following statement is easily verified.

Proposition. *If a formation $\mathcal{M}$ is not migration-proof, then amongst all possible defections from $\mathcal{M}$ there will be one that is emigration-proof.*

These observations serve to show that our notion of migrational stability is quite credible, provided that there are no interactions (externalities) *across* the communities in a formation. If this provision does not hold, the very *criterion* for migrational stability may depend on the actual situation on hand. In considering defection, individuals will have to take into account the future reactions of the remaining population. Presumably they will not defect for the temporary advantage based on the formation found at the instant of defection; nevertheless, the potential migrants may behave more or less conservatively. Various game-theoretic, "coalition-proof" equilibrium concepts may be invoked to formulate the requisite stability criterion. Also it should be mentioned that many studies on clubs, local public goods, congestions, matchings in the labour market, etc. may be thought of as migration models with interaction. It is likely to be a subject for continued future research. (For a review, see [2].)

Here we shall be content merely to point out that *interaction can be a source of migrational instability*, illustrated by a rather special example [4].

There is only one type of agent, with total population mass 1 (after normalization). Let us consider formations that consist of finitely many communities, with populations

$$0 < x_1 \leq \ldots \leq x_s \leq \ldots \leq x_m \leq 1, \qquad \textstyle\sum_s x_s = 1.$$

Denote this formation by x.

The *production function* of each community is $p(z) = z^\beta$, with $\beta > 1$, so that the gross product from a community with population z is $p(z)$ (in units of a numeraire good). The *per-capita cost* to an individual in this community is taken to be $Ap(z)/z$, where $0 < A < 1$. (For $z = 0$ the cost is defined to be 0; since $\beta > 1$, the cost function is continuous for $z \in [0, 1]$.)

Now suppose that the communities in the formation engage in a certain system of trade, which entitles every individual to an equal share of the total gross product. For an individual in the community with population x_s, the net payoff is

$$u_s(x) = p(x_1) + \ldots + p(x_m) - A\frac{p(x_s)}{x_s}.$$

We will not ask how such a "communistic" system of allocation may come about, but show that it gives individuals incentive to migrate whatever is the prevailing population distribution. Under such a system, migrational stability can only be imposed exogenously, if at all.

Proposition.
(a) There is a maximal level of payoff $C > 0$ (independent of x), so that

$$|u_s(x)| \leq C$$

in any formation x.
(b) There is a critical population $\delta > 0$ (independent of x), so that if

$$0 < x_s < \delta$$

for all the communities in a formation x, then there is incentive for the whole population to combine into a single community. In other words, the migration

$$(x_1, \ldots, x_m) \mapsto (0, \ldots, 0, 1)$$

[4] In order to allow for interactions, we have to re-formulate our model; see appendix. However, the following discussion will be self-explanatory.

will take place.

(c) In any formation x, there is incentive for a sub-population $\epsilon = \epsilon(x) > 0$ of the community x_m to emigrate. In other words, the migration

$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m - \epsilon, \epsilon)$$

will take place.

PROOF.

(a) First note that the function $p(z) = z^\beta$ ($\beta > 1$) is superadditive:

$$p(z) + p(w) \leq p(z+w) \qquad \text{for } z, w \geq 0.$$

Indeed, p is convex on the domain $[0, \infty)$. Hence for $0 \leq z \leq w \leq z + w$, $p(z) - p(0) \leq p(z+w) - p(w)$. But $p(0) = 0$, whence superadditivity. Now

$$\begin{aligned} |u_s(x)| &= \left| p(x_1) + \ldots + p(x_m) - A\frac{p(x_s)}{x_s} \right| \\ &\leq |p(x_1) + \ldots + p(x_m)| + |Ap(x_s)/x_s| \\ &\leq p(x_1) + \ldots + p(x_m) + A \\ &\leq p(1) + A \qquad \text{(by superadditivity)} \\ &\leq 1 + A. \end{aligned}$$

This proves the assertion, with $C = 1 + A$. Incidentally, the argument shows that if $A = 0$ (so there is no cost for production), then the formation consisting of a single community is migration-proof, as well as efficient (i.e. payoff-maximizing).

(b) Since $\beta > 1$, we can choose δ so that

$$0 < \delta < 1 \text{ and } \delta^{\beta - 1} < (1 - A)/3.$$

We need to verify that, when $0 < x_1 \leq \ldots \leq x_m \leq \delta$,

$$\begin{aligned} u_s(x) = p(x_1) + \cdots + p(x_m) - A\frac{p(x_m)}{x_m} &< p(1) - A\frac{p(1)}{1} \\ &= 1 - A. \end{aligned}$$

Since $0 < x_i < 1$ and $\sum_i x_i = 1$, by forming partial sums of the form $y_j = x_{i'} + \ldots + x_{i''}$ we can find a formation $\{y_1, \dots, y_k\}$ satifying

$$y_1 + \ldots + y_k = 1, \text{ and } \delta/2 \leq y_j \leq \delta \text{ for all but at most one j }.$$

Then the number of communities in this new formation is bounded as follows:

$$k \leq 1 + (2/\delta) \leq 3\delta^{-1}.$$

We have

$$\begin{aligned}
u_s(x) &\leq p(x_1) + \ldots + p(x_m) \\
&\leq p(y_1) + \ldots + p(y_k) && \text{(by superadditivity of } p) \\
&\leq kp(\delta) && \text{(by monotonicity of } p) \\
&\leq \frac{3p(\delta)}{\delta} \\
&= 3\delta^{\beta-1} \\
&\leq 1 - A, \quad \text{as desired.}
\end{aligned}$$

(c) We need to verify that, for sufficiently small ϵ,

$$\begin{aligned}
&p(x_1) + \cdots + p(x_m) - A\frac{p(x_m)}{x_m} \\
&\qquad < p(x_1) + \cdots + p(x_m - \epsilon) + p(\epsilon) - A\frac{p(\epsilon)}{\epsilon}.
\end{aligned}$$

This is equivalent to the inequality

$$\frac{p(x_m)}{x_m} > \frac{p(\epsilon)}{\epsilon} + \frac{1}{A}(p(x_m) - p(x_m - \epsilon) - p(\epsilon)).$$

We can choose $\epsilon > 0$ so small, that $1 > 1 - \epsilon \geq x_m > 0$ and

$$\frac{p(x_m)}{x_m} = x_m^{\beta-1} > \epsilon^{\beta-1} + \frac{1}{A}(\beta\epsilon - \epsilon^\beta).$$

Since p is convex,

$$p(x_m) - p(x_m - \epsilon) \leq p(1) - p(1 - \epsilon) \leq p'(1)\epsilon = \beta\epsilon.$$

Hence this choice of ϵ will yield the desired inequality, and the proof of the proposition is complete.

6 Notes

Certain resemblances will be noted between some of the ideas used in our proof of the Main Theorem and those employed by Shapley in his proof [8] of Scarf's theorem [6] (on the existence of the core). Thanks to the continuous nature of our model, the geometric method can be pursued more effectively, yielding stronger conclusions in more intuitive ways. Indeed, Scarf's Theorem itself will be seen to be a direct corollary of the Main Theorem (Part II).

Logically, the analysis can also be run in reverse, regarding the continuum model (and "large economies") as the limit of coalition games involving only discrete (or "small") populations. Historically, this is the path taken. Scarf's Theorem is the basis for most of the relevant results surveyed in [4, 10]. Close inspection of the argument in [7] shows that it again relies on Scarf's Theorem, and the "stable partition-allocation" (or migration-proof formation in our terminology) is obtained as a limit of the balanced sets provided by [6, Theorem 2].

References

[1] Danzer, L., Grünbaum, B., Klee, V. (1963) Helly's Theorem and its Relatives, in: Klee, V. (ed.), Convexity, Proceedings of Symposia in Pure Mathematics, vol.7, American Mathematical Society, Providence, Rhode Island, 101-180

[2] Greenberg, J. (1994) Coalition Structures, in: R.J.Aumann and S.Hart (eds.), Handbook of Game Theory, vol.2, Elsevier, Amsterdam, 1305-1337

[3] Hirsch, M.W. (1976) Differential Topology, Springer-Verlag, New York

[4] Kannai, Y. (1992) The Core and Balancedness, in: R.J.Aumann and S.Hart (eds.), Handbook of Game Theory, vol.1, Elsevier, Amsterdam, 355-395

[5] Roth, A.E. and M.Sotomayor (1992) Two-Sided Matching, in: R.J.Aumann and S.Hart (eds.), Handbook of Game Theory, vol.1, Elsevier, Amsterdam, 485-541

[6] Scarf, H.E. (1967) The Core of N-person Game, Econometrica 35, 50-67

[7] Shaked, A. (1982) Human Environment as a Local Public Good, Journal of Mathematical Economics 9, 275-283

[8] Shapley, L.S. (1973) On Balanced Games without Side Payments, in: T.C.Hu and S.M.Robinson (eds.), Mathematical Programming, Academic Press, New York, 261-290

[9] Spanier, E.H. (1966) Algebraic Topology, McGraw-Hill, New York

[10] Wooders, M.H. (1994) Large games and Economies with Effective Small Groups, in: J.-F.Mertens and S.Sorin (eds.), Game-Theoretic Methods in General Equilibrium Analysis, Kluwer Academic Publishers, Amsterdam

Appendix: Migration Models with Interaction

We give a general yet tentative formulation.

As before, there are n *types* of agents, indexed by the superscript $i \in N = \{1, \dots, n\}$; and we index the *communities* by the subscript $s \in \mathcal{S}$. Generally, $\mathcal{S}$ is an infinite set.

Let $x_s^i \geq 0$ be the population of individuals of type i in the community s, and $u_s^i \in \mathbb{R}$ be the per capita payoff for these individuals. Let $x = (x_s^i)_{i \in N, s \in \mathcal{S}}$, and $u = (u_s^i)_{i \in N, s \in \mathcal{S}}$. The array x is called a *formation*. The *payoff function* U specifies

$$u = U(x).$$

The *total population profile* $\Xi = (\Xi^1, \dots, \Xi^n)$ of the formation x is given by

$$\Xi^i = \textstyle\sum_s x_s^i.$$

Or we say that Ξ can be *distributed into* the formation x.

A *migration process* consists of a pair (x, ϵ), where $\epsilon = (\epsilon_{ss'}^i)_{i \in N, s, s' \in \mathcal{S}}$ is the population of type i migrating from the community s to s'. The array ϵ satisfies the condition $\epsilon_{ss'}^i + \epsilon_{s's}^i = 0$ for every i.

Thus the formation after the migration is given by

$$y_s^i = x_s^i + \textstyle\sum_{s'} \epsilon_{s's}^i.$$

Note that the total population profile is conserved.

The migration process (x, ϵ) is *incentive compatible* if: whenever $\epsilon_{ss'}^i > 0$ then $U_{s'}^i(y) > U_s^i(x)$.

We also say that (x, ϵ) is *voluntary* if: whenever $\epsilon_{ss'}^i > 0$ for some i then $x_{s'}^j = 0$ for all j. In other words, migrants are only allowed to form new communities — there cannot be any forced immigration without the consent of the indigenous population.

A formation x is *migration-proof* if there is no migration process (x, ϵ) which is voluntary and incentive compatible.

The model will be *scale-invariant* if $U(kx) = U(x)$ for all $k > 0$; and *non-interactive* if u_s^i depends only on $(x_s^1, \dots, x_s^n)$. The reader will have no difficulty reconciling this formulation with the one used in the paper.