Two-Sector Model

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Main reference: Ronald Jones, JPE 1965.

Consider an economy with two final goods and two factors of production. Each final good is produced under *constant return to scales* and perfect competition using both factors of productions We assume that the endowments of factors L and K are fixed and completely unelastically supplied. We do however have to decide how to allocate L and K across sectors.

TAKING COMMODITY PRICES AS GIVEN

This is the problem faced, for instance, by a "small open economy", which decisions have no impact on world prices.

The general problem and the resolution

The problem is to determine w, the wage rate, and r, the rental rate of capital, and $L_1, L_2, K_1, K_2,$ the allocation of factors across sectors, as well as the outputs $Y_1, Y_2.$ These are 8 unknowns. The 8 equations that allow to solve this problem are:

- the production functions (2 eqns):

$$Y_1 = F(K_1, L_1),$$

$$Y_2 = G(K_2, L_2).$$

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- the Value of Marginal Product = Marginal Cost conditions (4 eqns):

$$w = P_1 F_L (L_1, K_1),$$

$$w = P_2 G_L (L_2, K_2),$$

$$r = P_1 F_K (L_1, K_1),$$

$$r = P_2 G_K (L_2, K_2).$$

- the factor equilibrium conditions (2 eqns):

$$K_1 + K_2 = K,$$

$$L_1 + L_2 = L.$$

(The factor exhaustion equations, are redundant: $P_1 Y_1 = w L_1 + r K_1$ and $P_2 Y_2 = w L_2 + r K_2$). Note also that the VMP=MC equations imply:

$$\frac{w}{r} = \frac{F_L}{F_K} = \frac{G_L}{G_K},$$

i.e. capital and labor are allocated so as to equate the MRS across factors between sectors.

Log-Linear Approximations and Comparative Statics

This system is of course not solvable unless we assume functional forms. Instead of doing this, we will work with log-linear approximations. I write $d\log$ for the percentage change around a solution: $d\log X = \log(X/X^*)$. We need again 8 equations in $d\log$ of $Y_1, Y_2, w, r, K_1, K_2, L_1, L_2$.

- We first consider the production function (2 eqns):

$$d\log Y_1 = s_L^1 d\log L_1 + s_K^1 d\log K_1,$$

$$d\log Y_2 = s_L^2 d\log L_2 + s_K^2 d\log K_2,$$

where s_A^i is the cost share of factor $A \in \{K, L\}$ in sector $i = 1, 2$. Constant returns implies that $s_K^i + s_L^i = 1$ for $i = 1, 2$.

- Instead of using the VMP=MC equations, we use the definitions of the elasticities of substitutions, and then the pricing equations:

$$d \log K_1 - d \log L_1 = \sigma_1 (d \log w - d \log r),$$

$$d \log K_2 - d \log L_2 = \sigma_2 (d \log w - d \log r),$$

$$d \log P_1 = s_L^1 d \log w + s_K^1 d \log r,$$

$$d \log P_2 = s_L^2 d \log w + s_K^2 d \log r.$$

These equations hold because of constant returns.

- Finally we use the factor equilibrium conditions. To derive this equation, note that $dK_1 + dK_2 = dK$,hence

$$\frac{K_1}{K} d \log K_1 + \frac{K_2}{K} d \log K_2 = d \log K.$$

Let $\lambda_K^1 = \frac{K_1}{K}$ denote the % of the factor capital that is used in the production of commodity 1, and define similarly $\lambda_K^2, \lambda_L^1, \lambda_L^2$. Obviously we have $\lambda_K^2 + \lambda_K^1 = 1$ and $\lambda_L^1 + \lambda_L^2 = 1$. The log-linearization of the factor equilibrium conditions are finally written as:

$$\lambda_K^1 d \log K_1 + \lambda_K^2 d \log K_2 = d \log K,$$

$$\lambda_L^1 d \log L_1 + \lambda_L^2 d \log L_2 = d \log L.$$

This completes our system of 8 equations. The parameters are $s_K^1, s_K^2, \sigma_1, \sigma_2, \lambda_K^1, \lambda_L^1$. This system allows us to find the response of $L_1, L_2, K_1, K_2, Y_1, Y_2, r, w$ to a change in K, L, P_1, P_2.

The Factor-Price Equalization Theorem

A first remark: if there are no change in commodity prices, ie $d\log P_1 = d\log P_2 = 0,$ then factor prices are also unchanged, ie $d\log w = d\log r = 0.$ This follows directly from the two 'pricing' equations, assuming that the two sectors do not have the same factor intensities. This is a version of the *factor-price equalization theorem*: if two countries are trading two commodities and face the same output price, then the factor remuneration is the same even if the factors themselves are not traded. The intuition is that since final goods embody the factors, trading two final goods is equivalent to trading the factors.

This result holds therefore only if the different countries have the same technology (ie same F and G), and are not fully specialized (which is a corner solution; we are using the first-order conditions so our reasonning applies only to interior solutions where each country produces some of each commodity).

Factor-Intensity Assumption

We will from now on make the assumption that the sector 1 is always more capital-intensive than the sector 2; this means that for any ratio $\frac{w}{r},$ we have $s_K^1 > s_K^2$. This implies that $s_L^1 < s_L^2$ and so $\frac{s_K^1}{s_L^1} > \frac{s_K^2}{s_L^2}$ ie

$$\frac{K_1}{L_1} > \frac{K_2}{L_2}.$$

Note that this does not imply a higher capital per output in the capital-intensive sector). It is theoretically possible that one sector is more capital-intensive for some values of $\frac{w}{r}$ and less capital-intensive for other values ('factor intensity reversal'). We rule out this case here.

The Magnification effect

Two interesting exercises of comparative statics can now be conducted. First consider the effect of an increase in the supply of K greater than the increase in the supply of L, holding P_1 and P_2 fixed.

As we saw earlier, $d\log w = d\log r = 0$. This in turn implies $d\log K_i = d\log L_i = d\log Y_i$,for $i = 1, 2$.

To determine this effect, we use the factor equilibrium conditions which yield:

$$d\log Y_1 \left(\lambda_K^1 \lambda_L^2 - \lambda_L^1 \lambda_K^2\right) = \left(\lambda_L^2 d\log K - \lambda_K^2 d\log L\right).$$

Using the fact that the $\lambda' s$ sum to 1 for each factor:

$$d\log Y_1 \left(\lambda_K^2 - \lambda_L^1\right) = \left(\lambda_L^2 d\log K - \lambda_K^2 d\log L\right).$$

Notice finally that $s_K^1 > s_K^2$ implies that $\lambda_K^1 > \lambda_L^1$ (and so $\lambda_L^2 > \lambda_K^2$) since

$$\frac{K_1}{K} = \frac{K_1}{L_1}\frac{L_1}{K} > \frac{K_2}{L_2}\frac{L_1}{K} = \lambda_K^2 \frac{\lambda_L^1}{\lambda_L^2},$$

so

$$\frac{\lambda_K^1}{\lambda_K^2} > \frac{\lambda_L^1}{\lambda_L^2} \Leftrightarrow \frac{\lambda_K^1}{1 - \lambda_K^1} > \frac{\lambda_L^1}{1 - \lambda_L^1} \Leftrightarrow \lambda_K^1 > \lambda_L^1.$$

Hence $d\log Y_1 > 0$ and since $d\log K > d\log L$ we also have:

$$d\log Y_1 \left(\lambda_K^1 - \lambda_L^1\right) > \left(\lambda_L^2 - \lambda_K^2\right) d\log K,$$

$$\Rightarrow d\log Y_1 > d\log K.$$

since $\lambda_K^1 - \lambda_L^1 = \lambda_L^2 - \lambda_K^2$. Similarly we get $d\log Y_2 < d\log L$ and thus:

$$d\log Y_2 < d\log L < d\log K < d\log Y_1.$$

Increasing the supply of one factor at constant commodity prices will lead to an increase in the production of the commodity intensive in the use of that factor; this increase is *magnified* by the constant factor prices: there is a 'multiplicator'. (In International Trade, this is called the Rybcziynski theorem). The intuition is that since factor prices are fixed, an increase in quantity of capital must be allocated to the capital-intensive sector to clear the market for factors.

We have a dual relation that is obtained by considering the impact of the increase of commodity prices: $d\ln P_1 > d\ln P_2$ at constant factor supply. We get after similar computations:

$$d\ln r > d\ln P_1 > d\ln P_2 > d\ln w$$

ie the price of the factor in which the commodity whose price rise most is intensive rise, and the effect is more than 1 for 1. (In International Trade, this is called the Stopler-Samuelson theorem).

Extensions

It is easy to modify the model to incorporate taxes or technical change. For instance, if the firm in each sector are subsidized so that they receive $d_i p_i$ instead of p_i (or, alternatively, $d_i = 1/(1+t_i)$ where t_i is the VAT rate). The only modification we need to do is to change the 'pricing' equations:

$$d\log P_1 + d\log d_1 = s_L^1 d\log w + s_K^1 d\log r,$$

$$d\log P_2 + d\log d_2 = s_L^2 d\log w + s_K^2 d\log r.$$

At fixed commodity prices a very similar comparatitive statics exercise implies that subsidies are 'shifted backward' in a magnified fashion:

$$d\log w > d\log d_2 > d\log d_1 > d\log r,$$

ie if the labor-intensive sector is subsidized more (or taxed less) than the capital-intensive sector, wages increase more than one-for-one (but this effect may not be true when we close the model with a consumer demand).

We can also add technical change by modifying the following two equations:

$$d\log Y_1 = s_L^1 d\log L_1 + s_K^1 d\log K_1 + d\ln A_1,$$

$$d\log Y_2 = s_L^2 d\log L_2 + s_K^2 d\log K_2 + d\ln A_2,$$

$$d\log P_1 = s_L^1 d\log w + s_K^1 d\log r - d\ln A_1,$$

$$d\log P_2 = s_L^2 d\log w + s_K^2 d\log r - d\ln A_2.$$

CLOSING THE MODEL (GENERAL EQUILIBRIUM)

We close the model with the simple assumption that there is a representative consumer who has homothetic preferences; ie demand for each goods are proportional to his income: $\frac{X_1}{X_2} = D\left(\frac{P_1}{P_2}\right)$. We may log-linearize this equation and add it to our stack:

$$d\ln X_1 - d\ln X_2 = \sigma_D \left(d\ln P_2 - d\ln P_1\right),$$

where σ_d is the elasticity of substition of demand. Now P_1 and P_2 are endogenous too; so we added one equation. We can normalize one of the prices because this is a general equilibrium model. The commodity market equilibrium implies $d\ln X_i = d\ln Y_i$.

Jones (1965) shows how to define an elasticity of substitution on the supply side. Let $\delta_K = \sigma_1\lambda_K^1 s_L^1 + \sigma_2\lambda_K^2 s_L^2$, $\delta_L = \sigma_1\lambda_L^1 s_K^1 + \sigma_2\lambda_L^2 s_K^2$, and

$$\Lambda = \begin{pmatrix} \lambda_K^1 & \lambda_K^2 \\ \lambda_L^1 & \lambda_L^2 \end{pmatrix}, \qquad s = \begin{pmatrix} s_K^1 & s_K^2 \\ s_L^1 & s_L^2 \end{pmatrix}.$$

Then we can define σ_S as:

$$\sigma_S = \frac{1}{\det\Lambda}\frac{1}{\det s}\left(\delta_L + \delta_K\right),$$

and we have:

$$d\ln P_1 - d\ln P_2 = -\frac{1}{\det\Lambda}\frac{1}{(\sigma_S + \sigma_D)}\left(d\ln K - d\ln L\right),$$

whence:

$$d\ln Y_1 - d\ln Y_2 = \frac{1}{\det\Lambda}\frac{\sigma_D}{(\sigma_S + \sigma_D)}\left(d\ln K - d\ln L\right).$$

Now commodity prices adjust in response to a change in factor endowments. So the composition of commodities produced will not always ‘overreact’; it depends on whether the first term $\frac{1}{\det \Lambda}\frac{\sigma_D}{(\sigma_S+\sigma_D)}$ is bigger or smaller than 1. Large values of σ_S (ie large values of σ_1 and σ_2 underlying σ_S - substitution between and within sectors) will prevent 'magnification' since they imply that the factor price reactions will induce reallocation of factors. On the other hand, small values of σ_D will also tend to prevent 'magnification' because the consumer is unwilling to accept large relative changes in goods consumed.

Now consider the case of fixed factor endowments but we introduce a small subsidy; the result we get is:

$$d\ln w - d\ln r = \frac{1}{\det s}\frac{\sigma_D}{\sigma_D+\sigma_S}\left(d\ln d_2 - d\ln d_1\right).$$

If we subsidize the capital-intensive sector 1 less than the labor-intensive sector 2, part of the subsidy is shifted backward, affecting favourably the factor used intensively in the production of 2, ie labor. By how much labor is favoured depends on how much has been passed to the consumer; if σ_D is high the consumer's demand is relatively elastic and so price changes are small.

Assuming no subsidies, we can define the aggregate elasticity of substitution by

$$\sigma = (\sigma_S+\sigma_D)(\det\Lambda)(\det s),$$

since:

$$d\ln w - d\ln r = -\frac{d\ln K - d\ln L}{\sigma}.$$

σ is thus a composite of the three basic elasticities of substitution of the model - the demand elasticity and the supply elasticities.

A CASE WITH THREE FACTORS: EACH SECTOR HAS A SPECIFIC FACTOR

Sector 1 (or u) produces under CRS using unskilled labor L_u and capital; and sector 2 (or s) produces under CRS using skilled labor L_s and capital. Sector 1 is assumed to be more capital-intensive.

In this case we have to determine w_u, the wage rate of unskilled labor, w_s, the wage rate of skilled labor, and r, the rental rate of capital, and L_u, L_s, K_u, K_s, the allocation of factors across sectors, as well as the outputs Y_u, Y_s. There are 9 unknowns. The 9 equations that allow to solve this problem are:

- the production functions (2 eqns):

$$Y_u = F(K_u, L_u),$$

$$Y_s = G(K_s, L_s).$$

- the VMP=MC conditions (4 eqns):

$$w_u = P_u F_L(L_u, K_u),$$

$$w_s = P_s G_L(L_s, K_s),$$

$$r = P_u F_K(L_u, K_u),$$

$$r = P_s G_K(L_s, K_s).$$

- the factor equilibrium conditions (3 eqns):

$$K_s + K_u = \overline{K}.$$

$$L_u = \overline{L_u}.$$

$$L_s = \overline{L_s}.$$

The problem is in fact easier since we do not have to determine L_u and L_s (and note that of course if each industry has 2 specific factors, then the GE problem is immediatly solved; we know the quantities and infer the prices from the MVP).

Again it is possible to work through log-linear approximations:

$$d\log Y_u = s_L^u d\log L_u + s_K^u d\log K_u,$$

$$d\log Y_s = s_L^s d\log L_s + s_K^s d\log K_s,$$

$$d\log K_u - d\log L_u = \sigma_u \left(d\log w_u - d\log r\right),$$

$$d\log K_s - d\log L_s = \sigma_s \left(d\log w_s - d\log r\right),$$

$$d\log P_u = s_L^u d\log w_u + s_K^u d\log r,$$

$$d\log P_s = s_L^s d\log w_s + s_K^s d\log r,$$

$$\lambda_K^u d\log K_1 + \lambda_K^s d\log K_s = d\log K,$$

$$d\log L_u = d\log \overline{L_u},$$

$$d\log L_s = d\log \overline{L_s}.$$