Econ. 704 **François Gourio**

Problem Set 1
Due Thursday, March 29

Note: I will email today an additional problem that you will need to solve on the computer.

Problem 1 (Deterministic dynamic programing.)

This is an example that you have seen with Vladimir in class, but I put it here so that you check you can do it all by yourself now (without checking your notes!). Consider the following economy. Time is discrete and infinite, and there is a representative consumer who has utility

$$\sum_{t\geq 0} \beta^t \log c_t,$$

and a representative firm which operates the technology:

$$y_t = k_t^\alpha.$$

Capital depreciates at rate 100%:

$$k_{t+1} = i_t = y_t - c_t.$$

The initial capital stock $k_0 > 0$ is given.

(1) Write the social planner problem for this economy as a sequence problem.

(2) Derive the first-order conditions of this sequence problem.

(3) Write the Bellman equation corresponding to the sequence problem.

(4) Solve the Bellman equation, using a guess-and-verify method. Give the optimal policy rules $c(k)$ and $k'(k)$.

(5) Plot roughly (using paper and pencil) the value function and policy functions.

(6) Explain using the graph of question 5 the dynamics of capital and consumption, as a function of k_0. (Is there a steady-state? Is there convergence to the steady-state? Is this convergence monotonic?)

Problem 2 (Partial equilibrium model of firm investment)

Consider a firm operating a decreasing-return to scale technology, which accumulates capital to maximize the present discounted value of its profits:

$$\sum_{t=0}^{\infty} \beta^t \left(z k_t^\alpha - i_t\right),$$

$$\begin{aligned} s.t. &\quad : \\ k_{t+1} &= (1-\delta)k_t + i_t \\ k_0 &> 0 \text{ given,} \end{aligned}$$

where $z > 0$ and $\alpha > 0$ are parameters.

(1) Derive the first-order conditions of this problem.

(2) What is (are) the steady-state(s) of this system?

(3) Write the Bellman equation.

(4) Give the policy function. Explain briefly the dynamics implied by this model.

Problem 3 (Partial equilibrium model of firm investment with adjustment costs)

Consider a firm operating a decreasing-return to scale technology, which accumulates capital to maximize the present discounted value of its profits, but is now subject to adjustment costs:

$$\begin{aligned} v(k_0) &= \sum_{t=0}^{\infty} \beta^t \left(z k_t^{\alpha} - i_t - k_t c\left(\frac{i_t}{k_t}\right)\right), \\ s.t. &: \\ k_{t+1} &= (1-\delta)k_t + i_t \\ k_0 &> 0 \text{ given,} \end{aligned}$$

where $z > 0$ and $\alpha > 0$ are parameters. We assume that c is a function satisfying: c is convex, $c(\delta) = 0$ and $c'(\delta) = 0$, and $c'(x) < 0$ if $x < \delta$, and $c'(x) > 0$ if $x > \delta$. For instance, $c(x)$ could be given by the formula $c(x) = \frac{\psi}{2}(x-\delta)^2$. (Check that this formula satisfies the assumptions, and draw what c looks like.)

(1) Map the model into the sequence formulation (SP) of Handout 1. What is X, what is F, what is Γ and what is β?

(2) Given the assumptions that we made, is F increasing in the first argument? Is Γ increasing?

(3) Given the assumptions that we made, is F concave? Is Γ convex?

(4) Is the value function increasing in general?

(5) Is the value function concave in general?

(6) Write the Bellman equation for this problem.

(7) Assuming that the value function is differentiable, write the FOC of the Euler equation.

(8) Use the envelope theorem to find an expression for the derivative of the value function.

(9) Evaluate (7) and (8) at the steady-state and combine them to find a quadratic equation to solve for $g'(x^*)$.

(10) Solve the quadratic equation. Explain which parameters determine the speed of convergence to steady-state $g'(x^*)$.

Problem 4 (Writing some Bellman equations.)

(a) Consider again a firm but now with two capital stocks, k_1 and k_2, with different depreciation rates δ_1, δ_2; these capital stocks are accumulated subject to adjustment costs $c(k_1, k_2)$. The profit per period is $\pi(k_{1,t}, k_{2,t})$. Write the sequence problem and the Bellman equation.

(b) Consider the problem of a firm with profit function $\pi = zk^{\alpha}$, which maximizes discounted profits by accumulating capital as in Problem 2. However, the firm faces a time-to-build lag: the capital accumulation equation is $k_{t+2} = (1-\delta)k_{t+1} + i_t$, i.e. the investment takes two periods to be translated into capital rather than one as usual. Write the sequence problem and write the Bellman equation.

(c) Consider two firms, A and B, who compete in a market. The inverse demand curve is $p = P(Q)$. In each period, Firm A is the Stackelberg leader: it chooses how much to produce Q_t^A. Then firm B chooses its own production Q_t^B. Assume that production must evolve according to the following process, $Q_{t+1}^A = (1-\delta)Q_t^A + I_{t-1}^A$, and similarly for firm B. (I.e. production depends on installed capital, and investment is chosen one period before production). Write the Bellman equation for firm A and the Bellman equation for firm B.

(d) Consider a worker who has expected discounted utility $V = \sum_{t\geq 0} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}$. There is no way of saving so his consumption equals his wage w (which is constant) when he is employed, and his unemployment benefits b when he is unemployed. Assume $w > b$. The worker loses his job each period with probability $\lambda \in [0,1]$. Once he has lost his job, he becomes unemployed. Next he will find one new job with probability $\mu \in [0,1]$. Write two Bellman equations, linking the value V of being employed today and the value U of being unemployed today. Solve for these two equations. How do the values V and U depend on w, b, λ, μ?

Problem 5 (supplementary, not graded)

Explain why the state-controls formulation is 'usually' equivalent to the formulation of the sequence problem (SP). For this, you may assume that the feasibility correspondence Γ is such that: $\Gamma(x) = X$ is constant. State whatever additional assumptions you need to verify this equivalence. (You do not need to be overly rigorous.)

Econ. 704 **François Gourio**

Problem Set 1: Additional problem
Due Thursday, March 29 or Friday, March 30

Note: this problem is not very long - I wrote many questions to simplify things for you.

Numerical problem (Numerical deterministic dynamic programming.)

Consider the Bellman equation for a firm accumulating capital subject to convex adjustment costs:

$$\begin{aligned} V(k) &= \max_{c,k'} \left\{ zk^\alpha - i - \frac{\psi}{2} i^2 + \beta V(k') \right\} \\ s.t. &: \\ k' &= (1-\delta)k + i. \end{aligned}$$

This problem asks you to solve numerically (approximately) this equation using a procedure known as *value iteration*. I recommend using MATLAB. Learn how to write an "m-file" (i.e. a sheet where you can do all your MATLAB computations rather than typing them one line after the other in the command window).

We take the following parameter values: $\alpha = 0.5$, $\beta = 0.95$, $\psi = 2$, $\delta = 0.08$, $z = 1$ (a normalization). Follow the following steps:

(a) Compute the steady-state capital k^* (hint: write the first-order conditions of this model).

(b) Construct a grid for $k : k_1 < k_2 < ... < k_N$, with N equally spaced points between $k_1 = 0.5k^*$ and $k_N = 1.5k^*$. (Hint: you may want to use the LINSPACE command in MATLAB.) Set $N = 200$.

(c) Now our Bellman equation is, substituting in the capital accumulation equation:

$$V(k) = \max_{k' \geq 0} \left\{ zk^\alpha - (k' - (1-\delta)k) - \frac{\psi}{2} (k' - (1-\delta)k)^2 + \beta V(k') \right\}$$

Construct a matrix $N \times N$ with values $R(i,j) = zk_i^\alpha - (k_j - (1-\delta)k_i) - \frac{\psi}{2} (k_j - (1-\delta)k_i)^2$. For our discrete case the value function is a $N \times 1$ vector satisfying:

$$\forall i = 1...N : V(i) = \max_{j \in \{1,2,...N\}} (R(i,j) + \beta V(j)). \tag{1}$$

(d) Now the iteration step: Start with a guess for the value function, for instance $V(i) = 0$ for all $i = 1...N$. Compute the RHS of (1) for this guess, for any i. This defines a new value function as:

$$V^{new}(i) = \max_{j \in \{1,2,...N\}} (R(i,j) + \beta V^{guess}(j)).$$

Next set $V^{guess} = V^{new}$ and keep iterating this way until

$$\max_{i \in \{1 \ldots N\}} |V^{new}(i) - V^{guess}(i)| < \varepsilon,$$

where ε is a predetermined number (e.g. 10^{-5}). Plot the value function $V(i)$ [i.e. $V(k)$].

(e) Plot the policy rule: $k'(k)$. (Hint: use the MATLAB command MAX.)

(f) Assume $k_0 = 0.8k^*$. Using your programs, compute the sequence $\{k_t\}_{t=0}^{T}$ for $T = 100$ and plot it. Discuss briefly.

(g) With this computer program, can you check your theoretical results from problem 3 of PS1?

(h) (a bit harder). Suppose the firm starts in steady-state. What is the effect on the firm value of a permanent, unexpected one-percent increase in productivity z? What is the effect on the firm value of a permanent, unexpected one-percent increase in β? What is the firm is not in steady-state?

Problem Set 2
Due Thursday, April, 5th

Problem 1: Neoclassical Growth Model

Consider the Neoclassical Growth model:

$$\begin{aligned} v(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t\geq 0} \beta^t U(c_t) \\ s.t. &: c_t + k_{t+1} \leq H(k_t) \\ k_{t+1} &\geq 0, \quad k_0 \text{ given.} \end{aligned}$$

Here U is an increasing and concave function, and H is increasing and concave.

(1) Map this problem into the (SP) formulation: what is X? what is Γ? what is F? what is β?

(2) Write the Bellman equation.

(3) Assuming that the value function is differentiable, write the first-order condition for the Bellman equation and the Envelope condition. Obtain the Euler equation from combining these two equations.

(4) Find the steady-state (k^*, c^*).

(5) Is the policy function increasing? (You may give a graphical argument.)

(6) Compute the linear approximation of the policy function around the steady-state. Explain how the speed of convergence to the steady-state depends on the curvature of the utility function, on the curvature of the production function, and on the parameter β.

(7) Suppose at $t = 0$ the economy is in steady-state, i.e. $k_0 = k^*$. There is an unexpected, permanent increase in the production function H. Explain graphically and algebraically the dynamics of capital, consumption, and the marginal product of capital. In particular, give an approximation to the level of consumption at date 0 and of capital at time 1.

Problem 2: Housing market

Consider the following (deterministic, dynamic) model of the housing market. On the demand side, we have a relation between the demand for the stock of housing and the rent: $R_t = D(K_t)$. On the supply side, there is a construction industry which supplies new houses each period. This construction industry builds I_t house at time t, which are then added to the existing stock according to the standard capital accumulation equation $K_{t+1} = (1-\delta)K_t + I_t$. (Houses depreciate at rate δ.) The cost to this industry of building I_t house is $C(I_t)$, where C is an increasing and convex function. Let P_t the price of a house (i.e. the capital price, as opposed to the rent).

(1) Write the optimization problem of the construction sector. (Hint: this is a static problem). Obtain the FOC.

(2) Assume households are risk-neutral and discount the future at rate $\beta = \frac{1}{1+r}$. Write a relation between the current capital price P_t, the future capital price P_{t+1} and the rent R_t.

(3) Find the steady-state of this system. (Hint: it may be useful to draw a graph.)

(4) Write two first-order difference equations with only P_t, P_{t+1} and K_t, K_{t+1} which together determine the evolution of the system.

(5) Combine these two equations to obtain a second-order difference equation in K_t which gives us the solution.

(6) Write the linear approximation of this second-order linear difference equation around the steady-state.

(7) Find the solution of the linear approximation. (As usual, there will be two roots, one of which is less than one in absolute value and one which is greater than one - we will set the constant in front of that root equal to zero.) Also find the linear approximation for the price P_t around the steady-state. When the capital is above the steady-state, is the price above the steady-state too? As the capital grows towards the steady-state, does the price increase or fall?

(8) Using your answers to (1)-(7), consider the following experiment. We start at time 0 in steady-state, and there is an unexpected, permanent increase in demand. Plot the dynamic response of P_t, R_t, K_t and I_t as the economy converges to the new steady-state.

(9) Redo (8) when the shock is an increase in the supply, i.e. a downward shift in $c(.)$.

Problem 3: Homogeneity

Consider the following investment problem:

$$\begin{aligned} v(K_0) &= \max_{\{K_{t+1},I_t\}_{t=0}^{\infty}} \sum_{t=0} \beta^t \left(zK_t - I_t - K_t c\left(\frac{I_t}{K_t}\right)\right) \\ K_{t+1} &= (1-\delta)K_t + I_t, \\ &\quad \text{given } K_0. \end{aligned}$$

where $c(.)$ is the adjustment cost function. Assume that c is convex.

(1) Argue that $v(\lambda K_0) = \lambda v(K_0)$.

(2) Show that this implies that $v(K) = Kv(1)$.

(3) Write the Bellman equation for $v(K)$.

(4) Using (2) and (3), what can you say about the investment decision rule $I(K)$? Researchers often measure the relation between the investment-capital ratio of a firm and the ratio of its market value to its stock of capital (also known as Tobin's q). Why?

(5) Write a system of two equations in two unknowns (two numbers!) which determines the value function and the policy function.

(6) Is there a steady-state for this firm? If not what happens to its capital stock over time?

Problem 4: Markov chains

Consider the following model of worker flows. People are either employed, unemployed, or out of the labor force. Each period, the probability of finding a job for an unemployed person is f ; the probability of dropping out of the labor force if you are unemployed is d. For an employed worker, the probability of becoming unemployed is s (and there's a 0 probability of dropping directly out of the labor force). Finally, out of the labor force people become unemployed with probability p (and there's a 0 probability of becoming employed directly).

(1) Write down the transition matrix for the markov process "employment status". (Hint: it may be useful to draw a picture.)

(2) Find the steady-state shares of people employed, unemployed and out of the labor force. Explain how these shares vary with the parameters.

(3) Suppose that there is a recession, which we interpret as $x \stackrel{def}{=} f/s$ falling and $y \stackrel{def}{=} d/p$ falling. What happens to the three shares? Please give sufficient conditions for each share to be procyclical.

(4) Suppose for this question that there is no "out-of the labor force" status and the probabilities are simply f (finding a job), s (leaving his job). Also suppose that at time 0 the unemployment rate is a fraction $r \in [0, 1]$. What will be the unemployment rate tomorrow (i.e. $t = 1$)? Give the evolution of the unemployment rate u_t for all $t \geq 0$. What is the limit as $t \to +\infty$?

Problem 5: Markov chain

Consider the following two-state Markov Chain $\{z_t\}$. It can take two values, $z_t = 0$ or $z_t = 1$. The transition matrix is

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}.$$

This means of course that $\Pr(X_{t+1} = 0 \mid X_t = 0) = p$ and $\Pr(X_{t+1} = 1 \mid X_t = 0) = 1-p$, etc.

(a) Compute the invariant distributions of P.

(b) Compute the conditional mean for each state, i.e. the expected value next period conditional on being in state 0 or 1. Mathematically this is $E_t(X_{t+1}) = E(X_{t+1} \mid X_t = i)$ for $i = 0, 1$. Also compute the unconditional mean $E(X_t)$. How is it related to the conditional means?

(c) compute the conditional standard deviation:

$$\sigma_t(X_{t+1}) = E\left(X_{t+1}^2 \mid X_t = i\right) - E(X_{t+1} \mid X_t = i)^2.$$

Also compute the unconditional standard deviation. How is it related to the conditional standard deviation.

Problem 6: Markov chain

(1) Consider a consumer with a time-separable, expected utility function: $E\sum_{t=0}^{\infty}\beta^t u(c_t)$. Assume $\{c_t\}$ follows a Markov chain with transition matrix P. Let V_i be the expected discounted utility of the consumer if his current consumption is c_i. Write a system of equations to solve for V_i, for all $i = 1..S$. (There is an explicit solution in terms of matrices, but you do not need to show it.)

(2) Assume that the current utility also depends on past consumption, so that preferences are $E\sum_{t=0}^{\infty}\beta^t u(c_t - \theta c_{t-1})$. What does V (the expected discounted utility today) depend upon now? Write again a system of equations for the V's.

(3) Usually we prefer to think of consumption *growth* being a Markov chain. So assume that $\Delta \log c_t$ follows a Markov chain, and that preferences are $E\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$ for some $\gamma > 0$. Give a formula for the expected discounted value V as a function of today's consumption and today's consumption growth rate.

Econ. 704 Spring 2007 **François Gourio**

Problem Set 3
Due Thursday, April 12

Problem 1 (Markov Chain.)

For each of the following transition matrix, is it true that there is a unique ergodic distribution μ^* and that there is convergence to it from any initial μ_0? Prove your claim (e.g. using Result 1 or 2 from the Handout #2, or by exhibiting several ergodic distributions, or by showing that the limiting distribution does not exist or depends on the initial distribution μ_0).

$$(1) \quad : \quad P = \begin{pmatrix} .5 & .5 \\ .9 & .1 \end{pmatrix}$$

$$(2) \quad : \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(3) \quad : \quad P = \begin{pmatrix} .7 & .3 \\ 0 & 1 \end{pmatrix}$$

$$(4) \quad : \quad P = \begin{pmatrix} .5 & .5 & 0 \\ .3 & .3 & .4 \\ 0 & .5 & .5 \end{pmatrix}$$

$$(5) \quad : \quad P = \begin{pmatrix} .5 & .5 & 0 \\ .3 & .7 & 0 \\ 0 & .5 & .5 \end{pmatrix}$$

$$(6) \quad : \quad P = \begin{pmatrix} 1 & 0 & 0 \\ .3 & .4 & .3 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2 (AR processes.)

(a) Plot by hand the impulse response of the following time-series processes:

$$\begin{aligned} x_t &= \rho x_{t-1} + \varepsilon_t, \text{ for } \rho > 0 \text{ and for } \rho < 0, \\ x_t &= \varepsilon_t + \theta \varepsilon_{t-1}, \text{ for } \theta > 0 \text{ and for } \theta < 0. \end{aligned}$$

Note: you can do the computation by pencil-and-paper, and plot roughly the response.

(b) Consider the following process for consumption growth $\Delta \log c_t$. Plot the response of consumption growth ($\Delta \log c_t$) and the (log) consumption level ($\log c_t$) to a shock:

$$\begin{aligned} \Delta \log c_{t+1} &= \varepsilon_{t+1}, \\ \Delta \log c_{t+1} &= \rho \Delta \log c_t + \varepsilon_{t+1}. \end{aligned}$$

Extra (not required): what about the process $\Delta \log c_{t+1} = \rho \Delta \log c_t + \varepsilon_{t+1} - \theta \varepsilon_t$? (This one may be hard to do by hand, but easy with a computer - e.g. Excel).

Problem 3 (DP under uncertainty.)

Some motivation: in labor economics, a large literature measures the 'returns to experience' by a regression of the log wage on experience (= number of years you have worked up to today), holding some 'controls' fixed. For instance, a cross-sectional regression could be:

$$\log w_i = \alpha + \beta exp_i + x_i'\theta + \varepsilon_i,$$

where exp is experience, x is a vector of controls (schooling, etc.), and ε is an error term (e.g. measurement error in the wage). The coefficient β, which measures the returns to experience, is always significantly positive and usually large. This suggests that people should take into account that if they work more today, they will get higher wages in the future.

Problem: consider a worker who has to decide each period how many hours to work h_t. The hourly wage he gets is $w_t = w(e_{t-1})$ where e_{t-1} is his experience at the end of period $t-1$. Experience evolves according to $e_t = e_{t-1} + h_t$. The agent has utility $E\sum_{t=0}^{\infty} \beta^t (u(c_t) - v(h_t))$ where $v(.)$ is the disutility of working.

(1) Assuming that the agent consumes his current income, write the sequence problem and the Bellman equation for this worker.

(2) Assume for this question that the choice is not how many hours to work, but whether to work or not, i.e. $h \in \{0, 1\}$. If the agent does not work, his income is a benefit b. Write the Bellman equation.

(3) Now assume instead that the agent can save or borrow at rate r, what does his Bellman equation look like? (Hint: you can write the budget constraint as $a_{t+1} = Ra_t + w(e_{t-1})h_t - c_t$ where a_t = assets at the beginning of period t.)

(4) Consider simulating the model from question (1) and running the regression above on the data obtained from simulations. What would be the issues in running such a regression? How can you solve these issues?

(5) Consider introducing 'taste shocks' as follows: the disutility of working in period t is $\theta_t v(h_t)$, where θ_t is a $N-$state Markov chain. Write the Bellman equation. Does this solve the problem of question (4)?

Problem 4 (DP under uncertainty.)

Consider a firm which operators a labor-only decreasing return technology: $y = xn^{\alpha}$, with $0 < \alpha < 1$ and x is productivity. The firm decide each period how many workers n to hire at wage w (constant) and it sells its output at price P (constant). The firm also has to pay a fixed cost f as long as it stays in business (even with $n = 0$). The productivity x evolves according to a Markov process with transition matrix P. We assume that P satisfies the 'persistence' condition i.e. if $x \leq x'$, $P_{x'.}$ first-order stochastically dominates $P_{x.}$. Each period, the firm can decide to leave the industry, by paying a cost c (which may be negative), whereafter it gets a zero value.

(1) Write the profit function giving the revenue in a given period as a function of x, P, w.

(2) Write down the Bellman equation for the firm's problem.

(3) Show that the value function is increasing in productivity. Prove that the firm exit decision rule is to exit if productivity falls below a certain cutoff. Give the condition which determines this productivity cutoff.

(4) Give the employment policy function.

(5) Now suppose that there is an adjustment cost to labor, i.e. given you had n workers last period, if you want to hire n' workers this period, you have to pay $wn' + c(n, n')$ where c is some function. Write the new Bellman equation. Discuss intuitively, in terms of the value and policy functions, how this might change the solution.

Problem 5 (DP under uncertainty.)

Consider the following RBC model. Utility is:

$$U = E_0 \sum_{t \geq 0} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} v(L_t),$$

where v is an appropriate function (i.e., such that utility $\frac{C^{1-\gamma}}{1-\gamma}v(L)$ is increasing and concave in the two arguments).

The production function is Cobb-Douglas and the capital accumulation is as usual:

$$K_{t+1} = (1-\delta)K_t + I_t,$$

$$C_t + I_t = Y_t = F(K_t, Z_t N_t),$$

where F has constant returns to scale. The resource constraint for hours is:

$$L_t + N_t = 1.$$

Z_t follows a random walk with drift:

$$\ln Z_{t+1} = \ln Z_t + \mu + \sigma \varepsilon_{t+1},$$

where ε_t is iid $N(0,1)$.

Let $V(K, Z)$ be the value function given initial conditions $K_0 = K$ and $Z_0 = Z$.

(1) Write the Bellman equation for this problem.

(2) Guess and verify that the Bellman equation can be written as

$$V(K, Z) = Z^{1-\gamma} \phi\left(\frac{K}{Z}\right), \tag{1}$$

where ϕ satisfies itself a Bellman equation (which you should write).

Hint: plug the guess (1) on the right-hand side of the Bellman equation; show then that the left-hand side will also take the same form. Comment: this method allows to reduce the dimension of the state space: we now have to solve (numerically) a Bellman equation with only one state variable instead of two, which is a big help. This works here because of homogeneity of the utility, production function, and capital accumulation.

- Does this "trick" work if $\ln Z$ is a general AR(1), i.e. if

$$\ln Z_{t+1} = \rho \ln Z_t + \mu + \sigma \varepsilon_{t+1}?$$

- Does it work if ε is not normal?

(3) Consider the policy functions $C(K, Z)$, $K'(K, Z)$ and $N(K, Z)$, and define $Y(K, Z) = F(K, Z \times N(K, Z))$. Use the result of (2) to relate the impact of a reduction in capital and the impact of an improvement in technology, i.e. find a relation between

$$\varepsilon_{C,K} = \frac{K}{C}\frac{\partial C}{\partial K} \text{ and } \varepsilon_{C,Z} = \frac{Z}{C}\frac{\partial C}{\partial Z},$$

and similarly for the other functions.

Problem 6 (Complete markets, some special cases.)

Consider the complete markets model we studied in class. The social planner chooses consumption processes $\{c_{i,t}(s^t)\}$ for $i = 1...N$, all $t \geq 0$, all histories $s^t \in S^t$, to maximize the weighted sum of utilities:

$$\sum_{i=1}^{N} \lambda_i E \sum_{t\geq 0} \beta^t u_i(c_{i,t}(s^t)) = \sum_{i=1}^{N} \lambda_i \sum_{t\geq 0} \sum_{s^t \in S^t} \beta^t \pi_t \left(s^t\right) u_i(c_{i,t}(s^t)),$$

subject to the resource constraints

$$\forall t, \forall s^t : \sum_{i=1}^{N} c_{i,t}(s^t) \leq \sum_{i=1}^{N} e_{i,t}(s^t) \stackrel{def}{=} e_t\left(s^t\right). \tag{2}$$

(1) Assume that

$$u^i(c) = -\frac{1}{\gamma_i} e^{-\gamma_i c},$$

i.e. each agent has CARA utility, but agents have different degrees of risk aversions. Find the optimal consumption rule: $c_{i,t}(s^t)$ as a function of the aggregate endowment $e_t(s^t)$, the Pareto weigths $\{\lambda_i\}$ and the coefficients of risk aversion.

(2) Now assume instead that $u^i(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for all agents (i.e. CRRA with same risk aversion). Find the consumption rule. How does the average consumption of an agent relates to his Pareto weight?

(3) Assume that the economy has a storage technology, so that the resource constraint is now instead

$$S_t = S_{t-1} + e_{t-1}(s^{t-1}) - \sum_{i=1}^{I} c_{i,t-1}(s^{t-1}),$$

where S_t is the amount stored at the end of $t-1$, which can be used for consumption or storage at time t. Write the social planner problem and prove the optimal risk-sharing rule. Should you condition on aggregate consumption or aggregate income when running the regressions of Cochrane, Mace and Townsend?

Problem 7 (Open economy risk sharing)

This problem presents an important puzzle in international economics and studies one possible resolution. Consider two economies $i = A, B$ which trade together.

(1) Suppose there is only one good, which is traded between the two countries, each of which has preferences

$$U = E \sum_{t \geq 0} \beta^t U(c_t^i),$$

with $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Each country has a random endowment $\{e_{i,t}\}$. Assume there is no trade cost. Write the competitive equilibrium (or social planner) problem with complete markets. What should be the correlations of the countries' consumption growth rates? Should it be higher or lower than the correlations of the endowments?

This is not the case in the data. So here are a couple of possible fixes to this puzzle of "lack of international risk-sharing".

(2) Suppose now that there are "iceberg" trade costs, i.e. a fraction τ of the exports shipped from one country to the other are lost in transportation costs. The resource constraint is now:

$$\text{If } c_t^A < e_t^A : c_t^A + c_t^B \leq e_t^A + e_t^B - \tau \left(e_t^A - c_t^A \right),$$

$$\text{If } c_t^A > e_t^A : c_t^A + c_t^B \leq e_t^A + e_t^B - \tau \left(e_t^B - c_t^B \right),$$

$$\text{If } c_t^A = e_t^A : c_t^A + c_t^B \leq e_t^A + e_t^B.$$

Characterize as much as possible the optimal allocation. How might this help resolve the puzzle?

(3) Suppose now that there are two goods: a tradeable good and a non-tradeable good. The two countries have the same preferences:

$$U = E \sum_{t \geq 0} \beta^t U(c_{i,t}^T, c_{i,t}^N),$$

where c_t^T is consumption of tradeables at time t and c_t^N is consumption of non-tradeables at time t. Economy A has an endowment of tradeable goods $\left\{ e_{t,A}^T \right\}$ and an endowment of non-tradeable goods $\left\{ e_{t,A}^N \right\}$, and similarly for economy B. Assume that you can trade *costlessly* the tradeable good, characterize as much as possible the optimal allocation. How might this help resolve the puzzle?

Problem 8 (Computing Equilibria.)

Consider an economy without uncertainty, with two agents. One has preferences

$$\sum_{t\geq 0} \beta_1^t \frac{c_t^{1-\gamma_1}}{1-\gamma_1},$$

and the other has preferences

$$\sum_{t\geq 0} \beta_2^t \frac{c_t^{1-\gamma_2}}{1-\gamma_2}.$$

Each agent $i = 1, 2$ receives an endowment $\{e_{i,t}\}_{t=0}^{\infty}$. Agents can trade in complete markets.

(1) Assume now that $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. Let the endowments be $e_{1,t} = 3$ for all $t \geq 0$ and $e_{2,t} = 1$ for all $t \geq 0$. Define and compute the competitive equilibrium.

(2) Still assuming that $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$, let the endowments now be $e_{1,t} = 3$ if t is odd, and 1 if t is even; and let $e_{2,t} = 1$ if t is odd, and 3 if t is even. Hence $e_{1,t} + e_{2,t} = 4$ for all $t \geq 0$. Compute the competitive equilibrium.

(3) Assume $\gamma_1 = \gamma_2$ (but $\beta_1 \neq \beta_2$). Define a competitive equilibrium. Prove that $c_{1,t}/c_{2,t}$ converges either to 0 or infinity.

Problem Set 4
Due Thursday, April 19

Problem 1 (Asset Pricing with Linear Preferences)
Consider a representative agent economy with linear utility:

$$U = E\sum_{t\geq 0} \beta^t c_t.$$

Consider an asset with dividends $\{d_t\}$.

(1) Assume that dividends follow the AR(1) process:

$$d_t = \rho d_{t-1} + (1-\rho)\overline{d} + \varepsilon_t,$$

with ε_t iid $N(0,\sigma^2)$ and $0 \leq \rho < 1$. Find the asset price P_t. Show that in this model, prices are less volatile than dividends. In the data, it seems to be the opposite. This is the basis of Robert Shiller early (1981) claim that 'stock prices are too volatile in the data relative to the theory'. Discuss briefly.

(2) Assume that dividends follow the AR(1) process *in growth rates*:

$$\Delta d_t = \rho \Delta d_{t-1} + \varepsilon_t,$$

with ε_t iid $N(0,\sigma^2)$ and $-1 < \rho < 1$. Find the asset price P_t. Under what conditions are prices are less volatile than dividends? Comment briefly.

(3) [Optional.] More generally, express the innovation to prices as a function of the innovation to dividends, i.e. $P_t - E_{t-1}P_t$ as a function of $d_t - E_{t-1}d_t$. To do this, you can assume that $d_t = A(L)\varepsilon_t$ where $A(L)$ is an arbitrary (possibly infinite) lag polynomial, i.e. $d_t = a_0\varepsilon_t + a_1\varepsilon_{t-1} + ...$ where ε_t is *iid* $N(0,1)$. (And, if you want, you can also make the same computation when $\Delta d_t = A(L)\varepsilon_t$.) Note: these computations will also show up when we go over the Permanent Income Hypothesis.

Problem 2 (Basic Asset Pricing.)
Consider a representative agent economy with CRRA utility:

$$U = E \sum_{t \geq 0} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}.$$

Consider an asset with dividends $\{d_t\}$.

(1) Assume there is no uncertainty, but consumption and dividends grow at some constant rates:

$$\begin{aligned} \frac{d_{t+1}}{d_t} &= 1 + g_d, \\ \frac{c_{t+1}}{c_t} &= 1 + g_c. \end{aligned}$$

Write a formula for the price-dividend ratio of the asset. (This is called the Gordon model of stock prices.) How does this ratio depend on g_d, g_c and β? Explain briefly.

(2) Assume $\gamma = 1$ (log utility) and $d_t = c_t$, show that the price-dividend ratio is constant, no matter what the consumption process is. For $\gamma \neq 1$ (general case) and $d_t = c_t$, write the price-dividend ratio as a function of future consumption. Suppose we learn that consumption in the future is going to be higher than expected. Does this raise or decrease the price of the asset today? Explain your result.

(3) Assume $d_t = c_t^{\phi}$ for some real number ϕ. Assume also that $\Delta \log c_{t+1}$ is iid distributed $N(\mu, \sigma^2)$. What is the stochastic process for d_t? [i.e., how is d_t distributed?] Compute the price-dividend ratio of the asset, and compute the equity premium as a function of $\mu, \sigma^2, \phi, \gamma$.

Problem 3 (Equity Premium and Disasters.)

Consider a representative agent economy with CRRA utility:

$$U = E \sum_{t \geq 0} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}.$$

Assume the consumption process is the following.

- Each period, with probability $1-\lambda$, there is no disaster, and consumption growth is

$$\Delta \ln c_{t+1} = \mu + \sigma \varepsilon_{t+1},$$

where ε_{t+1} is *iid* $N(0,1)$.

- Each period, with probability λ, there is a disaster, and

$$\Delta \ln c_{t+1} = \mu + \sigma \varepsilon_{t+1} - b,$$

where ε_t is iid $N(0,1)$ and $b > 0$ is the size of the disaster.

(1) Compute the risk-free rate in this economy.

(2) Compute the price of an asset that pays consumption forever. (Hint: you may guess and verify that the price-dividend ratio is constant.)

(3) Plot roughly a time series path of consumption and of the equity price in this economy.

(4) Using question 2, find the average stock return in this economy and compute the equity premium. Discuss briefly.

Problem 4 (Equity Premium and the Concentration of Aggregate Shocks.)

Consider a two-period endowment economy. There is a measure one of agents. Agents all have the same income y in the first period. In period 2, they receive a random endowment $\widetilde{y}$. This income process has the following property: $\widetilde{y}$ will be equal to y for a proportion $1-\lambda$ of agents; and $\widetilde{y} = \frac{C-(1-\lambda)y}{\lambda}$ for the remainder of agents, where C is random aggregate consumption. (This equation is simply the resource constraint at time 2.) Hence, a proportion λ of agents will bear all the adjustment in consumption, up or down. But the agents do not know in period 1 if they are going to receive y tomorrow or the random $\frac{C-(1-\lambda)y}{\lambda}$. They cannot insure their income $\widetilde{y}$ since no markets exist for this.

Assume aggregate consumption can take two values, $C = y\,(1+\delta)$ or $C = y\,(1-\delta)$, each with probability $\frac{1}{2}$.

(1) Compute the price of an asset (in zero-supply) which pays aggregate consumption C tomorrow.

(2) Deduce the expected return on this asset (‘equity’).

(3) Compute the risk-free rate. How does it vary with λ? Deduce the equity premium.

(4) How does the equity premium vary with λ?

Problem 5 (Incomplete Markets)

Let's consider the example of incomplete market I gave in class. There are two agents, and 4 states. Each state has a probability 1/4 of occuring.

The endowments of the agents work as follows:

State:	1	2	3	4
Endowment of agent 1:	$y_l - \lambda$	y_l	$y_h - \lambda$	y_h
Endowment of agent 2:	y_l	$y_l - \lambda$	y_h	$y_h - \lambda$

Assume each agent has expected log utility: $U(c_1, c_2, c_3, c_4) = \pi_1 \log(c_1) + \pi_2 \log(c_2) + \pi_3 \log(c_3) + \pi_4 \log(c_4)$.

(1) Assume there are 4 assets available, given by the following matrix D :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Compute the competitive equilibrium, i.e. the consumption allocation and asset prices.

(2) Assume there are 4 assets available, given by the following matrix D :

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Compute the competitive equilibrium.

(3) Assume there are two assets available, given by the following matrix D :

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Compute the competitive equilibrium.

(4) Assume there are two assets available, given by the following matrix D :

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Compute the competitive equilibrium.

Problem 6: Term Structure

Consider a representative consumer with expected CRRA utility. Assume that $\Delta \log C_t = \rho \Delta \log C_{t-1} + \sigma \varepsilon_t$ with ε_t *iid* $N(0,1)$. We are interested in the price of a one-period bond and the price of a two-period bond in this economy. A two-period bond is a bond which pays out one good in two periods (and zero in the intermediate period). The term structure is a plot of maturity (x-axis) and expected return (y-axis), where maturity = in how many periods does the bond pay out one unit of good.[1]

(1) Show that when $\rho = 0$, the risk free rate is constant over time, and the term structure is flat (bonds of all maturities have the same expected return). Are two-period bonds risky in this economy?

(2) When $\rho \neq 0$, compute the price of a two-period bond and the price of a one-period bond and compute the implied expected returns. Discuss how the yield curve depends on the parameters of our model. Under what parameters can you obtain an upward-sloping yield curve? Explain.

[1] Technically, the yield curve is a plot of maturity against yields, but this is not very important for us.

Problem 7: Firm Cash Flows, Betas and Returns

Consider a representative consumer with expected CRRA utility. Assume $\Delta \log C_t = \mu_c + \sigma_c \varepsilon_t$ where ε_t is *iid* $N(0, \sigma^2)$. Consider N assets, $i = 1...N$ which pay respectively dividends D_{it} where each D_{it} follows a different process:

$$\Delta \log D_{it+1} = \mu_i + \lambda_i \varepsilon_{t+1} + \sigma_i \eta_{i.t+1},$$

with η_{it+1} *iid* $N(0, 1)$ and $\sigma_i, \lambda_i, \mu_i$ characterize this process. Assume that the process $\{\eta_{it+1}\}$ is independent of the proces $\{\varepsilon_{t+1}\}$.

(1) Compute the price-dividend ratio and expected return on each asset. Explain intuitively how it depends on μ_i, λ_i and σ_i.

(2) Compute the "consumption β" for each asset too.

Problem 8: Quizz: True / False / Uncertain (Explain briefly)

(1) To test for full risk sharing, I need to test if consumption is independent of income.

(2) If agents have utility over leisure and not only consumption, then even with full risk sharing, consumption will depend on whether agents work today or not.

(3) Labor economists often measure the "college premium" as the difference between the mean income of workers with college education and workers with a high-school degree (holding other characteristics fixed). This does not take risk into account. In which direction does this bias their results?

(4) Suppose that the equity premium is 6%, the beta of the stock market return on consumption is 200. What should be the expected return on a stock with a consumption beta of 150 if the risk-free rate is 2%?

(5) The equity premium puzzle is equivalent to saying that stock prices are very volatile.

(6) When markets are incomplete, the equilibrium is not efficient.

Problem Set 5
Due Thursday, April 26

Problem 1 (Heterogeneity in Risk Aversion and Portfolio Choices)

Consider an economy with two agents, which lasts two periods (periods 1 and 2). In the first period, agents trade together; in the second period, the uncertainty is resolved and people consume.

Agent A is risk-neutral, he has wealth w_A in period 1, and endowment $y_{A,s}$ in period 2 if state s is realized. Agent B is risk-averse, has wealth w_B in period 1, and endowment $y_{B,s}$ in period 2 if state s is realized. Denote agent B's utility by $u(.)$.

The two assets that they trade are:

- asset 1,a risk-free asset, which pays out one unit of consumption in period 2 whatever state is realized; there is a supply x_1 of asset 1 in period 1.

- asset 2, a risky asset, which pays out a dividend d_s in state s in period 2. (Denote the mean $Ed_s = d$.) There is a supply x_2 of asset 2 in period 1.

(Note: if you want to make this fully a general equilibrium model, you have to state who own the assets at the beginning. To simplify, let us assume that these assets are supplied inelastically in period 1 by some outside agent (e.g. a foreign economy)).

(0) Define a competitive equilibrium for this economy.

(1) Assume that agents are allowed to short assets, i.e. hold negative quantities of assets. Describe as much as you can the equilibrium: what is the average (i.e. expected) return on each asset? What portfolios do agents hold? (i.e. how much of their wealth is invested in each asset?).

(2) Now assume that agents have to hold weakly positive quantities of each asset. Again, describe as much as you can the equilibrium. Do the equilibrium returns on each asset depend on the wealth distribution? Explain.

(3) [Optional] Discuss how making this mode a GE model, as discussed above, may modify our analysis. (Hint: is there a potential for multiple equilibria?)

Problem 2 (Welfare cost of business cycles with unit root.)

Lucas (1987) assumes that log consumption follows a deterministic trend:

$$\log c_t = \mu t - \frac{\sigma^2}{2} + z_t,$$

with z_t normal $(0, \sigma^2)$. However, starting with Beveridge and Nelson (1982), it has been recognized that most macroeconomic series, including consumption, are better described as random walks, so let us assume that:

$$\log c_t = \log c_{t-1} + \mu - \frac{\sigma^2}{2} + \sigma \varepsilon_t,$$

with ε_t *iid* $N(0,1)$. Assume c_0 is known. The $\frac{\sigma^2}{2}$ correction ensures that $E\left(\frac{c_t}{c_{t-1}}\right) = e^{\mu}$ and thus $E(c_t) = c_0 e^{\mu t}$. How does this alternative assumption affect Lucas' calculation? Give a new estimate of the welfare cost of "business cycles" and discuss it briefly.

Problem 3 (Welfare cost of consumption volatility when markets are incomplete.)

This problem considers the idea, sketched in the class notes, that a few agents may suffer a lot during a recession. Suppose that the economy is populated by a measure one of agents with CRRA preferences:

$$U = E \sum_{t \geq 0} \beta^t \frac{c_{it}^{1-\gamma}}{1-\gamma}.$$

Agent i's income follows the process:

$$\log y_{it} = \mu_i t - \frac{\sigma_i^2}{2} - \frac{\eta_i^2}{2} + \sigma_i \varepsilon_{it} + \eta_i z_t,$$

where μ_i, σ_i, η_i are individual-specific parameters, ε_{it} is an idiosyncratic (i.e. individual-specific) shock which is *iid* across i (agents) and distributed $N(0,1)$, and z_t is the aggregate shock which is distributed $N(0,1)$. Agents which have a high σ_i have large idiosyncratic shokcs, while agents which have a high η_i have a high exposure to the business cycle.

Let's assume that the incompleteness is extreme: each agent must consume his income: $c_{it} = y_{it}$ (i.e. "autarky").

(1) What is the welfare cost of consumption volatility, i.e. how much is each consumer willing to pay to get rid of the risk in ε_{it} or in z_t? On average, η_i must be (roughly) equal to the standard deviation of aggregate consumption around its trend (why?). Hence an empirical order of magnitude is that $\sigma_i = 10\eta_i$. Comment on your results.

(2) An alternative model is that some people get ex-post zero exposure to the aggregate shock, and some people get a large exposure ex-post (e.g., unemployment). This would lead us to the following process:

$$\begin{aligned} \log y_{it} &= \mu_i t - \frac{\sigma_i^2}{2} + \sigma_i \varepsilon_{it}, \text{ with probability } 1-\pi, \\ &= \mu_i t - \frac{\sigma_i^2}{2} - \frac{\eta^2}{2} + \sigma_i \varepsilon_{it} + \eta z_t, \text{ with probability } \pi. \end{aligned}$$

i.e. with a probability π, you take on a big shock related to z_t. So ex-post π people share the aggregate risk z, and $1-\pi$ are not affected. Assume the event 'being affected by z_t' is independent of ε_{it} and of z_t. Compute the welfare cost of consumption volatility and discuss.

Problem 4 (Time Aggregation and the PIH.)

In practice the data that we have measures consumption *flows* over certain periods (e.g. a month or a quarter). This question explores what implications of PIH remain if we use such time-averaged data.

Assume that consumption follows the PIH:

$$c_t = E_t c_{t+1},$$

which it is convenient in this case to rewrite as $c_{t+1} = c_t + e_{t+1}$ with $E_t e_{t+1} = 0$.

(1) Assume we observe the following measure:

$$g_k = \frac{c_{2k} + c_{2k+1}}{2}.$$

(i.e. we observe consumption monthly flow, but the decision period of the individual is 2 weeks.) Compute the covariance of $g_{k+1} - g_k$ with $g_k - g_{k-1}$. Does g_k satisfies the PIH property, i.e. is measured consumption g_k a random walk?

(2) Compute the covariance of $g_{k+1} - g_k$ with $g_{k-1} - g_{k-2}$, and use this to suggest a test of the PIH with time-averaged data.

(3) Suppose instead that measured consumption is the last two weeks, i.e. $g_k = c_{2k+1}$. Does measured consumption g_k satisfy the PIH?

Problem 5 (Some examples of PIH.)

Consider a PIH-consumer:

$$\begin{aligned} c_t &= E_t c_{t+1}, \\ b_{t+1} &= R b_t + y_t - c_t, \\ & b_0 \text{ given.} \end{aligned}$$

$b_t =$asset holdings at the end of period $t-1$.

$R = 1 + r$ is the gross risk-free interest rate.

We will consider the following processes for $\{y_t\}$. In each case ε_t is an iid process with mean 0 and variance σ^2, and ρ and θ lie between 0 and 1 (strictly).

(1) $y_t = \overline{y} + \varepsilon_t$,

(2) $y_t = \rho y_{t-1} + (1-\rho)\overline{y} + \varepsilon_t$,

(3) $\Delta y_t = \rho \Delta y_{t-1} + \varepsilon_t$.

Questions:

(a) Compute for each case (1) to (3) the *impulse response* of consumption to an innovation to income, i.e. what happens to c_t and c_{t+k} for $k = 1, 2, \ldots$ if $\varepsilon_t = 1$ and $\varepsilon_{t+k} = 0$, for all $k \geq 1$. Plot (roughly) the responses of income, consumption and assets to an income shock, i.e. plot y_{t+k}, b_{t+k} and c_{t+k} if $\varepsilon_t = 1$ and $\varepsilon_{t+k} = 0$ for $k \geq 1$.

(b) Compute for each case (1) to (3) the ratio of standard deviations: $\sigma(c)/\sigma(y)$. How does ρ change this ratio?

Problem 6 Quizz: True/False/Uncertain - explain.

(1) The high correlation between consumption and current income disproves the PIH.

(2) If people have habits (i.e. they care about their last period's consumption as well as the current one), then the PIH will not hold even if $\beta R = 1$.

(3) Suppose people have families and they can share risk together; and each family can save or borrow a risk-free asset. The the PIH would not hold at the family level.

(4) [Same setup as 3]. The PIH would not hold at the individual level.

(5) Under the PIH, the amount of uncertainty that the consumer faces does not affect his savings choice.

(6) Suppose that I receive today a 1$ windfall (which won't happen in the future again). How much of this would I consume if markets were complete? If the PIH was true?

Problem Set 6, At Last
Due Thursday, May 3

Problem 1 (Numerical Solution of a Savings Problem)

This problem asks you to solve on the computer a model with uncertain income and a borrowing constraint. *I will go over the solution of this problem in class on Thursday, May 3.*

Individual income follows a Markov chain. Assume income can take one of two values: $y_t = 0.1$ (unemployed) or $y_t = 1$ (employed). The transition matrix is

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

with $p = \Pr(\text{unemployed at } t+1 | \text{unemployed at } t) = 0.5$ and $q = \Pr(\text{employed at } t+1 | \text{employed at } t) = 0.8$.

Preferences are CRRA:

$$E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

with $\gamma = 2$ and $\beta = 0.96$. The asset accumulation equation is:

$$a_{t+1} = (1+r)a_t + y_t - c_t,$$

with $r = 0.02$.

(1) What is the 'natural borrowing constraint' in this economy?

(2) Write the Bellman equation for a given borrowing constraint $\overline{a}$.

(3) Solve the Bellman equation on the computer using value iteration when $\overline{a} = 0$. To do this, follow these steps: (note: this is very similar to what you did in PS1 for a case without uncertainty)

(a) define a grid for assets: $A = \{a_1 = \overline{a}, a_2, ..., a_N\}$ where $N = \#$ of points in the grid, and the upper bound a_N is set to 10. (You can set $N = 200$ or so.) [The upper bound is arbitrary here, but if it is big enough it should not affect the results.]

(b) Define a guess for the value $V(a, y)$ for any a in the grid and any $y \in \{y_l, y_h\}$. So V will be a $N \times 2$ matrix.

(c) Use the Bellman equation to update your guess: for any (a, y), compute

$$\max_{a' \in A} \{u(c) + \beta E_{y'} V(a', y')\},$$

ie given the budget constraint and the markov assumption for y :

$$\max_{a' \in A} \left\{ u((1+r)a + y - a') + \beta \sum_{y'} P(y, y') V(a', y') \right\}.$$

To compute this max, you have to compute the value of $u((1+r)a + y - a') + \beta \sum_{y'} P(y, y')V(a', y')$ for any a', and pick the a' which gives you the greater value. This max value, call it $V^{new}(a, y)$, is our update for $V(a, y)$. (So you need to perform this maximization $2N$ times, for each possible value of (a, y)).

(d) If the new guess $V^{new}(a, y)$ is close to $V(a, y)$ for all values of a and y, i.e.

$$\max_{a,y} |V^{new}(a, y) - V(a, y)| < 0.0001,$$

then stop the iteration. Otherwise go back to c with your new guess V^{new}.

(e) Plot the value functions.

(f) Compute the policy function $a'(a, y)$ and $c(a, y)$. (That is, find the $a'(a, y)$ which solves the max in the step (c)). Plot these functions. Under the PIH consumption depends linearly on assets. How is the result different in this case? Does the borrowing constraint bind?

(g) [optional but recommended]

- Compare your solution when $p = .5$ and $q = .8$ to the solution when $p = .2$ and $q = .8$ i.e. the process is *iid*.

- Compare your solutoin when $\overline{a} = 0$, when $\overline{a} =$ the natural borrowing constraint, and when $\overline{a} = 2$.

- Intuitively, what do you think is the effect of changing β, r or γ on the solution? (Of course you can try!...)

(h) [optional] simulate the savings and consumption choice of the individual for 20 periods, i.e. start with some asset level a_0, and draw income randomly from the markov chain. Then plot the path for consumption, income and assets that is generated by these random draws. Is it true that consumption is smoother than income?

Problem 2 (Savings problem in equilibrium)

Consider the savings problem in equilibrium of Handout 8. Each agent has an income which follows a Markov chain with possible values $y_1 < y_2 < ... < y_N$ and transition matrix P. Assume that the minimum possible income y_1 is 0, so that the natural borrowing constraint is zero:

$$\begin{aligned} v(a,y) &= \max_{a'\geq 0}\left\{u(c)+\beta\sum_{y'=1}^{N}P(y,y')v(a',y')\right\} \\ a' &= (1+r)a+y-c. \end{aligned}$$

There is no outside supply of bond. Compute the competitive equilibrium, i.e. find the allocation and interest rate.

Problem 3 (Consumption-Savings and portfolio choice)

Consider the following portfolio/savings problem: the consumer chooses how much to save, and how to allocate these savings b/w a risky and a risk-free asset:

$$\begin{aligned} &\max_{\{c_t,\alpha_t\}} E\sum_{t=0}^{\infty}\beta^t u(c_t) \\ s.t. \quad : \quad & W_{t+1} = \left(\alpha_t R_{t+1} + (1-\alpha_t)R^f\right)(W_t - c_t), \end{aligned}$$

with $\log R_{t+1}$ is iid normal (μ,σ^2) and R^f is constant. Assume $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

(1) Write the Bellman equation.

(2) Write the FOCs and Envelope condition.

(3) Find the value function, up to a multiplicative constant, by guess and verify.

(4) Find the consumption rule (up to a multiplicative constant) and characterize as much as possible the portfolio savings rule.

Problem 4 (PIH)

Consider a PIH consumer who has the following income process:

$$\begin{aligned} y_t &= y_t^P + u_t, \\ y_t^P &= y_{t-1}^P + v_t, \end{aligned}$$

with (u_t, v_t) iid and independent of each other, $Eu_t = Ev_t = 0$ and $Var(u_t) = \sigma_u^2$ and $Var(v_t) = \sigma_v^2$.

(1) What is the response of consumption to an increase of u today? What is the response of consumption to an increase in v today?

(2) Compute the standard deviation of consumption as a function of the standard deviation of the shocks. Also compute the standard deviation of income as a function of the standard deviation of the shocks.

(3) Suppose you run the regression in cross-section:

$$\Delta c_i = \beta_0 + \beta_1 \Delta y_i + \varepsilon_i.$$

What would you estimate for β_1 and β_0? (You can assume for this question that r is close to zero.)

(4) As you saw in problem, the standard deviation of consumption is ex-post the cross-sectional standard-deviation of consumption. Over the last 25 years income inequality has risen quite a bit in the US (e.g. the cross-sectional standard deviation of income has risen by about 25%), but consumption inequality seem to have risen far less (less than 10%). What would you conclude from this model?

Problem 5 (PIH and the cross-sectional distribution of income and consumption.)

Consider an economy with a large number N consumers which are identical at time 0. The consumers satisfy the PIH: they have the utility

$$U_i = -\frac{1}{2} E \sum_{t=0}^{\infty} \beta^t \left(c^* - c_{i,t}\right)^2,$$

$$\begin{aligned} \beta R &= 1 \\ b_{i,t+1} &= R b_{i,t} + y_{i,t} - c_{i,t}. \end{aligned}$$

The income process is the same for everyone and is iid: $y_{i,t} = \overline{y} + \varepsilon_{i,t}$, with $\varepsilon_{i,t}$ uncorrelated across individuals and across time, mean 0, and variance σ^2. Assume all agents start with the same $b_{i,0} = 0$ and the same $y_{i,0} = \overline{y}$.

(1) Find the consumption of each agent at time 0.

(2) What is the cross-sectional variance of consumption and of income at time 0? (Hint: the cross-sectional variance at time t is $\frac{1}{N}\sum_{i=1}^{N}\left(c_{i,t} - \overline{c}_t\right)^2$ where $\overline{c}_t = \frac{1}{N}\sum_{i=1}^{N} c_{i,t}$ is the mean of consumption at time t.)

(3) At the beginning of the period $t = 1$, the shock $\varepsilon_{i,1}$ is realized for each agent i. Find the consumption of each agent as a function of his shock. What is the cross-sectional variance of consumption at time $t = 1$? What is the cross-sectional variance of income at $t = 1$?

(4) Answer the same questions as in (3) for $t = 2$. Can you prove (or state) something general about how the cross-sectional variance of consumption and income evolve over time for a given cohort of identical individuals? Comment briefly.

Problem 6 Quizz: True/False/Uncertain - explain.

(1) Countries with higher growth should have higher interest rates.

(2) Consider a consumer who has access to a risk-free asset, faces no borrowing constraint and has a utility function with $u''' > 0$. Will this consumer have smoother consumption than a PIH consumer (i.e. one with quadratic utility)?

(3) If a person is more 'patient', he will be willing to pay more to get rid of consumption volatility.

(4) There has been an increase in individual uncertainty in the US. This should make the interest rate higher.

(5) There is also empirical evidence that macroeconomic volatility has declined in the past 20 years. This would make the interest rate higher too.

(6) Under what conditions is the short-term (i.e. one year) risk-free real interest rate procyclical?

Econ. 704, Spring 2007 **François Gourio**

Final Exam

Answer all the questions on your blue book.

This exam is 2 hours long and will be graded on a 120-point scale.

1pt = 1mn

Short Questions: True/False/Uncertain - please explain your answers, if possible with one or two equations (6 pts for each, 60pts total).

(1) An asset which return is more volatile will have a higher mean (ie expected) return.

(2) An increase in the growth rate of its dividends increases the price of an asset.

(3) An increase in the growth rate of the economy increases asset prices.

(4) Under the permanent income model (quadratic utility with $\beta R = 1$ and no borrowing constraints), what is the effect on consumption and assets of a permanent increase in income starting today? Please plot the paths of income, consumption and assets.

(5) Under the permanent income model (quadratic utility with $\beta R = 1$ and no borrowing constraints), what is the effect on consumption and assets of an anticipated, permanent increase in income starting next year? Please plot the paths of income, consumption and assets.

(6) What is the equity premium puzzle? Explain in two sentences.

(7) The 'excess sensitivity' puzzle is that in a regression of consumption on lagged consumption and current income, $C_t = \alpha + \beta C_{t-1} + \gamma Y_t + \varepsilon_t$, we find $\gamma \neq 0$.

(8) Consider the choice between going to college and not. There are two deterministic income flows, one that you get if you go to college $\{Y_{c,t}\}$ and the other that you get if you do not go to high-school: $\{Y_{h,t}\}$. What implicit assumption(s) do labor economists make when they assume that the decision is made by comparing

$$\sum_{t=0}^{\infty} \frac{Y_{c,t}}{(1+r)^t} \text{ and } \sum_{t=0}^{\infty} \frac{Y_{h,t}}{(1+r)^t}?$$

What if instead they assume that the decision is made by comparing

$$\sum_{t=0}^{\infty} \frac{\log Y_{c,t}}{(1+r)^t} \text{ and } \sum_{t=0}^{\infty} \frac{\log Y_{h,t}}{(1+r)^t}?$$

(9) Many people think that borrowing constraints are now less tight than they used to be 20 years ago in the US. What is the effect of a relaxation of borrowing constraints on the interest rate and on the stock of capital?

(10) Harder: assume that there is a measure one of agents who have CARA utility and save or borrow in a risk-free asset (no borrowing constraints). The individual productivity is y and evolves according to an AR(1): $y_{t+1} = \overline{y}(1-\rho) + y_t\rho + \sigma\varepsilon_{t+1}$, where ε_{t+1} is *iid* and $N(0,1)$. There is an aggregate production function $Y = F(K,N)$,, so that the wage is $w = F_2(K,N)$ and the interest rate is $r = F_1(K,N) - \delta$ where K = total capital and N = total labor. We know from handout 7 that the consumption rule is:

$$c(a,y) = ra + \frac{r}{r+1-\rho}w(y-\overline{y}) + w\overline{y} - k_0\log(\beta R) - k_1\sigma^2,$$

where $k_0 > 0$ and $k_1 > 0$ depend on the parameters (this is not exactly true, they depend on r too, but this simplifies a bit). Consider thus an economy with many agents like this one, and assume that the income of any agent is independent of the income of any other agent. Consider a 'stationary equilibrium' without aggregate shocks but only micro-level shocks, where the economy has a constant stock of capital K, investment is constant equal to δK (and there is no government debt). Write the market-clearing condition for capital. What is the effect of an increase in risk σ on the stock of capital and the interest rate?

Extra Space for Short Questions

Problem 1 (Consumption/Savings choice: 30 pts)

Consider the following economy: there is a representative consumer with log utility:

$$E \sum_{t=0}^{\infty} \beta^t \log(C_t),$$

and there is a representative firm with an AK technology: $Y_t = A_t K_t$. The technology A_t is stochastic, and it follows the following process:

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1},$$

where ε_{t+1} is an iid random variable with mean 0 and $-1 < \rho < 1$. The resource constraint is $K_{t+1} = A_t K_t - C_t$. There is an initial stock of capital K_0 and an initial technology level A_0.

(1) Write the social planner problem for this economy as a sequence problem.

(2) Write the corresponding Bellman equation.

(3) Is the social planner problem equivalent to a competitive equilibrium for this economy?

(4) Find the value function of this problem by guess-and-verify. (**Hint:** the guess is $V(K, A) = a_0 + a_1 \log K + a_2 \log A$, and you do not need to determine the exact value of a_0.)

(5) Give the time series relation between $\log K_{t+1}$ (resp. $\log Y_{t+1}$) and $\log A_t, \log K_t$ (resp. $\log A_t, \log K_t$ and $\log A_{t+1}$).

(6) What is the average growth rate of capital (i.e. $E\Delta \log K_{t+1}$) and of output (i.e. $E(\Delta \log Y_{t+1})$)? Explain your result.

(7) What is the variance of the growth rate of capital (i.e. $Var(\Delta \log K_{t+1})$) and of output ($Var(\Delta \log Y_{t+1})$)? Explain your result.

(8) Under what conditions is capital growth $\Delta \log K_t$ positively serially correlated in this model? (i.e. $Corr(\Delta \log K_{t+1}, \Delta \log K_t) > 0$)? Under what conditions is output growth $\Delta \log Y_t$ positively serially correlated in this model? (i.e. $Corr(\Delta \log Y_{t+1}, \Delta \log Y_t) > 0$)?

Extra Space for Problem 1

Problem 2 (Asset Pricing when Leisure enters the Utility Function: 30pts)

When we discussed asset pricing in class, we assumed usually that there was representative consumer with utility

$$E\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma}.$$

Suppose instead that the agent has utility over consumption c_t and leisure l_t:

$$E\sum_{t=0}^{\infty}\beta^t u(c_t,l_t).$$

(1) Write the Euler equation for any asset i, i.e. for any return $R_{i,t+1}$, assuming an interior solution.

(2) Rewrite the Euler equation using the following first-order Taylor approximation around (c_t, l_t) of $\log u_c$:

$$\log u_c(c_{t+1}, l_{t+1}) \simeq \log u_c(c_t, l_t) - \eta_1 \Delta \log c_{t+1} - \eta_2 \Delta \log l_{t+1}$$

with

$$\eta_1 = -\frac{u_{cc}(c_t, l_t)c_t}{u_c(c_t, l_t)} \text{ and } \eta_2 = -\frac{u_{cl}(c_t, l_t)l_t}{u_c(c_t, l_t)}.$$

and we assume that η_1 and η_2 are constant.

Hint: write $u_c(c_{t+1}, l_{t+1}) = \exp(\log u_c(c_{t+1}, l_{t+1}))$ and $R_{i,t+1} = \exp(\log R_{i,t+1})$.

(3) Assume that the log return on asset i, consumption growth $\Delta \log c_{t+1}$, and leisure growth $\Delta \log l_{t+1}$ are jointly normally distributed.[2] Derive an equation for the risk-free rate of return $R_{f,t+1}$, given the variance of consumption and leisure growth, their covariance, and the preference parameters. Comment briefly.

[2]If you wish, you can assume that they are iid over time; this will mean that the conditional expectations are equal to unconditional expectations, but it will not simplify the computations.

(4) Find an equation for the (log) risk premium on any asset i, i.e. $E_t \log\left(\frac{R_{it+1}}{R_{f,t+1}}\right)$. Interpret this equation; explain in particular the role of η_2.

(5) In which special case(s) is this model equivalent to the consumption CAPM that we studied in class?

(6) An empirical puzzle is, roughly, to explain why stocks that do poorly in recessions have high expected returns. What condition on η_2 would help you explain this puzzle? Is this condition "reasonable"?