Econ 745 **François Gourio**

Problem Set 1 - Due Tuesday, Nov 2

Question 1: Asset pricing basics (the "standard model")

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = E\sum_{t\geq 0}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows the process

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_c^2}{2} + \sigma_c \varepsilon_{t+1},$$

where ε_{t+1} is *iid* $N(0,1)$.

Consider N assets, $i = 1...N$ which pay respectively dividends D_{it} where each D_{it} follows a different process:

$$\Delta \log D_{it+1} = \mu_i - \frac{\lambda_i^2}{2} - \frac{\chi_i^2}{2} + \lambda_i \varepsilon_{t+1} + \chi_i \eta_{i.t+1},$$

with η_{it+1} *iid* $N(0,1)$ and χ_i, λ_i, μ_i characterize this process. $\eta_{i,t+1}$ is uncorrelated with ε_{t+1} at all leads and lags, i.e.

$$E(\varepsilon_{t+1-k}\eta_{i,t+1}) = 0 \text{ for all } k \geq 0, \text{ and all } k \leq 0$$

Note: you will need the log-normal formula, i.e. if X is $N(\mu,\sigma^2)$ then $E\left(\exp(X)\right) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$.

(a) Compute the mean of $\frac{C_{t+1}}{C_t}$ and $\frac{D_{it+1}}{D_{it}}$. (So you can see why I added the terms $-\frac{\sigma_c^2}{2}$, $-\frac{\lambda_i^2}{2}$, etc.).

(b) Compute the risk-free rate.

(c) Compute the price-dividend ratio on asset i. Explain intuitively how it depends on μ_i, λ_i and χ_i.

(d) Compute the expected return and expected excess return (i.e. return less the risk-free rate) on asset i. Explain intuitively how it depends on μ_i, λ_i and χ_i. Discuss the statement "idiosyncratic risk is not priced."

(e) Is it true that more volatile assets ("more risky assets") have higher average returns?

(f) Plot (roughly) the effect of a shock ε_{t+1} on consumption, dividends (both for a low λ and a high λ asset), returns, and the price-dividend ratio. (This is akin to an "impulse response function")

(g) Define the asset i's "consumption beta" $\beta_{i,c}$ as the slope of the time-series regression of the asset return on consumption growth:

$$R_{t+1}^i = \alpha_i + \beta_{i,c}\Delta \log C_{t+1} + \nu_{i,t+1}.$$

Compute $\beta_{i,c}$. What is the cross-sectional (i.e., across i) relation between $\beta_{i,c}$ and expected returns? [Hint: you can use the following approximation: if (U,V) is jointly normal, and g is a smooth function, then: $Cov(g(U),V) = E(g'(U)) \times Cov(U,V)$. (this is known as Stein's lemma).]

(h) In the US, starting around 1985, the volatility of consumption growth fell (the "Great Moderation"). What would a standard asset pricing model predict for the risk-free rate and the equity premium, if people realize immediately in 1985 the decrease in volatility?

(i) True, false or uncertain: according to theory, countries with more volatile consumption and dividends should have more volatile stock prices.

Question 2: Endowment economy with Markov chain

This exercise asks you to solve a standard endowment economy asset pricing model numerically. This is the Mehra-Prescott (1985, JME) setup, but you don't need to read the paper. (The computations are fairly simple so you could almost do them by hand or with Excel, but it will be useful for you to learn to use Matlab or a similar software to solve this.)

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = E\sum_{t\geq 0}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows a Markov chain with two states:

$$\Delta \log C_t = \mu_h \text{ or } \mu_l,$$

and

$$\begin{aligned}
\Pr(\Delta \log C_t = \mu_h | \Delta \log C_{t-1} = \mu_h) &= p, \\
\Pr(\Delta \log C_t = \mu_l | \Delta \log C_{t-1} = \mu_h) &= 1-p, \\
\Pr(\Delta \log C_t = \mu_h | \Delta \log C_{t-1} = \mu_l) &= 1-q, \\
\Pr(\Delta \log C_t = \mu_l | \Delta \log C_{t-1} = \mu_l) &= q.
\end{aligned}$$

We assume the "stock" pays a dividend equal to consumption, forever. We will write this setup for a general Markov chain with S states, here $S = 2$. Hence, there is a transition matrix $Q(s, s') = \Pr(\Delta \log C_t = \mu(s') | \Delta \log C_{t-1} = \mu(s))$ and a vector $\mu = [\mu_h, \mu_l]$.

See my notes on Markov chains if you are lost, http://people.bu.edu/fgourio/markov.pdf

(a) Write the equation which determines the risk-free rate. Note that it will depend on the current state s, call it $R^f(s)$.

(b) Write the equation which determines the price–dividend ratio. (Note: this is a recursion.) Call it $q(s)$.

(c) Write the equation which determines the expected return on the stock: $E_t R^e_{t+1} = ER(s)$.

(d) Consider the following parameters:

$$\begin{aligned}
S &= 2, \mu_h = 1.02, \mu_l = .98, \\
p &= q = .95, \beta = .98, \gamma = 2.
\end{aligned}$$

Compute the expected return and the expected excess return conditional on each state.

(e) Compute the unconditional expected return and excess return on the stock. To do so, you need to use the invariant (or ergodic) distribution to compute the probability that each state is reached. This measure is a $S \times 1$ vector whose components sum to 1, call it υ, such that $\upsilon = \upsilon Q'$. (See the Markov notes if you are unfamiliar with that.)

(f) Produce a plot, as a function of γ, of the unconditional excess return on the stock (i.e. the equity premium).

(g) Calibration: for a two-state Markov chain with transition

$$\begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

and means (μ_h, μ_l), with $\mu_h = \mu + \delta$ and $\mu_l = \mu - \delta$, compute the mean, variance, and serial correlaton of $\Delta \log C_t$. We can measure these statistics in U.S. data and then find p, q, δ, to replicate these statistics, so that our implied time series process for $\Delta \log C_t$ is similar to the data. In US data, typical numbers would be for annual consumption growth: $E(\Delta \log C_t) = 2\%$, $var(\Delta \log C_t) = 0.0004$, and $\rho(\Delta \log C_t) = -.1$. What are the implied δ, p, q?

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Problem Set 2 - Due Tuesday, Nov 16 by 9pm in my mailbox

Question 1: Time-varying volatility, Time-varying growth rates, and Asset Prices

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = E\sum_{t\geq 0}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows the process:

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_t^2}{2} + \sigma_t \varepsilon_{t+1},$$

where ε_{t+1} is *iid* $N(0,1)$, and σ_{t+1} follows a Markov chain with transition probabilities $Q(\sigma_{t+1}|\sigma_t)$. We assume that the two processes $\{\varepsilon_{t+1}\}$ and $\{\sigma_{t+1}\}$ are independent. Note that σ_t is drawn at time t, and ε_{t+1} is drawn at time $t+1$.(Note: This problem does not require a lot of computations - write the simple version of the equations, and compute only when necessary.)

(1) Compute the risk-free rate, as a function of σ_t. What is the effect of σ_t on the risk-free rate? Explain briefly the intuition.

(2) Consider an asset with dividends equal to consumption. Give an equation which determines the price-dividend ratio, as a function of σ_t. (You do not need to solve for this equation.)

(3) Using (2), solve for the expected equity return $E_t\left(R^e_{t+1}\right)$, as a function of σ_t, and for the log equity premium $\log\frac{E_t R^e_{t+1}}{R^f_{t+1}}$. What is the effect of σ_t on each of these two objects? Explain briefly the intuition.

(4) Assuming for this question only that σ_t is *iid*, what is the effect of σ_t on the price-dividend ratio? Explain briefly the intuition. [Note: this result can in fact be generalized to non-iid σ_t, as long as σ_t is positively serially correlated.]

(5) (a) Does the model generate the time-series predictability evidence, i.e. if you run the regression

$$R^e_{t+1} - R^f_{t+1} = \alpha + \beta\frac{D_t}{P_t} + \varepsilon_{t+1}$$

in the model, do you get a positive sign? [You do not need a formal proof, and can consider the case σ_t is iid only.]

(b) In the data, the regression $R^e_{t+1} = \alpha + \beta\frac{D_t}{P_t} + \varepsilon_{t+1}$ gives the same positive sign as the regression above. Is it the case in this model?

(6) Answer questions (1)-(5) again, but now assume that the process of consumption is

$$\Delta \log C_{t+1} = \boldsymbol{\mu}_t - \frac{\sigma^2}{2} + \sigma\varepsilon_{t+1},$$

i.e. instead of a time-varying risk we have a time-varying growth rate of consumption: μ_t follows a Markov chain with transition $Q(\mu_{t+1}|\mu_t)$.

(7) Answer questions (1)-(5) again, but now assume that the process of consumption is

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_c^2}{2} + \sigma_c\varepsilon_{t+1},$$

while dividends are not equal to consumption anymore, and satisfy

$$\Delta \log D_{t+1} = \mu_d - \frac{\lambda_t^2}{2} + \lambda_t \varepsilon_{t+1},$$

so that there is a time-varying risk of dividends, but not consumption.

(8) Same as (7), but now dividends satisfy

$$\Delta \log D_{t+1} = \boldsymbol{\mu}_t - \frac{\sigma_d^2}{2} + \sigma_d \varepsilon_{t+1},$$

so that there is a time-varying growth rate of dividends, but not consumption.

Discussion

(9) A journalist tells you, "Markets are inefficient, since prices often rise then crash without changes in dividends". What do you answer (4 sentences max).

(10) There is a lot of mention in the financial press of "bubbles". How would you define a "bubble"? By this measure and given the survey of empirical evidence in class in the Cochrane paper (1st lecture), are there bubbles in the stock market? Perhaps in other markets?

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Problem Set 3 - Due Tuesday, November 30 by 6pm in my mailbox

Solving and simulating an asset pricing model

This problem is motivated by the idea that "rich people" are the marginal investors in asset markets. Hence, we need to think about the risk that these rich people face; in particular, their income may be more risky than that of "poor people".

Consider an endowment economy model with two (classes of) agents, and two trees. The first tree is tradable, call it "capital income", and produces a dividend flow $\{y_t^K\}$. The second tree is not tradable, call it "labor income", and has a dividend flow $\{y_t^L\}$. Agent w (for worker) receives a share $1-\alpha$ of the labor income tree's dividend each period. Agent k (for capitalist) receives a share α of the labor income tree's dividend and the entire capital income tree's dividend each period.

Let $y_t = y_t^L + y_t^K$ be total income and let $s_t = y_t^L / y_t$ be the share of labor income in the economy. We assume that s_t is a mean-reverting, AR(1) process:

$$s_{t+1} = \rho s_t + (1-\rho)\overline{s} + \sigma_1 \varepsilon_{t+1},$$

with ε_{t+1} iid $N(0,1)$. (Note: s_t has to lie in (0,1), so implicitely we assume that ε_{t+1} is a truncated normal. Alternatively, we could write a time-series process for $\log\left(\frac{s_t}{1-s_t}\right)$.)

Second, we assume that aggregate income growth is

$$\Delta \log y_{t+1} = \mu + \sigma_2 \varepsilon_{t+1} + \sigma_3 \eta_{t+1},$$

with η_{t+1} iid $N(0,1)$ independent of ε_{t+1}. Clearly, the parameter σ_1 determines the variance of the labor income share, and the choice of σ_2 and σ_3 determines the variance of $\Delta \log y$ and its correlation with the share.

We assume that workers and capitalists cannot trade with each other. (Perhaps workers just don't participate in asset markets because of high fixed costs.) Hence, in the asset market only capitalists trade the capital income tree, and a risk-free asset. Capitalists have expected discounted utility (CRRA). In equilibrium, the consumption of the capitalists is equal to their total income,

$$c_t^k = \alpha y_t^L + y_t^K.$$

<u>Questions:</u>

(1) Write an equation for the price of the capital tree. Show that the price-dividend ratio at time t is a function of s_t.

(2) Derive a functional equation for the price-dividend ratio, $\frac{P_t}{D_t} = \frac{P_t}{y_t^K} = \Phi(s_t)$.

(3) Solve numerically the model, assuming $\gamma = 4$, $\beta = .98, \rho = .9$, $\alpha = .5$, $\overline{s} = .8, \mu = .02, \sigma_1 = .01$, $\sigma_2 = -.01$, and $\sigma_3 = .02$. You will have to solve for the value $\Phi(s)$, for any $s \in (0,1)$. The simplest way to solve this equation is to discretize it, i.e. solve for this price for any value in a grid $\{s_1, s_2, ..., s_N\}$. See the note below on how to calculate integrals.

Report the following four statistics: (unconditional) mean risk-free rate, the mean equity premium, the standard deviation of the equity return and the risk-free-rate. To obtain these statistics, you will have to simulate the model.

(4) Plot the P-D ratio, the conditional equity premium $E_t\left(R^e_{t+1}\right)$, and the risk-free rate, as a function of s_t. Using these pictures, discuss whether the model can reproduce the evidence that low P/D ratios forecast negatively future returns, and future excess returns.

(5) How do the model results change as you increase σ_3? Report the same four statistics, and interpret.

(6) How do the model results change as you make σ_2 positive rather than negative? Report the same four statistics, and interpret.

Not on how to compute integrals numerically: To solve the model, you need to compute an integral over the shocks ε_t and η_t. One possibility is to approximate these shocks with discrete distributions. For instance, approximate the expectation of a standard normal random variable $\varepsilon \hookrightarrow N(0,1)$ as

$$E_\varepsilon\left(f(\varepsilon)\right) = \sum_{i=1}^{10} \pi_i f(\varepsilon_i),$$

where the ε_i are linearly spaced between -3 and 3, and $\pi_i = \phi\left(\varepsilon_i\right)$ where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the pdf of $N(0,1)$. This is not the best way to do it, but it should be good enough.

Problem Set 4 - Due Thursday, December 9, by 6pm in my mailbox

Risk of deflation and the pricing of nominal bonds and TIPS

Consider a representative agent economy endowment economy. The RC has expected utility and a CRRA felicity function:

$$E\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\gamma}}{1-\gamma}.$$

The process for aggregate consumption is,

$$\Delta\log C_t=\mu_c-\frac{\sigma_c^2}{2}+\sigma_c\varepsilon_t,$$

where ε_t is *i.i.d.* $N(0,1)$, and inflation follows the process:

$$\pi_t=\Delta\log P_t=\log\left(\frac{P_t}{P_{t-1}}\right)=\mu_p-\frac{\sigma_p^2}{2}-\frac{\chi^2}{2}+\sigma_p\varepsilon_t+\chi\eta_t,$$

where P_t is the price level (the CPI), and π_t the (log) inflation rate, and η_t is *i.i.d.* $N(0,1)$ and independent of $\{\varepsilon_{t'}\}$.

Recall from the lectures that a *nominal* stochastic discount factor in this case is

$$M_{t+1}^{nom}=\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{P_t}{P_{t+1}},$$

and that the dollar price of a claim to 1\$ at time $t+n$, call it $Q_{t,n}$, satisfies

$$\begin{aligned}Q_{t,n}&=E_t\left(M_{t+1}^{nom}Q_{t+1,n-1}\right),\\Q_{t,0}&=1.\end{aligned}$$

(1) Risk-free assets. Compute the term structure of purely risk-free assets. That is, find $Q_{t,n}^{rf}$ the *dollar* price of a claim to 1 good in period $t+n$, i.e. a claim to P_{t+n}\$ at time $t+n$, for any $n=1,2,....$ (You can solve this in closed form.) Give an expression for the yield $y_{t,n}^{rf}$, defined implicitely as

$$Q_{t,n}^{rf}=\frac{1}{\left(1+y_{t,n}^{rf}\right)^n}.$$

Plot the real yield curve, i.e. $y_{t,n}^{rf}$ as a function of n. Deduce the average return on risk-free assets, as a function of maturity n.

(2) Nominal risk-free assets. Compute the term structure of nominal risk-free assets. That is, find $Q_{t,n}^{nom}$ the dollar price of a claim to 1 dollar in period $t+n$, for any $n=1,2,....$ (You can solve this in closed form.) Deduce the yield, plot the nominal yield curve, and compute the average return on nominal risk-free assets, as a function of maturity n.

(3) Inflation expectations. Using (1) and (2), discuss under which conditions is the.average return on the risk-free assets larger than the average return on nominal assets. The difference between the yield on real and nominal bonds is often interpreted as a measure of the market's inflation expectation. Under what condition is that a good measure?

(4) TIPS (Treasury Inflation Protected Securities). A feature of TIPS that has been discussed recently is that TIPS coupon payments are adjusted up or down according to inflation, but the principal repayment is adjusted only up, and never down. (The US government basically says, we'll pay you back more principal if there is inflation, and not less if there is deflation.) Given that the likelihood of deflation has gone up recently, this feature of TIPS may have started to impact prices.

For simplicity assume that TIPS are zero-coupon bonds. The nominal payoff at $t+n$ of a TIPS that was purchased at time t for a nominal price $Q^{tp}_{t,n}$ is

$$D^{tp}_{t,t+n} = \max\left(1, \frac{P_{t+n}}{P_t}\right),$$

i.e. it pays off the 1$ face value times the inflation adjustment $\frac{P_{t+n}}{P_t}$, if inflation is positive over the lifetime of the bond, and if inflation is negative it simply pays off 1$. Hence, unlike standard bonds, *holding maturity fixed, the date at which the bond was issued matters.* (Since P_t shows up in the payoff.)

(a) Write an equation for the value at time t of a TIPS maturing in n periods, that was issued when the price level was P. Call this $Q^{tip}_{t,n}(P)$.

(b) You will need to compute this numerically. You can do it as you like, but there is a "smart" approach which makes the computation a bit nicer. Define for any $u > 0$, $g_{t,n}(u) = Q^{tip}_{t,n}(uP_t)$. Write an equation for $g_{t,n}(u)$, as a function of $g_{t+1,n-1}(u')$ for some u'. Use this equation to compute the function $g_{t,n}$.

Parameters: assume $\mu_c = .02, \mu_p = .02, \sigma_c = .02, \sigma_p = .01, \chi = .01, \beta = .99, \gamma = 4$. See below for a note on how to compute integrals.

(c) The questions here are a bit open-ended. Basically I want you to think, how do I use the model solution to illustrate the effect of deflation risk on the price of TIPS. Produce plots or a table to illustrate the effect of inflation volatility on the relative price of TIPS and (nominal) Treasuries. Suppose we have a few bad realizations of inflation, how would this affect the prices? Can you use the model to illustrate the importance of the cash flow effect and the risk premium effect? (e.g. by setting risk aversion to be very small). How could you check in the data to see if deflation risk matters for TIPS pricing?

Note: With everything iid as above the yield curves not very interesting of course! But it is enough to show the effect of the asymmetric adjustment of TIPS to inflation vs. deflation. Still, it would be interesting to consider more complex dynamics for consumption and/or inflation, e.g. an AR(1) for consumption growth, $\Delta \log C_t = \mu_c(1-\rho) + \rho \Delta \log C_{t-1} + \sigma_c \varepsilon_t$, or for inflation. Another possibility would be to use Epstein-Zin utility. This may be a good topic for a second-year paper.

Note: to solve it, you need to compute an integral over the shocks ε_t and η_t. One possibility is to approximate these shocks with discrete distributions. For instance, approximate the expectation of a standard normal random variable $\varepsilon \hookrightarrow N(0,1)$ as

$$E_\varepsilon\left(f(\varepsilon)\right) = \sum_{i=1}^{10} \pi_i f(\varepsilon_i),$$

where the ε_i are linearly spaced between -3 and 3, and $\pi_i = \phi(\varepsilon_i)$ where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the pdf of $N(0,1)$. This is not the best way to do it, but it should be good enough.