Computational Appendix on Disasters and Asset Pricing

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This appendix details the calculations required to solve the disaster model with Epstein-Zin utility. This is done with very standard techniques. I will assume the following processes for consumption growth and dividend growth:

$$\begin{aligned}\Delta \log C_t &= \mu_c(s_t) + \sigma_c(s_t)\varepsilon_t, \\ \Delta \log D_t &= \mu_d(s_t) + \sigma_d(s_t)\varepsilon_t,\end{aligned}$$

with s_t a Markov chain with transition matrix Q and ε_t an *iid* $N(0,1)$ random variable, and ε_t and s_t are independent. This encompasses the standard Mehra-Prescott model as well as the disaster model, and can easily accommodate additional dynamics. It can also accomodate the specification of Barro (2006) who use the historical distribution of disasters.

Normalize utility by consumption to write the utility-consumption ratio V_t/C_t :

$$\frac{V_t}{C_t} = \left(1 - e^{-\rho} + e^{-\rho} E_t \left(\left(\frac{V_{t+1}}{C_{t+1}}\right)^{1-\theta} \left(\frac{C_{t+1}}{C_t}\right)^{1-\theta} \right)^{\frac{1-\alpha}{1-\theta}} \right)^{\frac{1}{1-\alpha}}. \tag{1}$$

Let $f(s_t) = \left(\frac{V_t}{C_t}\right)^{1-\alpha}$. Then f solves the functional equation

$$f(s) = 1 - e^{-\rho} + e^{-\rho} E_{\varepsilon', s'|s} \left(f(s')^{\frac{1-\theta}{1-\alpha}} \left(\frac{C_{t+1}}{C_t}\right)^{1-\theta} \right)^{\frac{1-\alpha}{1-\theta}},$$

i.e. if we make the expectations explicit:

$$f(s) = 1 - e^{-\rho} + e^{-\rho} \left[\sum_{s' \in S} Q(s,s') f(s')^{\frac{1-\theta}{1-\alpha}} \exp\left((1-\theta)\mu_c(s') + (1-\theta)^2 \frac{\sigma_c(s')^2}{2} \right) \right]^{\frac{1-\alpha}{1-\theta}}.$$

I solve this fixed point problem by iteration, starting with the guess $f(s) = 1$.

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Next, let $h(s_t) = V_t/C_t = f(s_t)^{1/(1-\alpha)}$. The SDF is:

$$\begin{aligned} M_{t+1} &= e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha}\left(\frac{V_{t+1}}{E_t(V_{t+1}^{1-\theta})^{\frac{1}{1-\theta}}}\right)^{\alpha-\theta}, \\ &= e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\theta}\frac{h(s_{t+1})^{\alpha-\theta}}{E_t\left(h(s_{t+1})^{1-\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\theta}\right)^{\frac{\alpha-\theta}{1-\theta}}}, \\ &= e^{-\rho}\exp(-\theta\mu_c(s_{t+1}) - \theta\sigma_c(s_{t+1})\varepsilon_{t+1})h(s_{t+1})^{\alpha-\theta} \\ &\quad \times\left[\sum_{s'\in S} Q(s_t,s')f(s')^{\frac{1-\theta}{1-\alpha}}\exp\left((1-\theta)\mu_c(s') + (1-\theta)^2\frac{\sigma_c(s')^2}{2}\right)\right]^{\frac{\theta-\alpha}{1-\theta}}. \end{aligned}$$

To compute the bond price, I use the condition:

$$P_t^b = E_t\left(M_{t+1}\times x_{t+1}\right),$$

with x_{t+1} =payoff of bond, which given the default is 1 if there is no disaster and $(1-def)\,prdef + (1-prdef)$ if there is a disaster, where def = amount of default and $prdef$ = probability of default conditional on disaster. Hence,

$$\begin{aligned} P^b(s) &= \zeta(s)e^{-\rho}\sum_{s'\in S} Q(s,s')E_\varepsilon\exp(-\theta\mu_c(s') - \theta\sigma_c(s')\varepsilon)h(s')^{\alpha-\theta}x(s'), \\ &= \zeta(s)e^{-\rho}\sum_{s'\in S} Q(s,s')\exp\left(-\theta\mu_c(s') + \frac{\theta^2}{2}\sigma_c^2(s')\right)h(s')^{\alpha-\theta}x(s'), \end{aligned}$$

where

$$\zeta(s) = \left[\sum_{s'\in S} Q(s,s')f(s')^{\frac{1-\theta}{1-\alpha}}\exp\left((1-\theta)\mu_c(s') + (1-\theta)^2\frac{\sigma_c(s')^2}{2}\right)\right]^{\frac{\theta-\alpha}{1-\theta}}.$$

The realized bond return is $x_{t+1}/P_t^b = x(s_{t+1})/P^b(s_t)$; the expected bond return conditional on the current state is $cbr(s) = E_t\left(x_{t+1}\right)/P^b(s_t) = \sum_{s'\in S} Q(s_t,s')x(s')/P^b(s_t)$; and the unconditional bond return is $E\left[E_t\left(x_{t+1}\right)/P^b(s_t)\right] = \sum_{s\in S}\mu(s)cbr(s)$ where μ is the ergodic distribution of the markov chain $\{s_t\}$.

Finally, I calculate the value of equity with the standard recursion:

$$\frac{P_t}{D_t} = E_t\left(M_{t+1}\left(1+\frac{P_{t+1}}{D_{t+1}}\right)\frac{D_{t+1}}{D_t}\right),$$

which shows that $\frac{P_t}{D_t}$ is a function of the state variable $s_t : \frac{P_t}{D_t} = g(s_t)$. I again find g by iterating on this recursion:

$$g(s_t) = E_t\left(e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\theta}\left(\frac{D_{t+1}}{D_t}\right)\left(\frac{h(s_{t+1})}{E_t\left(h(s_{t+1})^{1-\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\theta}\right)^{\frac{1}{1-\theta}}}\right)^{\alpha-\theta}.\left(g(s_{t+1})+1\right)\right),$$

$$g(s) = e^{-\rho}\zeta(s)\sum_{s'\in S} Q(s,s')\exp\left(\mu_d(s') - \theta\mu_c(s') + \frac{(\sigma_d(s') - \theta\sigma_c(s'))^2}{2}\right)h(s')^{\alpha-\theta}\left(g(s')+1\right).$$

The conditional equity return is then:

$$ecr(s) = \frac{E_{\varepsilon',s'|s}\left(\left(g(s')+1\right)\frac{D_{t+1}}{D_t}\right)}{g(s)} = \frac{\sum_{s'\in S} Q(s,s')\left(g(s')+1\right)\exp(\mu_d(s') + \sigma_d(s')^2/2)}{g(s)},$$

and the unconditional equity return is simply $\sum_{s\in S}\mu(s)ecr(s)$. The standard deviations of the returns are similarly easily computed.