Shadow prices and Short-run vs Long-run Demand

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These notes formalize the concept of shadow prices, using a simple example of short vs long-run demand curves. It is based on Kevin. Murphy's lectures.

1 The (Hicksian) Long-run demand

Note: a similar analysis is doable with Marshallian demand but income effects make results sometimes less sharp.

First, we define the vector of long-run demand $X^H(P_1,...,P_n,U) = \left(X_1^H(P_1,...,P_n,U),...,X_n^H(P_1,...,P_n,U)\right)$, which is defined by the standard cost-minimization problem:

$$\min_{X_1,...,X_n} \sum_{k=1}^{n} P_k X_k,$$

$$st: U(X_1,...,X_n) = U.$$

The FOCs are:

$$\forall k = 1...n, \quad \mu \frac{\partial U}{\partial X_k}(X_1,...,X_n) = P_k,$$

$$U(X_1,...,X_n) = U.$$

This is a system of $n+1$ equations in $n+1$ unknowns $(\mu, X_1,...,X_n)$. I will assume that these FOCs are necessary and sufficient, and that they define a unique solution for any price vector and utility level.

2 The Short-run demand

We assume that some choices have been made in the past and cannot be altered today (under any circumstance). This is an extreme example of adjustment costs. Assume that good 1 is the commodity we can't adjust. The short-run demand is defined as the solution $X^{SR}\left(P_1,...,P_n,U,\overline{X_1}\right) = \left(X_1^{SR}\left(P_1,...,P_n,U,\overline{X_1}\right),...,X_n^{SR}\left(P_1,...,P_n,U,\overline{X_1}\right)\right)$ to the problem:

$$\min_{X_1,...,X_n} \sum_{k=1}^{n} P_k X_k,$$

$$st: U(X_1,...,X_n) = U,$$

$$\text{and } X_1 = \overline{X_1}.$$

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This can be rewritten as:

$$\min_{X_1,...,X_n} \sum_{k=2}^{n} P_k X_k + P_1 \overline{X_1},$$
$$st : U\left(\overline{X_1},..., X_n\right) = U.$$

The FOCs that characterize the solution of this program are:

$$\forall k = 2...n, \quad \widetilde{\mu} \frac{\partial U}{\partial X_k}\left(\overline{X_1}, X_2, ..., X_n\right) = P_k,$$
$$U\left(\overline{X_1},..., X_n\right) = U.$$

which is a system of n equations in n unknowns $(\widetilde{\mu}, X_2, ..., X_n)$.

3 Shadow Price

Define now the shadow price P_1^* of commodity 1 as the marginal value of the first commodity:

$$P_1^* = \frac{1}{\widetilde{\mu}} \frac{\partial U}{\partial X_1}\left(\overline{X_1}, X_2^{SR}, ..., X_n^{SR}\right).$$

Note that since the short-run demand depends on all prices and utility as well as $\overline{X}_1$ (i.e. $X_i^{SR} = X_i^{SR}\left(P_1, ..., P_n, U, \overline{X_1}\right)$), P_1^* is itself a function of $P_1, ..., P_n, M, U, \overline{X_1}$.

Now rewrite the system of FOCs characterizing short-run demand, using this definition of P_1^* :

$$\frac{\partial U}{\partial X_1}\left(\overline{X_1}, X_2^{SR}, ..., X_n^{SR}\right) = \widetilde{\mu} P_1^*$$
$$\forall k = 2...n, \quad \frac{\partial U}{\partial X_k}\left(\overline{X_1}, X_2^{SR}, ..., X_n^{SR}\right) = \widetilde{\mu} P_k$$
$$U\left(\overline{X_1},..., X_n\right) = U.$$

Note that this system of equations is the same as the one of the long-run demand. Thus, given our assumption of uniqueness:

$$\forall k = \mathbf{1}...n, \quad X_k^{SR}\left(P_1, ..., P_n, U, \overline{X_1}\right) = X_k^H\left(P_1^*, ..., P_n, U\right). \tag{1}$$

4 Some consequences

Suppose we want to know how the fact that commodity 1 cannot be adjusted changes the **elasticity of demand of commodity** j. Differentiate the equation (1) for good j with respect to its own price:

$$\frac{\partial X_j^{SR}}{\partial P_j} = \frac{\partial X_j^H}{\partial P_1} \frac{\partial P_1^*}{\partial P_j} + \frac{\partial X_j^H}{\partial P_j}. \tag{2}$$

To find $\frac{\partial P_1^*}{\partial P_j}$ use the demand for good 1:

$$\overline{X_1} = X_1^H\left(P_1^*, ..., P_n, U\right).$$

Hence:

$$0 = \frac{\partial X_1^{SR}}{\partial P_j} = \frac{\partial X_1^H}{\partial P_1}\frac{\partial P_1^*}{\partial P_j} + \frac{\partial X_1^H}{\partial P_j},$$

$$\frac{\partial P_1^*}{\partial P_j} = -\frac{\frac{\partial X_1^H}{\partial P_j}}{\frac{\partial X_1^H}{\partial P_1}}. \tag{3}$$

Since the denominator is negative, the shadow price increases if 1 and j are substitutes and decreases if they are complements. Example: if you increase the price of gas and you cannot adjust your car consumption, a complement, the shadow price of cars decreases.

Plugging 3 into 2 yields:

$$\frac{\partial X_j^{SR}}{\partial P_j} = \frac{\partial X_j^H}{\partial P_j} - \frac{\frac{\partial X_j^H}{\partial P_1}\frac{\partial X_1^H}{\partial P_j}}{\frac{\partial X_1^H}{\partial P_1}}.$$

This is the "second law of demand": the own-price elasticity of demand is always smaller in the short run. Since $\frac{\partial X_j^H}{\partial P_1}\frac{\partial X_1^H}{\partial P_j} > 0$, the short-run (Hicksian) elasticity of demand is smaller, when 1 and j are either good substitutes or good complements. The intuition is that an increase in the price of good j decreases the demand for complements, which in turn feeds back into a lower demand for good j. If the complement is not adjustable in the short-run, the feed back does not occur, so the consumption of good j falls by less in the short-run than in the long-run.

Let's now analyze the impact of having good 1 fixed on the *cross elasticity* of demand:

$$\frac{\partial X_j^{SR}}{\partial P_k} = \frac{\partial X_j^H}{\partial P_1}\frac{\partial P_1^*}{\partial P_k} + \frac{\partial X_j^H}{\partial P_k}.$$

Thus:

$$\frac{\partial X_j^{SR}}{\partial P_k} = \frac{\partial X_j^H}{\partial P_k} - \frac{\frac{\partial X_1^H}{\partial P_k}\frac{\partial X_j^H}{\partial P_1}}{\frac{\partial X_1^H}{\partial P_1}}.$$

The last term $-\frac{\frac{\partial X_1^H}{\partial P_k}\frac{\partial X_j^H}{\partial P_1}}{\frac{\partial X_1^H}{\partial P_1}}$ is positive if and only 1 and k, and 1and j, are substitutes or if 1 and k, and 1 and j are complements. For instance, assume we're thinking of 1 as cars, j as tires and k as gasoline, and assume all these goods are complements. An increase in the price of tires will not reduce the demand for gas as much in the short-run as it will in the long-run, because in the long-run car consumption will fall.

5 Additional Examples

This simple method to introduce shadow prices can also be used to analyze several other cases:

- corner solutions;
- rationing;
- taxes (taking prices as exogenous).