Technical Appendix for
"Investment Spikes: New Facts and a General Equilibrium Exploration"
Preliminary Version

François Gourio and Anil K. Kashyap*

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This technical appendix presents all the computations required to solve the model of business cycle with fixed costs to investment proposed by Julia Thomas in "Is Lumpy Investment Relevant for the Business Cycle?" (*Journal of Political Economy*, 2002, 110(3), 508:534). The appendix then extends the model in some directions.

1 Baseline Thomas Model

This section considers the baseline Thomas model.

A. Deriving the first-order conditions

In this section I derive the first-order conditions. The equations that I obtain are the same as in Thomas (2002). The social planner problem is to choose $\{c_t, n_{jt}, i_{jt}, \alpha_{jt}, \theta_{jt+1}, k_{jt+1}\}$ to maximize:

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(N_t\right) \right],$$

$$s.t. \quad : \quad c_t + \sum_{j=0}^{J} \theta_{jt} \alpha_{jt} i_{jt} \leq \sum_{j=0}^{J} \theta_{jt} A_t k_{jt}^{\gamma} n_{jt}^{v},$$

*Respectively Boston University and University of Chicago Graduate School of Business and NBER. Contact: *fgourio@bu.edu* and *anil.kashyap@chicagogsb.edu*.

$$
\begin{aligned}
N_t &= \sum_{j=0}^{J} \theta_{jt} \left(n_{jt} + R(\alpha_{jt}) \right), \\
\forall j &= 0...J-1 : k_{j+1,t+1} = (1-\delta) \, k_{j,t}, \\
\forall j &= 0...J : k_{0,t+1} = (1-\delta) k_{jt} + i_{jt}, \\
\forall j &= 0...J-1 : \theta_{j+1,t+1} = (1-\alpha_{jt}) \theta_{jt}, \\
\theta_{0,t+1} &= \sum_{j=0}^{J} \theta_{jt} \alpha_{jt}.
\end{aligned}
$$

I form the Lagrangean L, with obvious notations for the multipliers:

$$
\begin{aligned}
L \;=\; & \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - v\left(\sum_{j=0}^{J} \theta_{jt} \left(n_{jt} + R(\alpha_{jt}) \right) \right) \right] + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\sum_{j=0}^{J} \theta_{jt} A_t k_{jt}^{\gamma} n_{jt}^{\upsilon} - c_t - \sum_{j=0}^{J} \theta_{jt} \alpha_{jt} i_{jt} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \left(\sum_{j=0}^{J-1} \zeta_{j+1t} \left((1-\delta) \, k_{j,t} - k_{j+1,t+1} \right) \right) + \sum_{t=0}^{\infty} \beta^t \sum_{j=0}^{J-1} \eta_{j+1t} \left((1-\alpha_{jt}) \theta_{jt} - \theta_{j+1t+1} \right) \\
& + \sum_{t=0}^{\infty} \beta^t \eta_{0t} \left(\sum_{j=0}^{J} \theta_{jt} \alpha_{jt} - \theta_{0t+1} \right) + \sum_{t=0}^{\infty} \beta^t \sum_{j=0}^{J} \pi_{jt} \left((1-\delta) k_{jt} + i_{jt} - k_{0,t+1} \right).
\end{aligned}
$$

Now take the first-order conditions:

FOC wrt c_t :

$$u'(c_t) = \lambda_t$$

FOC wrt n_{jt} $(j = 0...J)$:

$$v'(N_t) \theta_{jt} = \upsilon \lambda_t \theta_{jt} A_t k_{jt}^{\gamma} n_{jt}^{\upsilon - 1}$$

Let $w_t = v'(N_t)/\lambda_t$ denote the wage, this FOC can be rewritten as the usual MPL = wage condition,

$$w_t = \upsilon A_t k_{jt}^{\gamma} n_{jt}^{\upsilon - 1}$$

FOC wrt i_{jt} $(j = 0...J)$:

$$-\lambda_t \theta_{jt} \alpha_{jt} + \pi_{jt} = 0$$

FOC wrt k_{jt} $(j = 1...J-1)$:

$$\lambda_t \theta_{jt} MPK_{jt} + (1-\delta) \pi_{jt} + \zeta_{j+1,t} (1-\delta) - \zeta_{j,t-1} \frac{1}{\beta} = 0$$

$$\zeta_{j,t-1} = \beta \zeta_{j+1,t} (1-\delta) + \beta (1-\delta) \pi_{jt} + \beta \lambda_t \left(\theta_{jt} MPK_{jt} \right)$$

Define $s_{jt} = \frac{\zeta_{jt}}{\lambda_t \theta_{j-1t}}$, then we have:

$$s_{j,t-1} \lambda_{t-1} \theta_{j-1t-1} = \beta \theta_{jt} \lambda_t s_{j+1,t} (1-\delta) + \beta (1-\delta) \alpha_{jt} \lambda_t \theta_{jt} + \beta \lambda_t \left(\theta_{jt} MPK_{jt} \right)$$

$$\frac{s_{j,t-1}}{1-\alpha_{j-1,t-1}} = \frac{\beta\lambda_t}{\lambda_{t-1}}\left[s_{j+1,t}(1-\delta)+(1-\delta)\alpha_{jt}+MPK_{jt}\right]$$

$$\frac{s_{j,t-1}}{1-\alpha_{j-1,t-1}} = \frac{\beta\lambda_t}{\lambda_{t-1}}\left[\frac{s_{j+1,t}}{1-\alpha_{j,t}}(1-\alpha_{jt})(1-\delta)+(1-\delta)\alpha_{jt}+MPK_{jt}\right]$$

Let $\mu_{jt} = \frac{s_{jt}}{1-\alpha_{j-1t}}$, then:

$$\forall j = 1...J-1 : \mu_{j,t-1} = \frac{\beta\lambda_t}{\lambda_{t-1}}\left[MPK_{jt}+(1-\delta)\left(\alpha_{jt}+\mu_{j+1,t}(1-\alpha_{jt})\right)\right]$$

FOC wrt k_{0t} :

$$\lambda_t\left(\theta_{0t}\gamma A_t k_{0t}^{\gamma-1}n_{0t}^{v}\right)+\zeta_{1t}(1-\delta)+(1-\delta)\pi_{0t}-\frac{1}{\beta}\sum_{j=0}^{J}\pi_{jt-1}=0$$

$$\gamma A_t k_{0t}^{\gamma-1}n_{0t}^{v}+\frac{\zeta_{1t}}{\lambda_t\theta_{0t}}(1-\delta)+(1-\delta)\alpha_{0t}-\frac{\lambda_{t-1}}{\beta\lambda_t}\frac{\sum_{j=0}^{J}\theta_{jt-1}\alpha_{jt-1}}{\theta_{0t}}=0$$

$$\frac{\beta\lambda_t}{\lambda_{t-1}}\left(\gamma k_{0t}^{\gamma-1}n_{0t}^{v}+\mu_{1t}(1-\alpha_{0t})(1-\delta)+(1-\delta)\alpha_{0t}\right)=1$$

Define $\mu_{0t} = \frac{\beta\lambda_t}{\lambda_{t-1}}\left(\gamma k_{0t}^{\gamma-1}n_{0t}^{v}+(1-\delta)\left(\mu_{1t}(1-\alpha_{0t})+\alpha_{0t}\right)\right)$, which is the same recursion as above. Then the FOC reads $\mu_{0t}=1$.

FOC wrt k_{Jt} :

$$\lambda_t\theta_{Jt}A_t\gamma k_{Jt}^{\gamma-1}n_{Jt}^{v}-\frac{1}{\beta}\zeta_{J,t-1}+\pi_{Jt}(1-\delta)=0,$$

leading to

$$\begin{aligned} A_t\gamma k_{Jt}^{\gamma-1}n_{Jt}^{v}+\frac{\pi_{Jt}}{\lambda_t\theta_{Jt}}(1-\delta) &= \frac{\zeta_{J,t-1}}{\lambda_{t-1}\theta_{J-1t}}\frac{\theta_{J-1t}}{\theta_{Jt}}\frac{\lambda_{t-1}}{\beta\lambda_t} \\ \frac{\beta\lambda_t}{\lambda_{t-1}}\left(A_t\gamma k_{Jt}^{\gamma-1}n_{Jt}^{v}+(1-\delta)\right) &= \mu_{J,t-1}, \end{aligned}$$

which is actually the specialization of the equation for any j, given $\alpha_{Jt}=1$.

FOC wrt α_{jt} ($j = 0...J-1$):

$$-v'(N_t)\theta_{jt}R'(\alpha_{jt})-\lambda_t\theta_{jt}i_{jt}-\theta_{jt}\eta_{j+1t}+\eta_{0t}\theta_{jt}=0$$

$$-w_tR'(\alpha_{jt})-i_{jt}-\frac{\eta_{j+1t}}{\lambda_t}+\frac{\eta_{0t}}{\lambda_t}=0$$

and since $R'(x)=G^{-1}(x)$, denoting $v_{jt}=\frac{\eta_{jt}}{\lambda_t}$ (i.e. transforming from utils units into goods units):

$$v_{0t}-w_tG^{-1}(\alpha_{jt})-i_{jt}=v_{j+1t}.$$

FOC wrt θ_{jt}, for $j \neq 0$ and $j \neq J$:

$$-v'(N_t)\left(n_{jt}+R(\alpha_{jt})\right)+\lambda_t\left[A_t k_{jt}^{\gamma} n_{jt}^{\upsilon}-\alpha_{jt} i_{jt}\right]+\eta_{j+1t}(1-\alpha_{j,t})-\frac{\eta_{jt-1}}{\beta}+\eta_{0,t}\alpha_{jt}=0$$

Which yields

$$v_{jt-1}=\frac{\beta\lambda_t}{\lambda_{t-1}}\left[-w_t\left(n_{jt}+R(\alpha_{jt})\right)+A_t k_{jt}^{\gamma} n_{jt}^{\upsilon}-\alpha_{jt} i_{jt}+v_{j+1t}(1-\alpha_{j,t})+v_{0,t}\alpha_{jt}\right],$$

FOC wrt θ_{0t} similarly yields:

$$v_{0,t-1}=\frac{\beta\lambda_t}{\lambda_{t-1}}\left[-w_t\left(n_{0t}+R(\alpha_{0t})\right)+A_t k_{0t}^{\gamma} n_{0t}^{\upsilon}-\alpha_{0t} i_{0t}+v_{1t}(1-\alpha_{0,t})+v_{0,t}\alpha_{0t}\right].$$

For $\theta_{Jt,}$ we get the general equation, given that $\alpha_{J,t}=1$:

$$v_{jt-1}=\frac{\beta\lambda_t}{\lambda_{t-1}}\left[-w_t\left(n_{jt}+R(1)\right)+A_t k_{jt}^{\gamma} n_{jt}^{\upsilon}-i_{jt}+v_{0,t}\right].$$

B. List of all the equations

Technological shocks:

$$\begin{aligned} A_t &= X_t z_t, \\ X_{t+1} &= \Theta_a X_t, \\ \log z_t^a &= \rho^a \log z_{t-1}^a + \varepsilon_t^a, \\ & \quad \varepsilon_t^a \text{ iid } N(0,\sigma_a^2). \end{aligned}$$

Production functions, Law of motion for capital stocks:

$$\begin{aligned} \forall j &= 0...J : y_{jt} = A_t k_{jt}^{\gamma} n_{jt}^{\upsilon}, \\ \forall j &= 0...J-1 : k_{j+1,t+1} = (1-\delta)k_{j,t}, \\ \forall j &= 0...J : k_{0,t+1} = (1-\delta)k_{j,t} + i_{jt}. \end{aligned}$$

Evolution of the cross-sectional distribution:

$$\begin{aligned} \theta_{0,t+1} &= \sum_{j=0}^{J} \alpha_{jt}\theta_{jt}, \\ \forall j &= 1...J : \theta_{j,t+1} = \theta_{j-1,t}(1-\alpha_{j-1,t}). \end{aligned}$$

Resource constraints:

$$\begin{aligned} c_t + \sum_{j=0}^{J} \theta_{jt}\alpha_{jt} i_{jt} &= \sum_{j=0}^{J} \theta_{jt} y_{jt}, \\ \sum_{j=0}^{J} \theta_{jt} n_{jt} + \sum_{j=0}^{J} \theta_{jt} R(\alpha_{jt}) &= N_t, \end{aligned}$$

$$\text{where } R(\alpha_{jt}) = \int_0^{G^{-1}(\alpha_{jt})} x dG(x).$$

First-order conditions for optimization:

Consumer optimization:

$$\begin{aligned} \lambda_t &= u_1(c_t, 1 - N_t), \\ w_t &= \frac{u_2(c_t, 1 - N_t)}{u_1(c_t, 1 - N_t)}. \end{aligned}$$

Labor demand:

$$n_{jt} = \left(\frac{\upsilon A_t k_{jt}^{\gamma}}{w_t} \right)^{\frac{1}{1-\upsilon}}.$$

Indifference of marginal firm of vintage j between adjusting and not:

$$\forall j = 0...J - 1 : v_{0t} - v_{j+1,t} = B_t i_{jt} + w_t G^{-1}(\alpha_{jt}).$$

Ex-dividend value of firm recursion:

$$\begin{aligned} \forall j &= 0...J - 1 : \\ v_{jt} &= \mathbb{E}_t \left(\frac{\beta \lambda_{t+1}}{\lambda_t} \left(y_{j,t+1} - w_{t+1} n_{j,t+1} - \alpha_{j,t+1} i_{j,t+1} + \alpha_{j,t+1} v_{0,t+1} + (1 - \alpha_{j,t+1}) v_{j+1,t+1} - w_{t+1} R(\alpha_{j,t+1}) \right) \right) \end{aligned}$$

(This equation is also true for $j = J$, with $\alpha_{J,t} = 1$ for all $t \geq 0$.)

Marginal value of capital:

$$\forall j = 0...J - 1 : \mu_{j,t} = \mathbb{E}_t \left(\frac{\beta \lambda_{t+1}}{\lambda_t} \left(\frac{\partial y_{j,t+1}}{\partial k_{j,t+1}} + (1 - \delta) \left(\alpha_{j,t+1} + (1 - \alpha_{j,t+1}) \mu_{j+1,t+1} \right) \right) \right)$$

with

$$\mu_{J,t} = \mathbb{E}_t \left(\frac{\beta \lambda_{t+1}}{\lambda_t} \left(\frac{\partial y_{J,t+1}}{\partial k_{J,t+1}} + (1 - \delta) \right) \right),$$

$$\mu_{0t} = 1.$$

<u>(i) Relation between Thomas' first order condition for the choice of capital and my equations</u>

Of course, the two are equivalent. This paragraph derives it. Thomas writes the first order condition as:

$$\lambda_t B_t = \mathbb{E}_t \begin{pmatrix} \beta\lambda_{t+1}\left(\frac{\partial y_{0,t+1}}{\partial k_{0,t+1}} + (1-\delta)\alpha_{0,t+1}B_{t+1}\right) \\ +\beta^2\lambda_{t+2}(1-\delta)\varphi_{0,t+1}\left(\frac{\partial y_{1,t+2}}{\partial k_{1,t+2}} + (1-\delta)\alpha_{1,t+2}B_{t+2}\right) \\ +... \\ +\beta^J\lambda_{t+J}(1-\delta)^{J-1}\varphi_{J-2,t+J-1}\left(\frac{\partial y_{J-1,t+J}}{\partial k_{J-1,t+J}} + (1-\delta)\alpha_{J-1,t+J}B_{t+J}\right) \\ +\beta^{J+1}\lambda_{t+J+1}(1-\delta)^J\varphi_{J-1,t+J}\left(\frac{\partial y_{J,t+J+1}}{\partial k_{J,t+J+1}} + (1-\delta)B_{t+J+1}\right) \end{pmatrix},$$

where $\varphi_{j,t+1+j}$ is the probability of not adjusting between t and $t+j$ if you adjusted at time t :

$$\forall j = 0...J-1 : \varphi_{j,t+1+j} = \prod_{i=0}^{j}(1-\alpha_{i,t+1+i}).$$

Using the recursion for the μ_{jt} defined above, one can obtain the Thomas first order condition:

$$\mu_{0,t} = \mathbb{E}_t\left(\frac{\beta\lambda_{t+1}}{\lambda_t}\left(\frac{\partial y_{0,t+1}}{\partial k_{0,t+1}} + (1-\delta)\left(\alpha_{0,t+1}B_{t+1} + (1-\alpha_{0,t+1})\mu_{1,t+1}\right)\right)\right)$$

$$\mu_{0,t} = \mathbb{E}_t\left(\frac{\beta\lambda_{t+1}}{\lambda_t}\left(\begin{matrix} \frac{\partial y_{0,t+1}}{\partial k_{0,t+1}} + (1-\delta)\times ... \\ \left(\alpha_{0,t+1}B_{t+1} + (1-\alpha_{0,t+1})\frac{\beta\lambda_{t+2}}{\lambda_{t+1}}\left(\frac{\partial y_{1,t+2}}{\partial k_{1,t+2}} + (1-\delta)\left(\alpha_{1,t+2}B_{t+2} + (1-\alpha_{1,t+2})\mu_{2,t+2}\right)\right)\right) \end{matrix}\right)\right)$$

Thus by iterating:

$$\lambda_t\mu_{0,t} = \mathbb{E}_t \begin{pmatrix} \beta\lambda_{t+1}\left(\frac{\partial y_{0,t+1}}{\partial k_{0,t+1}} + (1-\delta)\alpha_{0,t+1}B_{t+1}\right) \\ +\beta^2\lambda_{t+2}(1-\alpha_{0,t+1})(1-\delta)\left(\frac{\partial y_{1,t+2}}{\partial k_{1,t+2}} + (1-\delta)\alpha_{1,t+2}B_{t+2}\right) \\ +...+ \\ +\beta^{J+1}\lambda_{t+J+1}\varphi_{J-1,t+J})(1-\delta)^J\left(\frac{\partial y_{J,t+J+1}}{\partial k_{J,t+J+1}} + (1-\delta)B_{t+J+1}\right) \end{pmatrix}$$

Given that $\mu_{0,t} = B_t$, this is the same as Thomas.

C. Balanced growth path and stationarized model

Thereafter I denote the (gross) growth rate for a variable x with Θ_x. We look for BGP with A growing at rate Θ_a, B at rate Θ_b and J constant. Standard arguments then imply that c, y, v, w grow at a same rate Θ_y and the growth rate of i_j is rate $\frac{\Theta_c}{\Theta_b}$, and N_t and n_{jt} are constant. Finally k_{jt} grow at rate $\frac{\Theta_c}{\Theta_b}$. To determine these growth rates, observe that $\Theta_c = \Theta_a\left(\frac{\Theta_c}{\Theta_b}\right)^\gamma$, whence $\Theta_c = \left(\Theta_a\Theta_b^{-\gamma}\right)^{\frac{1}{1-\gamma}}$, and we can thereafter compute the growth rates of all the variables.

Let's now rewrite the equation in terms of detrended variables. A variable with a $\widetilde{\ }$ tilde is divided by its trend growth rate, e.g. $\widetilde{c_t} = \frac{C_t}{\Theta_c^t}$. I do not put a tilde above variables that are trendless. Note also that $\widetilde{A_t} = z_t^A$ and $\widetilde{B_t} = z_t^B$.

$$\begin{aligned}
\forall j &= 0...J : \widetilde{y}_{jt} = z_t^a \widetilde{k}_{jt}^{\gamma} n_{jt}^{\upsilon} \\
\forall j &= 0...J-1 : \widetilde{k}_{j+1,t+1} = \left(\frac{1-\delta}{\Theta_k}\right)\widetilde{k}_{j,t} \\
\widetilde{k}_{0,t+1} &= \left(\frac{1-\delta}{\Theta_k}\right)\widetilde{k}_{j,t} + \frac{1}{\Theta_k}\widetilde{i}_{jt}
\end{aligned}$$

$$\begin{aligned}
\theta_{0,t+1} &= \sum_{j=0}^{J} \alpha_{jt}\theta_{jt} \\
\forall j &= 1...J : \theta_{j,t+1} = \theta_{j-1,t}(1-\alpha_{j-1,t})
\end{aligned}$$

$$\begin{aligned}
\widetilde{c}_t &= \sum_{j=0}^{J} \theta_{jt}\widetilde{y}_{jt} - z_t^b \sum_{j=0}^{J} \theta_{jt}\alpha_{jt}\widetilde{i}_{jt} \\
N_t &= \sum_{j=0}^{J} \theta_{jt} n_{jt} + \sum_{j=0}^{J} \theta_{jt} R(\alpha_{jt})
\end{aligned}$$

$$\forall j = 0...J : R(\alpha_{jt}) = \int_0^{G^{-1}(\alpha_{jt})} x dG(x)$$

$$\begin{aligned}
\widetilde{\lambda}_t &= u_1(\widetilde{c}_t, 1-N_t) \\
\widetilde{w}_t &= \frac{u_2(\widetilde{c}_t, 1-N_t)}{u_1(\widetilde{c}_t, 1-N_t)} \\
\widetilde{n}_{jt} &= \left(\frac{\upsilon z_t^a \widetilde{k}_{jt}^{\gamma}}{\widetilde{w}_t}\right)^{\frac{1}{1-\upsilon}}
\end{aligned}$$

$$\widetilde{v}_{0t} - \widetilde{v}_{j+1t} = z_t^b \widetilde{i}_{jt} + \widetilde{w}_t G^{-1}(\alpha_{jt})$$

$$\widetilde{v}_{jt} = \mathbb{E}_t\left(\Theta_y\Theta_\lambda \frac{\beta\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left(\begin{array}{c} \widetilde{y}_{j,t+1} - \widetilde{w}_{t+1} n_{j,t+1} - \alpha_{j,t+1} z_{t+1}^b \widetilde{i}_{j,t+1} \\ +\alpha_{j,t+1}\widetilde{v}_{0,t+1} + (1-\alpha_{j,t+1})\widetilde{v}_{j+1,t+1} - \widetilde{w}_{t+1} R(\alpha_{j,t+1}) \end{array}\right)\right)$$

$$\widetilde{\mu}_{j,t} = \beta\Theta_\lambda\Theta_b\mathbb{E}_t\left(\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left(MPK_{j,t+1} + (1-\delta)\left(\alpha_{j,t+1}\widetilde{z}_{t+1}^b + (1-\alpha_{j,t+1})\widetilde{\mu}_{j+1,t+1}\right)\right)\right)$$

where $MPK_{j,t+1} = \gamma\left(z_{t+1}^a\right)^{\frac{1}{1-\upsilon}} \widetilde{w}_{t+1}^{-\frac{\upsilon}{1-\upsilon}} \widetilde{k}_{j,t+1}^{\gamma-1+\frac{\gamma\upsilon}{1-\upsilon}} \upsilon^{\frac{\upsilon}{1-\upsilon}}$.

Given that $\alpha_{J,t} = 1$, we have

$$\widetilde{\mu}_{J,t} = \beta\Theta_\lambda\Theta_b E_t\left(\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left(MPK_{J,t+1} + (1-\delta)\widetilde{z}^b_{t+1}\right)\right),$$

$$\widetilde{\mu}_{0,t} = \widetilde{z}^b_t.$$

D. Computation of the Nonstochastic Steady-State

Now we look for the steady-state values of the nondetrended variables when all shocks $(\varepsilon^A_t, \varepsilon^B_t)$ are set equal to 0 (or equivalently $z^a_t = z^b_t = 1$). I denote the nonstochastic steady-state variables with *. The program takes J as given and assumes $\alpha_J = 1$, then computes the steady-state values. One can then iterate over J, starting from a low value, until one finds $\alpha_{J-1} = 1$. This shows that $J-1$ is the correct number of vintages.

Unknown variables: $\{y_j, k_j, i_j, n_j, \theta_j, v_j, \mu_j\}_{j=0}^{J}$, $\{\alpha_j\}_{j=0}^{J-1}$ and $\{c, n, w, \lambda\}$, whence a total of $7(J+1) + J + 4 = 8J + 11$ unknowns. Note that $\alpha_J = 1$.

$$\forall j = 0...J : y^*_j = k^{*\gamma}_j n^{*\upsilon}_j \tag{1.1}$$
$$\forall j = 0...J-1 : k^*_{j+1} = \left(\frac{1-\delta}{\Theta_k}\right) k^*_j \tag{1.2}$$
$$\forall j = 0...J : k^*_0 = \left(\frac{1-\delta}{\Theta_k}\right) k^*_j + \frac{1}{\Theta_k} i^*_j \tag{1.3}$$

$$\theta^*_0 = \sum_{j=0}^{J} \alpha^*_j \theta^*_j \tag{1.4}$$
$$\forall j = 1...J : \theta^*_j = \theta^*_{j-1}(1-\alpha^*_{j-1}) \tag{1.5}$$

$$c^* = \sum_{j=0}^{J} \theta^*_j y^*_j - \sum_{j=0}^{J} \theta^*_j \alpha^*_j i^*_j \tag{1.6}$$
$$N^* = \sum_{j=0}^{J} \theta^*_j n^*_j + \sum_{j=0}^{J} \theta^*_j R(\alpha^*_j) \tag{1.7}$$

$$\lambda^* = u_1(c^*, 1-N^*) \tag{1.8}$$
$$w^* = \frac{u_2(c^*, 1-N^*)}{u_1(c^*, 1-N^*)} \tag{1.9}$$
$$\forall j = 0...J : n^*_j = \left(\frac{\upsilon k^{*\gamma}_j}{w^*}\right)^{\frac{1}{1-\upsilon}} \tag{1.10}$$

$$\forall j = 0...J-1 : v_0^* - v_{j+1}^* = i_j^* + w^* G^{-1}(\alpha_j^*) \tag{1.11}$$

$$\forall j = 0...J-1 : v_j^* = \Theta_y \Theta_\lambda \beta \left(y_j^* - w^* n_j^* - \alpha_j^* i_j^* + \alpha_j^* v_0^* + (1-\alpha_j^*) v_{j+1}^* - w^* R(\alpha_j^*) \right) \tag{1.12}$$

$$v_J^* = \Theta_y \Theta_\lambda \beta \left(y_J^* - w^* n_J^* - i_J^* + v_0^* - w^* R(1) \right) \tag{1.13}$$

$$\forall j = 0...J : \mu_j^* = \beta \Theta_\lambda \Theta_b \left(MPK_j^* + (1-\delta) \left(\alpha_j^* + (1-\alpha_j^*) \mu_{j+1}^* \right) \right) \tag{1.14}$$

$$\mu_0^* = 1 \tag{1.15}$$

We have a system of:

$$\begin{aligned} & (J+1) + J + (J+1) + 1 + J + 1 + 1 + 1 + 1 + (J+1) + J + J + 1 + J + 1 + 1 \\ = \;& 8J + 11 \end{aligned}$$

equations in $8J+11$ unknowns.[1]

I solve this system by the following algorithm: given a guess for (k_0, w, α_{J-1}), I use (1.2) to deduce all the k_j's and use (1.3) to find the i_j's. From (1.10) compute the n_j's and then the y_j's from (1.1). Next, given α_{J-1}, I use (1.11) and (1.13) to solve for v_0, v_J. Next, I iterate in the following manner to find the v_j's and α_j's: use (1.12) for $j = J-1$ to find v_{J-1}. Given v_{J-1}, use (1.11) for $j = J-2$ to find α_{J-2}. Go back to (1.12) and find v_{J-2}. And so on you can get all the v_j's and α_j's. Given the α's now use (1.4) and (1.5) to compute the θ's: make a guess for θ_0, then infer all the θ_j's from (1.5), then compute θ_0. Given the θ's, compute c^*, N^*, λ^* and $\frac{u_2}{u_1}$. Finally, given μ_J given by equation 1.14 for $j = J$, work backward to find all the μ_j's. To conclude, we check whether our guess (k_0, w, α_{J-1}) was correct or not: for this we use equations 1.12, 1.9 and 1.15. This implicitely defines a mapping from $R^{J+2} \rightarrow R^{J+2}$ for which we find a zero numerically using FSOLVE in MATLAB(c). For this model and the extensions that we consider, this method works very well: we are able to find the steady-state with a precision of 1e-13, often less, as maximum error for each of the $J+2$ equations.

ADDING A CONSTANT TO LOOK AT PERMANENT TFP CHANGES

$$\begin{aligned} \forall j &= 0...J : y_j^* = A k_j^{*\gamma} n_j^{*\upsilon} \\ \forall j &= 0...J-1 : k_{j+1}^* = \left(\frac{1-\delta}{\Theta_k} \right) k_j^* \\ \forall j &= 0...J : k_0^* = \left(\frac{1-\delta}{\Theta_k} \right) k_j^* + \frac{1}{\Theta_k} i_j^* \end{aligned}$$

[1] Note that the equation $\sum_{j=0}^{J} \theta_j^* = 1$ is actually redundant given (1.4).

$$\theta_0^* = \sum_{j=0}^{J} \alpha_j^* \theta_j^*$$
$$\forall j = 1...J : \theta_j^* = \theta_{j-1}^*(1-\alpha_{j-1}^*)$$

$$c^* = \sum_{j=0}^{J} \theta_j^* y_j^* - \sum_{j=0}^{J} \theta_j^* \alpha_j^* i_j^*$$
$$N^* = \sum_{j=0}^{J} \theta_j^* n_j^* + \sum_{j=0}^{J} \theta_j^* R(\alpha_j^*)$$

$$\lambda^* = u_1(c^*, 1-N^*)$$
$$w^* = \frac{u_2(c^*, 1-N^*)}{u_1(c^*, 1-N^*)}$$
$$\forall j = 0...J : n_j^* = \left(\frac{\upsilon A k_j^{*\gamma}}{w^*}\right)^{\frac{1}{1-\upsilon}}$$

$$\forall j = 0...J-1 : v_0^* - v_{j+1}^* = i_j^* + w^* G^{-1}(\alpha_j^*)$$
$$\forall j = 0...J-1 : v_j^* = \Theta_y \Theta_\lambda \beta \left(y_j^* - w^* n_j^* - \alpha_j^* i_j^* + \alpha_j^* v_0^* + (1-\alpha_j^*) v_{j+1}^* - w^* R(\alpha_j^*)\right)$$
$$v_J^* = \Theta_y \Theta_\lambda \beta \left(y_J^* - w^* n_J^* - i_J^* + v_0^* - w^* R(1)\right)$$
$$\forall j = 0...J : \mu_j^* = \beta \Theta_\lambda \Theta_b \left(MPK_j^* + (1-\delta)\left(\alpha_j^* + (1-\alpha_j^*)\mu_{j+1}^*\right)\right)$$
$$\mu_0^* = 1$$

E. Log-Linearization of the equations around the nonstochastic steady-state

I now compute the log-linearized first-order conditions. I take the equations from step 2 and log-linearize them around the nonstochastic steady-state we computed in step 3. A ^ denotes a first-order term i.e. $\widehat{c_t} = \log\left(\frac{\widetilde{c_t}}{c^*}\right) \simeq \frac{\widetilde{c}_t - c^*}{c^*}$ is the % deviation from the steady-state. I use the following fact: $\widehat{f(x_t)} = \frac{f(x_t)-f(x^*)}{f(x^*)} = f'(x^*)\frac{x_t - x^*}{x^*}\frac{x^*}{f(x^*)} = \widehat{x_t}\frac{f'(x^*)x^*}{f(x^*)}$.

The system of log-linearized equations is the following:

$$\forall j = 0...J : \widehat{y}_{jt} = \gamma \widehat{k}_{jt} + \upsilon \widehat{n}_{jt} + \widehat{z}_t^a \quad (1.16)$$
$$\forall j = 0...J-1 : \widehat{k}_{j+1,t+1} = \widehat{k}_{j,t} \quad (1.17)$$
$$\forall j = 0...J : \widehat{k}_{0,t+1} = \frac{\left(\frac{1-\delta}{\Theta_k}\right) k_j^*}{k_0^*} \widehat{k}_{j,t} + \frac{\frac{1}{\Theta_k} i_j^*}{k_0^*} \widehat{i}_{jt} \quad (1.18)$$

$$\widehat{\theta}_{0,t+1} = \sum_{j=0}^{J} \left(\widehat{\alpha}_{jt} + \widehat{\theta}_{jt}\right) \frac{\alpha_j^* \theta_j^*}{\theta_0^*} \tag{1.19}$$

$$\forall j = 1...J : \widehat{\theta}_{j,t+1} = \widehat{\theta}_{j-1,t} + \frac{-\alpha_{j-1}^*}{1-\alpha_{j-1}^*} \widehat{\alpha}_{j-1,t} \tag{1.20}$$

$$\frac{c^*}{y^*}\widehat{c}_t + \frac{i^*}{y^*}\sum_{j=0}^{J} \frac{\theta_j^* \alpha_j^* i_j^*}{i^*} \left(\widehat{z}_t^B + \widehat{\theta}_{jt} + \widehat{\alpha}_{jt} + \widehat{i}_{jt}\right) = \sum_{j=0}^{J} \frac{\theta_j^* y_j^*}{y^*} \left(\widehat{\theta}_{jt} + \widehat{y}_{jt}\right) \tag{1.21}$$

$$\rightarrow$$

$$c^*\widehat{c}_t + \sum_{j=0}^{J} \theta_j^* \alpha_j^* i_j^* \left(\widehat{z}_t^B + \widehat{\theta}_{jt} + \widehat{\alpha}_{jt} + \widehat{i}_{jt}\right) = \sum_{j=0}^{J} \theta_j^* y_j^* \left(\widehat{\theta}_{jt} + \widehat{y}_{jt}\right)$$

$$\sum_{j=0}^{J} \theta_j^* n_j^* \left(\widehat{\theta}_{jt} + \widehat{n}_{jt}\right) + \sum_{j=0}^{J} \theta_j^* R(\alpha_j^*) \left(\widehat{\theta}_{jt} + \frac{R'(\alpha_j^*)\alpha_j^*}{R(\alpha_j^*)}\widehat{\alpha}_{jt}\right) = N^* \widehat{N}_t \tag{1.22}$$

Given the utility function of the Hansen-Rogerson form,

$$\widehat{\lambda}_t = -\widehat{c}_t \tag{1.23}$$

$$\widehat{w}_t = \widehat{c}_t \tag{1.24}$$

These equations need to be adapted if we change the utility function. In our modification #2, we use a general utility function defined by its intertemporal elasticity of substitution $1/\sigma$ and its Frisch elasticity ε_{nw}. Following Rotemberg and Woodford (1995; in Cooley "Frontiers of Business Cycle Research) we have the system of first-order equations

$$\widehat{c}_t = \varepsilon_{c\lambda}\widehat{\lambda}_t + \varepsilon_{cw}\widehat{w}_t,$$

$$\widehat{n}_t = \varepsilon_{n\lambda}\widehat{\lambda}_t + \varepsilon_{nw}\widehat{w}_t,$$

where $\varepsilon_{n\lambda} = \varepsilon_{nw}/\sigma$, $\varepsilon_{cw} = \varepsilon_{nw}(\sigma-1)/\sigma \times wN/C$, and $\varepsilon_{c\lambda} = (\varepsilon_{cw} - 1)/\sigma$. (Note that to use this parametrization, we solve the steady-state to obtain a labor supply of 0.2 in the steady-state, rather than

$$\widehat{n}_{jt} = \frac{1}{1-\upsilon}\left(\gamma\widehat{k}_{jt} - \widehat{w}_t + \widehat{z}_t^A\right) \tag{1.25}$$

The next computations are slightly more intricate, so I decompose them. Start with:

$$\forall j = 0...J-1 : \widetilde{v}_{0t} - \widetilde{v}_{j+1t} = z_t^B \widetilde{i}_{jt} + \widetilde{w}_t G^{-1}(\alpha_{jt})$$

The loglinear corresponding equation is

$$\begin{aligned}\forall j &= 0...J-1: \\ \frac{v_0^*}{v_0^*-v_{j+1}^*}\widehat{v}_{0t} - \frac{v_{j+1}^*}{v_0^*-v_{j+1}^*}\widehat{v}_{j+1} &= \frac{i_j^*}{v_0^*-v_{j+1}^*}\left(\widehat{z}_t^b + \widehat{i}_{jt}\right) + \left(1-\frac{i_j^*}{v_0^*-v_{j+1}^*}\right)\left(\widehat{w}_t + \frac{G^{-1\prime}(\alpha_j)^*\alpha_j^*}{G^{-1}(\alpha_j)^*}\widehat{\alpha}_{jt}\right)\end{aligned}$$

which we can simplify to

$$v_0^*\widehat{v}_{0t} - v_{j+1}^*\widehat{v}_{j+1} = i_j^*\left(\widehat{z}_t^b + \widehat{i}_{jt}\right) + \left(v_0^* - v_{j+1}^* - i_j^*\right)\left(\widehat{w}_t + \frac{G^{-1\prime}(\alpha_j)^*\alpha_j^*}{G^{-1}(\alpha_j)^*}\widehat{\alpha}_{jt}\right) \tag{1.26}$$

I use $G^{-1\prime} = \frac{1}{G'(G^{-1}(.))}$ to compute the last term.

Next we have the equation:

$$\widetilde{v}_{jt} = \mathbb{E}_t\left(\frac{\beta\Theta_y\Theta_\lambda\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left(\begin{array}{c}\widetilde{y}_{j,t+1} - \widetilde{w}_{t+1}n_{j,t+1} - \alpha_{j,t+1}z_{t+1}^b\widetilde{i}_{j,t+1} \\ +\alpha_{j,t+1}\widetilde{v}_{0,t+1} + (1-\alpha_{j,t+1})\widetilde{v}_{j+1,t+1} - \widetilde{w}_{t+1}R(\alpha_{j,t+1})\end{array}\right)\right) \tag{1.27}$$

leading to:

$$\widehat{v}_{jt} = \mathbb{E}_t\left(\begin{array}{c}\widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\beta\Theta_y\Theta_\lambda y_j^*}{v_j^*}\widehat{y}_{j,t+1} + \frac{-\beta\Theta_y\Theta_\lambda w^* n_j^*}{v_j^*}\left(\widehat{w}_{t+1} + \widehat{n}_{j,t+1}\right) + \frac{-\beta\Theta_y\Theta_\lambda i_j^*\alpha_j^*}{v_j^*}\left(\widehat{\alpha}_{j,t+1} + \widehat{z}_{t+1}^b + \widehat{i}_{j,t+1}\right) \\ +\frac{\beta\Theta_y\Theta_\lambda v_0^*\alpha_j^*}{v_j^*}\left(\widehat{\alpha}_{j,t+1} + \widehat{v}_{0,t+1}\right) + \frac{\beta\Theta_y\Theta_\lambda v_{j+1}^*(1-\alpha_j^*)}{v_j^*}\left(-\frac{\alpha_j}{1-\alpha_j}\widehat{\alpha}_{j,t+1} + \widehat{v}_{j+1,t+1}\right) \\ +\frac{-\beta\Theta_y\Theta_\lambda w^* R(\alpha_j^*)}{v_j^*}\left(\widehat{w}_{t+1} + \frac{R'(\alpha_j^*)\alpha_j^*}{R(\alpha_j^*)}\widehat{\alpha}_{j,t+1}\right)\end{array}\right) \tag{1.28}$$

The equation for $j = J$ is the same except that $\alpha_{J,t+1} = 1$ and thus $\widehat{\alpha}_{J,t+1} = 0$.

Finally

$$\begin{aligned} j &= 0...J-1 : \widehat{\mu}_{j,t} = \\ & E_t\left(\begin{array}{c}\widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\gamma w^{*-\frac{v}{1-v}} k_j^{*\gamma-1+\frac{\gamma v}{1-v}} v^{\frac{v}{1-v}}}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\left(\frac{1}{1-v}\widehat{z}_{t+1}^a - \frac{v}{1-v}\widehat{w}_{t+1} + \left(\gamma - 1 + \frac{\gamma v}{1-v}\right)\widehat{k}_{j,t+1}\right) \\ +\left(1 - \frac{\gamma w^{*-\frac{v}{1-v}} k_j^{*\gamma-1+\frac{\gamma v}{1-v}} v^{\frac{v}{1-v}}}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\right) \times .. \\ ..\left(\frac{\alpha_j^*}{\alpha_j^*+(1-\alpha_j^*)\mu_{j+1}^*}\left(\widehat{\alpha_{j,t+1}} + \widehat{z}_{t+1}^b\right) + \left(1 - \frac{\alpha_j^*}{\alpha_j^*+(1-\alpha_j^*)\mu_{j+1}^*}\right)\left(\frac{-\alpha_j^*}{1-\alpha_j^*}\widehat{\alpha}_{j,t+1} + \widehat{\mu}_{j+1,t+1}\right)\right)\end{array}\right)\end{aligned} \tag{1.29}$$

and similarly

$$\widehat{\mu}_{J,t} = E_t\left(\begin{array}{c}\widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\gamma w^{*-\frac{v}{1-v}} k_J^{*\gamma-1+\frac{\gamma v}{1-v}} v^{\frac{v}{1-v}}}{\frac{\mu_J^*}{\beta\Theta_\lambda\Theta_b}}\left(\frac{1}{1-v}\widehat{z}_{t+1}^a - \frac{v}{1-v}\widehat{w}_{t+1} + \left(\gamma - 1 + \frac{\gamma v}{1-v}\right)\widehat{k}_{J,t+1}\right) \\ +\left(1 - \frac{\gamma w^{*-\frac{v}{1-v}} k_J^{*\gamma-1+\frac{\gamma v}{1-v}} v^{\frac{v}{1-v}}}{\frac{\mu_J^*}{\beta\Theta_\lambda\Theta_b}}\right)\widehat{z}_{t+1}^b\end{array}\right) \tag{1.30}$$

And:

$$\widehat{\mu}_{0,t} = \widehat{z}_t^b \tag{1.31}$$

Finally we have the shocks law of motions:

$$\widehat{z}_{t+1}^i = \rho_i \widehat{z}_t^i + \varepsilon_{t+1}^i, i \in \{A, B\} \tag{1.32}$$

This system of equations can be fed in a linear system solver. I use Paul Klein's programs for this.

2 Extensions: Random Breakdowns, Embodied Technology

Addition #1 as compared to Thomas: we add an investment price term B_t to the resource constraint. $B_t = X_t^B z_t^B$ with $\log z_t^B = \rho^b \log z_{t-1}^B + \varepsilon_t^B$ and $\frac{X_{t+1}^B}{X_t^B} = \Theta_b$, and ε_t^B iid $N(0, \sigma_b^2)$.

Addition #2: we add the random breakdowns.

I assume that each period, each unit is subject to a probability of breakdown λ. A breakdown requires an investment ψk_{jt} to replace lost capital; this occurs before the decision to invest is made. i_{jt} is the investment not including the possible breakdown.

Shock:	Breakdown		No Breakdown	
Decision:	Adjust	Not Adjust	Adjust	Not Adjust
Probability:	$\lambda\theta_{jt}\alpha_{jt}$	$\lambda\theta_{jt}(1-\alpha_{jt})$	$(1-\lambda)\theta_{jt}\alpha_{jt}$	$(1-\lambda)\theta_{jt}(1-\alpha_{jt})$
Investment Rate:	$\frac{i_{jt}}{k_{jt}} + \psi$	ψ	$\frac{i_{jt}}{k_{jt}}$	0

[Addition #3: not used in the paper and not transcribed here: convex adjustment to changing k_0]

A. Deriving the First-Order Conditions

The planner problem is now to choose $\{c_t, n_{jt}, \theta_{jt+1}, \alpha_{jt}, k_{jt+1}, i_{jt}\}$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(\sum_{j=0}^{J} \theta_{jt}\left(n_{jt} + R(\alpha_{jt})\right) \right) \right]$$

$$\begin{aligned}
s.t. \quad &: \quad c_t + \sum_{j=0}^{J} \theta_{jt}\alpha_{jt} i_{jt} + \psi\lambda \sum_{j=0}^{J} \theta_{jt} k_{jt} \leq \sum_{j=0}^{J} \theta_{jt} A_t k_{jt}^{\gamma} n_{jt}^{\upsilon} \\
k_{j+1,t+1} &= (1-\delta)\, k_{j,t} \text{ for } j = 0...J-1 \\
k_{0,t+1} &= (1-\delta) k_{jt} + i_{jt} \text{ for } j = 0...J \\
\theta_{j+1,t+1} &= (1-\alpha_{jt})\theta_{jt} \text{ for } j = 0...J-1 \\
\theta_{0,t+1} &= \sum_{j=0}^{J} \theta_{jt}\alpha_{jt}
\end{aligned}$$

The Lagrangean is:

$$\begin{aligned}
L \;=\; & \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(\sum_{j=0}^{J} \theta_{jt}\left(n_{jt} + R(\alpha_{jt})\right) \right) \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\sum_{j=0}^{J} \theta_{jt} A_t k_{jt}^{\gamma} n_{jt}^{\upsilon} - c_t - \sum_{j=0}^{J} \theta_{jt}\alpha_{jt} i_{jt} - \psi\lambda \sum_{j=0}^{J} \theta_{jt} k_{jt} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \left(\sum_{j=0}^{J-1} \zeta_{j+1t}\left((1-\delta)\, k_{j,t} - k_{j+1,t+1} \right) \right) + \sum_{t=0}^{\infty} \beta^t \sum_{j=0}^{J-1} \eta_{j+1t}\left((1-\alpha_{jt})\theta_{jt} - \theta_{j+1t+1} \right)
\end{aligned}$$

$$+\sum_{t=0}^{\infty}\beta^t\eta_{0t}\left(\sum_{j=0}^{J}\theta_{jt}\alpha_{jt}-\theta_{0t+1}\right)+\sum_{t=0}^{\infty}\beta^t\sum_{j=0}^{J}\pi_{jt}\left((1-\delta)k_{jt}+i_{jt}-k_{0,t+1}\right)$$

The first-order conditions are:

FOC wrt c_t :

$$u'(c_t)=\lambda_t$$

FOC wrt n_{jt} :

$$v'(N_t)\theta_{jt}=\upsilon\theta_{jt}A_tk_{jt}^{\gamma}n_{jt}^{\upsilon-1}\lambda_t$$

Let $w_t=v'(N_t)/\lambda_t$, then $\upsilon A_tk_{jt}^{\gamma}n_{jt}^{\upsilon-1}=w_t$.

FOC wrt i_{jt} :

$$-\lambda_t\theta_{jt}\alpha_{jt}+\pi_{jt}=0$$

FOC wrt α_{jt} :

$$-v'(N_t)\theta_{jt}R'(\alpha_{jt})-\lambda_t\theta_{jt}i_{jt}-\theta_{jt}\eta_{j+1t}+\eta_{0t}\theta_{jt}=0$$

$$-w_tR'(\alpha_{jt})-i_{jt}-\frac{\eta_{j+1t}}{\lambda_t}+\frac{\eta_{0t}}{\lambda_t}=0$$

and since $R'(x)=G^{-1}(x)$, denoting $v_{jt}=\frac{\eta_{jt}}{\lambda_t}$ (i.e. transforming from utils units into goods units):

$$v_{0t}-w_tG^{-1}(\alpha_{jt})-i_{jt}=v_{j+1t}$$

FOC wrt θ_{jt}, for $j\neq 0$:

$$-v'(N_t)\left(n_{jt}+R(\alpha_{jt})\right)+\lambda_t\left[A_tk_{jt}^{\gamma}n_{jt}^{\upsilon}-\alpha_{jt}i_{jt}-\psi\lambda k_{jt}\right]+\eta_{j+1t}(1-\alpha_{j,t})-\frac{\eta_{jt-1}}{\beta}+\eta_{0,t}\alpha_{jt}=0$$

and as before this equation also holds for $j=J$. Defining $v_{jt}=\eta_{jt}/\lambda_t$ yields

$$v_{jt-1}=\frac{\beta\lambda_t}{\lambda_{t-1}}\left(-w_t\left(n_{jt}+R(\alpha_{jt})\right)+A_tk_{jt}^{\gamma}n_{jt}^{\upsilon}-\alpha_{jt}i_{jt}-\psi\lambda k_{jt}+v_{j+1t}(1-\alpha_{j,t})+v_{0,t}\alpha_{jt}\right).$$

FOC wrt θ_{0t} :

$$-v'(N_t)\left(n_{0t}+R(\alpha_{0t})\right)+\lambda_t\left[A_tk_{0t}^{\gamma}n_{0t}^{\upsilon}-\alpha_{0t}i_{0t}-\psi\lambda k_{0t}\right]+\eta_{1t}(1-\alpha_{0,t})-\frac{\eta_{0,t-1}}{\beta}+\eta_{0,t}\alpha_{0t}=0.$$

which can be rewritten as:

$$v_{0t-1}=\frac{\beta\lambda_t}{\lambda_{t-1}}\left(-w_t\left(n_{0t}+R(\alpha_{0t})\right)+A_tk_{0t}^{\gamma}n_{0t}^{\upsilon}-\alpha_{0t}i_{0t}-\psi\lambda k_{0t}+v_{1t}(1-\alpha_{0,t})+v_{0,t}\alpha_{0t}\right).$$

FOC wrt k_{jt}, for $j \neq 0$:

$$\lambda_t \left(\theta_{jt} MPK_{jt} - \psi\lambda\theta_{jt}\right) + (1-\delta)\pi_{jt} + \zeta_{j+1,t}(1-\delta) - \zeta_{j,t-1}\frac{1}{\beta} = 0$$

This can be rewritten as

$$\zeta_{j,t-1} = \beta\zeta_{j+1,t}(1-\delta) + \beta(1-\delta)\pi_{jt} + \beta\lambda_t \left(\theta_{jt} MPK_{jt} - \psi\lambda\theta_{jt}\right)$$

Define $s_{jt} = \frac{\zeta_{jt}}{\lambda_t \theta_{j-1t}}$, then we have:

$$s_{j,t-1}\lambda_{t-1}\theta_{j-1t-1} = \beta\theta_{jt}\lambda_t s_{j+1,t}(1-\delta) + \beta(1-\delta)\alpha_{jt}\lambda_t\theta_{jt} + \beta\lambda_t \left(\theta_{jt} MPK_{jt} - \psi\lambda\theta_{jt}\right)$$

$$\frac{s_{j,t-1}}{1-\alpha_{j-1,t-1}} = \frac{\beta\lambda_t}{\lambda_{t-1}} \left[s_{j+1,t}(1-\delta) + (1-\delta)\alpha_{jt} + MPK_{jt} - \psi\lambda\right]$$

$$\frac{s_{j,t-1}}{1-\alpha_{j-1,t-1}} = \frac{\beta\lambda_t}{\lambda_{t-1}} \left[\frac{s_{j+1,t}}{1-\alpha_{j,t}}(1-\alpha_{jt})(1-\delta) + (1-\delta)\alpha_{jt} + MPK_{jt} - \psi\lambda\right]$$

Let $\mu_{jt} = \frac{s_{jt}}{1-\alpha_{j-1t}}$, then:

$$\mu_{j,t-1} = \frac{\beta\lambda_t}{\lambda_{t-1}} \left[\mu_{j+1,t}(1-\alpha_{jt})(1-\delta) + (1-\delta)\alpha_{jt} + MPK_{jt} - \psi\lambda\right],$$

and as before this equation also holds for $j = J$.

FOC wrt k_{0t} :

$$\lambda_t \left(\theta_{0t} MPK_{0t} - \psi\lambda\theta_{0t}\right) + (1-\delta)\pi_{0t} + \zeta_{1,t}(1-\delta) - \frac{1}{\beta}\sum_{j=0}^{J}\pi_{jt-1} = 0$$

$$\left(\theta_{0t} MPK_{0t} - \psi\lambda\theta_{0t}\right) + (1-\delta)\frac{\pi_{0t}}{\lambda_t} + \frac{\zeta_{1,t}}{\lambda_t}(1-\delta) = \frac{\lambda_{t-1}}{\lambda_t\beta}\sum_{j=0}^{J}\theta_{j-1t}\alpha_{j-1t}$$

$$1 = \frac{\beta\lambda_t}{\lambda_{t-1}} \left[(MPK_{0t} - \psi\lambda) + (1-\delta)\alpha_{0t} + \mu_{1,t}(1-\alpha_{0t})(1-\delta)\right]$$

B. List of all the equations

The equations which change are the following:

$$k_{j+1,t+1} = (1-\delta)\, k_{j,t}$$

$$c_t + \sum_{j=0}^{J}\theta_{jt}\alpha_{jt}i_{jt} + \psi\lambda\sum_{j=0}^{J}\theta_{jt}k_{jt} \leq \sum_{j=0}^{J}\theta_{jt}A_t k_{jt}^{\gamma} n_{jt}^{v}$$

$$N_t = \sum_{j=0}^{J}\theta_{jt}\left(n_{jt} + R(\alpha_{jt})\right)$$

$$v_{0t} - w_t G^{-1}(\alpha_{jt}) - i_{jt} = v_{j+1t}.$$

$$\mu_{j,t-1} = \frac{\beta\lambda_t}{\lambda_{t-1}} \left[\mu_{j+1,t}(1-\alpha_{jt})(1-\delta) + (1-\delta)\alpha_{jt} + MPK_{jt} - \psi\lambda\right],$$

$$1 = \frac{\beta\lambda_t}{\lambda_{t-1}} \left[(MPK_{0t} - \psi\lambda) + (1-\delta)\alpha_{0t} + \mu_{1,t}(1-\alpha_{0t})(1-\delta)\right]$$

$$v_{jt-1} = \frac{\beta\lambda_t}{\lambda_{t-1}} \left(-w_t\left(n_{jt} + R(\alpha_{jt})\right) + A_t k_{jt}^{\gamma} n_{jt}^{\upsilon} - \alpha_{jt} i_{jt} - \psi\lambda k_{jt} + v_{j+1t}(1-\alpha_{j,t}) + v_{0,t}\alpha_{jt}\right).$$

C. Balanced growth

I skip this step which is very close to the previous section.

D. Nonstochastic steady-state

The steady-state is characterized again by $8J + 11$ unknowns:

$$\begin{aligned}
\forall j &= 0...J : y_j^* = k_j^{*\gamma} n_j^{*\upsilon} \\
\forall j &= 0...J-1 : k_{j+1}^* = \left(\frac{1-\delta}{\Theta_k}\right) k_j^* \\
\forall j &= 0...J : k_0^* = \left(\frac{1-\delta}{\Theta_k}\right) k_j^* + \frac{1}{\Theta_k} i_j^*
\end{aligned}$$

$$\begin{aligned}
\theta_0^* &= \sum_{j=0}^{J} \alpha_j^* \theta_j^* \\
\forall j &= 1...J : \theta_j^* = \theta_{j-1}^*(1-\alpha_{j-1}^*)
\end{aligned}$$

$$\begin{aligned}
c^* &= \sum_{j=0}^{J} \theta_j^* y_j^* - \sum_{j=0}^{J} \theta_j^* \alpha_j^* i_j^* - \psi\lambda \sum_{j=0}^{J} \theta_j^* k_j^* \\
N^* &= \sum_{j=0}^{J} \theta_j^* n_j^* + \sum_{j=0}^{J} \theta_j^* R(\alpha_j^*) + \theta_0^* f(0) = \sum_{j=0}^{J} \theta_j^* n_j^* + \sum_{j=0}^{J} \theta_j^* R(\alpha_j^*)
\end{aligned}$$

$$\begin{aligned}
\lambda^* &= u_1(c^*, 1-N^*) \\
w^* &= \frac{u_2(c^*, 1-N^*)}{u_1(c^*, 1-N^*)} \\
\forall j &= 0...J : n_j^* = \left(\frac{\upsilon k_j^{*\gamma}}{w^*}\right)^{\frac{1}{1-\upsilon}}
\end{aligned}$$

$$\forall j = 0...J-1 : v_0^* - v_{j+1}^* = i_j^* + w^* G^{-1}(\alpha_j^*)$$

$$\forall j = 0...J-1 : v_j^* = \Theta_y \Theta_\lambda \beta \left(y_j^* - w^* n_j^* - \alpha_j^* i_j^* - \lambda \psi k_j^* + \alpha_j^* v_0^* + (1-\alpha_j^*) v_{j+1}^* - w^* R(\alpha_j^*) \right)$$

$$v_J^* = \Theta_y \Theta_\lambda \beta \left(y_J^* - w^* n_J^* - \lambda \psi k_J^* - i_J^* + v_0^* - w^* R(1) \right)$$

$$\forall j = 0...1 : \mu_j^* = \beta \Theta_\lambda \Theta_b \left(MPK_j^* - \psi \lambda + (1-\delta)\alpha_j^* + (1-\delta)(1-\alpha_j^*)\mu_{j+1}^* \right)$$

$$\begin{aligned} \mu_0^* &= \beta \Theta_\lambda \Theta_b \left[(MPK_0^* - \psi\lambda) + (1-\delta)\alpha_0^* + \mu_1^*(1-\alpha_0^*)(1-\delta) \right] \\ &= \beta \Theta_\lambda \Theta_b \left[(MPK_0^* - \psi\lambda) + (1-\delta)\alpha_0^* + \mu_1^*(1-\alpha_0^*)(1-\delta) \right] \end{aligned}$$

$$\mu_0^* = 1$$

Note the simplications that arise because I assume that $f(0) = f'(0) = 0$. This is common in adjustment cost specifications.

Note that the equations

$$v_J^* = \Theta_y \Theta_\lambda \beta \left(y_J^* - w^* n_J^* - i_J^* - \psi \lambda k_J^* + v_0^* - w^* R(1) \right)$$

$$v_0^* - v_J^* = i_{J-1}^* + w^* G^{-1}(\alpha_{J-1}^*)$$

imply that

$$v_J^* = \Theta_y \Theta_\lambda \beta \left(y_J^* - w^* n_J^* - i_J^* + v_J^* + i_{J-1}^* + w^* G^{-1}(\alpha_{J-1}^*) - \psi \lambda k_J^* - w^* R(1) \right)$$

$$v_J^* \left(1 - \Theta_y \Theta_\lambda \beta\right) = \Theta_y \Theta_\lambda \beta \left(y_J^* - w^* n_J^* - i_J^* + i_{J-1}^* + w^* G^{-1}(\alpha_{J-1}^*) - \psi \lambda k_J^* - w^* R(1) \right)$$

$$v_J^* = \frac{\Theta_y \Theta_\lambda \beta}{1 - \Theta_y \Theta_\lambda \beta} \left(y_J^* - w^* n_J^* - i_J^* + i_{J-1}^* + w^* G^{-1}(\alpha_{J-1}^*) - \psi k_{J-1}^* - w^* R(1) \right)$$

E. Log-linearization of equations

I only note the equations for which there is a change.

$$c^* \widehat{c}_t + \sum_{j=0}^{J} \theta_j^* \alpha_j^* i_j^* \left(z_t^B + \widehat{\theta}_{jt} + \widehat{\alpha}_{jt} + \widehat{i}_{jt} \right) + \psi \lambda \sum_{j=0}^{J} \theta_j^* k_j^* \left(\widehat{\theta}_{jt} + \widehat{k}_{jt} \right) = \sum_{j=0}^{J} \theta_j^* y_j^* \left(\widehat{\theta}_{jt} + \widehat{y}_{jt} \right)$$

$$\begin{aligned} \forall j &= 0...J-1 : \\ \frac{v_0^*}{v_0^* - v_{j+1}^*} \widehat{v}_{0t} - \frac{v_{j+1}^*}{v_0^* - v_{j+1}^*} \widehat{v}_{j+1} &= \frac{i_j^*}{v_0^* - v_{j+1}^*} \left(\widehat{z}_t^b + \widehat{i}_{jt} \right) + \left(1 - \frac{i_j^*}{v_0^* - v_{j+1}^*} \right) \left(\widehat{w}_t + \frac{G^{-1\prime}(\alpha_j)^* \alpha_j^*}{G^{-1}(\alpha_j)^*} \widehat{\alpha}_{jt} \right) \end{aligned}$$

which we can simplify to

$$v_0^*\widehat{v}_{0t} - v_{j+1}^*\widehat{v}_{j+1} = i_j^*\left(\widehat{z}_t^b + \widehat{i}_{jt}\right) + \left(v_0^* - v_{j+1}^* - i_j^*\right)\left(\widehat{w}_t + \frac{G^{-1\prime}(\alpha_j)^*\alpha_j^*}{G^{-1}(\alpha_j)^*}\widehat{\alpha}_{jt}\right)$$

Note that when G is uniform, $G(x) = x/B$ for $0 \leq x \leq B$ and $G^{-1}(y) = By$ for $0 \leq y \leq 1$. Thus

$$\frac{G^{-1\prime}(\alpha)\alpha}{G^{-1}(\alpha)} = \frac{B \times \alpha}{B\alpha} = 1,$$

which is independent of B or α.

Next:

$$\widehat{v}_{jt} = \mathbb{E}_t\left(\begin{array}{c} \widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\beta\Theta_y\Theta_\lambda y_j^*}{v_j^*}\widehat{y}_{j,t+1} + \frac{-\beta\Theta_y\Theta_\lambda w^* n_j^*}{v_j^*}\left(\widehat{w}_{t+1} + \widehat{n}_{j,t+1}\right) + \frac{-\beta\Theta_y\Theta_\lambda \boldsymbol{\lambda\psi}\mathbf{k}_j}{v_j}\widehat{k}_{jt+1} \\ \frac{-\beta\Theta_y\Theta_\lambda i_j^*\alpha_j^*}{v_j^*}\left(\widehat{\alpha}_{j,t+1} + \widehat{z}_{t+1}^b + \widehat{i}_{j,t+1}\right) \\ +\frac{\beta\Theta_y\Theta_\lambda v_0^*\alpha_j^*}{v_j^*}\left(\widehat{\alpha}_{j,t+1} + \widehat{v}_{0,t+1}\right) + \frac{\beta\Theta_y\Theta_\lambda v_{j+1}^*(1-\alpha_j^*)}{v_j^*}\left(-\frac{\alpha_j}{1-\alpha_j}\widehat{\alpha}_{j,t+1} + \widehat{v}_{j+1,t+1}\right) \\ +\frac{-\beta\Theta_y\Theta_\lambda w^* R(\alpha_j^*)}{v_j^*}\left(\widehat{w}_{t+1} + \frac{R'(\alpha_j^*)\alpha_j^*}{R(\alpha_j^*)}\widehat{\alpha}_{j,t+1}\right) \end{array}\right)$$

Note that if G is uniform, $R(x) = \int_0^{G^{-1}(x)} sg(s)ds = \int_0^{Bx}\frac{s}{B}ds = \frac{1}{2B}(Bx)^2 = \frac{Bx^2}{2}$ so

$$\frac{R'(\alpha)\alpha}{R(\alpha)} = \frac{B\alpha \times \alpha}{B\frac{\alpha^2}{2}} = 2,$$

again independent of B or α.

The equation for $j = J$ is the same except that $\alpha_{J,t+1} = 1$ and thus $\widehat{\alpha}_{J,t+1} = 0$.

Finally

$$\forall j = 0...J-1 : \widehat{\mu}_{j,t} = E_t\left(\begin{array}{c} \widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\mu_{j+1}^*(1-\alpha_j^*)(1-\delta+\psi)}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\left(\widehat{\mu}_{j+1t+1} + \frac{-\alpha_j^*}{1-\alpha_j^*}\widehat{\alpha}_{j,t+1}\right) + \frac{(1-\delta)\alpha_j^*}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\widehat{\alpha}_{jt+1} \\ +\frac{MPK_j^*}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\left(\frac{1}{1-\upsilon}\widehat{z}_{t+1}^a - \frac{\upsilon}{1-\upsilon}\widehat{w}_{t+1} + \left(\gamma - 1 + \frac{\gamma\upsilon}{1-\upsilon}\right)\widehat{k}_{j,t+1}\right) \end{array}\right).$$

$$\forall j = 0...J-1 : \widehat{\mu}_{j,t} = E_t\left(\begin{array}{c} \widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\mu_{j+1}^*(1-\alpha_j^*)(1-\delta+\psi)}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\left(\widehat{\mu}_{j+1t+1} + \frac{-\alpha_j^*}{1-\alpha_j^*}\widehat{\alpha}_{j,t+1}\right) + \frac{(1-\delta)\alpha_j^*}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\widehat{\alpha}_{jt+1} \\ +\frac{MPK_j^*}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\left(\frac{1}{1-\upsilon}\widehat{z}_{t+1}^a - \frac{\upsilon}{1-\upsilon}\widehat{w}_{t+1} + \left(\gamma - 1 + \frac{\gamma\upsilon}{1-\upsilon}\right)\widehat{k}_{j,t+1}\right) + \frac{-\psi\left(1-\alpha_j^*\right)}{\frac{\mu_j^*}{\beta\Theta_\lambda\Theta_b}}\left(\frac{-\alpha_j^*}{1-\alpha_j^*}\widehat{\alpha}_{j,t+1}\right) \end{array}\right).$$

$$\widehat{\mu}_{0,t} = E_t\left(\begin{array}{c} \widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\mu_1^*(1-\alpha_0^*)(1-\delta+\psi)}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}}\left(\widehat{\mu}_{1t+1} + \frac{-\alpha_0^*}{1-\alpha_0^*}\widehat{\alpha}_{0,t+1}\right) + \frac{(1-\delta)\alpha_0^*}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}}\widehat{\alpha}_{0t+1} \\ +\frac{MPK_0^*}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}}\left(\frac{1}{1-\upsilon}\widehat{z}_{t+1}^a - \frac{\upsilon}{1-\upsilon}\widehat{w}_{t+1} + \left(\gamma - 1 + \frac{\gamma\upsilon}{1-\upsilon}\right)\widehat{k}_{0t+1}\right) + \frac{-\psi\left(1-\alpha_0^*\right)}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}}\left(\frac{-\alpha_0^*}{1-\alpha_0^*}\widehat{\alpha}_{0t+1}\right) \end{array}\right)$$

$$\left[-\frac{w_t}{\Theta_k^t}f'\left(\frac{k_{0,t}}{\Theta_k^t} - 1\right) + \left(MPK_{0t} - \psi(1-\alpha_{0t})\right) + (1-\delta)\alpha_{0t} + \mu_{1,t}(1-\alpha_{0t})(1-\delta+\psi)\right]$$

$$
\widehat{\mu}_{0,t} = E_t \left(\begin{array}{c} \widehat{\lambda}_{t+1} - \widehat{\lambda}_t + \frac{\mu_1^*(1-\alpha_0^*)(1-\delta+\psi)}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}} \left(\widehat{\mu}_{1t+1} + \frac{-\alpha_0^*}{1-\alpha_0^*} \widehat{\alpha}_{0,t+1} \right) + \frac{(1-\delta)\alpha_0^*}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}} \widehat{\alpha}_{0t+1} \\ +\frac{MPK_0^*}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}} \left(\frac{1}{1-\upsilon} \widehat{z}_{t+1}^a - \frac{\upsilon}{1-\upsilon} \widehat{w}_{t+1} + \left(\gamma - 1 + \frac{\gamma\upsilon}{1-\upsilon} \right) \widehat{k}_{0t+1} \right) + \frac{-\psi\left(1-\alpha_0^*\right)}{\frac{\mu_0^*}{\beta\Theta_\lambda\Theta_b}} \left(\frac{-\alpha_0^*}{1-\alpha_0^*} \widehat{\alpha}_{0t+1} \right) \end{array} \right)
$$

$$
\left[-\frac{w_t}{\Theta_k^t} f' \left(\frac{k_{0,t}}{\Theta_k^t} - 1 \right) + (MPK_{0t} - \psi(1-\alpha_{0t})) + (1-\delta)\alpha_{0t} + \mu_{1,t}(1-\alpha_{0t})(1-\delta+\psi) \right]
$$