

Problem Set 1 - Due Friday Jan 22

Question 1: Asset pricing basics

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = E \sum_{t \geq 0} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows the process

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_c^2}{2} + \sigma_c \varepsilon_{t+1},$$

where ε_{t+1} is *iid* $N(0, 1)$.

Consider N assets, $i = 1 \dots N$ which pay respectively dividends D_{it} where each D_{it} follows a different process:

$$\Delta \log D_{it+1} = \mu_i - \frac{\lambda_i^2}{2} - \frac{\chi_i^2}{2} + \lambda_i \varepsilon_{t+1} + \chi_i \eta_{i,t+1},$$

with η_{it+1} *iid* $N(0, 1)$ and χ_i, λ_i, μ_i characterize this process. $\eta_{i,t+1}$ is uncorrelated with ε_{it+1} at all leads and lags, i.e.

$$E(\varepsilon_{t+1-k} \eta_{i,t+1}) = 0 \text{ for all } k \geq 0, \text{ and all } k \leq 0$$

Note: you will need the log-normal formula, i.e. if X is $N(\mu, \sigma^2)$ then $E(\exp(X)) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$.

(a) Compute the mean of $\frac{C_{t+1}}{C_t}$ and $\frac{D_{it+1}}{D_{it}}$. (So you can see why I added the terms $-\frac{\sigma_c^2}{2}$, $-\frac{\lambda_i^2}{2}$, etc.)

(b) Compute the risk-free rate.

(c) Compute the price-dividend ratio on asset i . Explain intuitively how it depends on μ_i, λ_i and χ_i .

(d) Compute the expected return and expected excess return (i.e. return less the risk-free rate) on asset i . Explain intuitively how it depends on μ_i, λ_i and χ_i . Discuss the statement “idiosyncratic risk is not priced.”

(e) Define the asset i 's “consumption beta” $\beta_{i,c}$ as the slope of the time-series regression of the asset return on consumption growth:

$$R_{t+1}^i = \alpha_i + \beta_{i,c} \Delta \log C_{t+1} + \nu_{i,t+1}.$$

Compute $\beta_{i,c}$. What is the cross-sectional (i.e., across i) relation between $\beta_{i,c}$ and expected returns? [Hint: you can use the following approximation: if (U, V) is jointly normal, and g is a smooth function, then: $Cov(g(U), V) = E(g'(U)) \times Cov(U, V)$. (this is known as Stein's lemma).]

Small note on this problem: I did not specify if the assets D_{it} were in zero net supply or not. If they are not, then we must have some other asset(s), which pay a total dividend $C_t - \sum_i D_{it}$, such that the sum of all the “dividends” in the economy is C_t (since it is an endowment economy, there is no physical investment). This last asset could be human capital. Technically, note that the process for the dividend of this asset is complicated - it cannot be a random walk too, since a sum of random walks is generally not a random walk...

Question 2: Endowment economy with Markov chain

This exercise asks you to solve a standard endowment economy asset pricing model numerically. This is the Mehra-Prescott (1985, JME) setup, but you don't need to read the paper. (The computations are fairly simple so you could almost do them by hand, but it will be useful for you to learn to use Matlab or a similar software to solve this.)

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = E \sum_{t \geq 0} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows a Markov chain with two states:

$$\Delta \log C_t = \mu_h \text{ or } \mu_l,$$

and

$$\begin{aligned} \Pr(\Delta \log C_t = \mu_h | \Delta \log C_{t-1} = \mu_h) &= p, \\ \Pr(\Delta \log C_t = \mu_l | \Delta \log C_{t-1} = \mu_h) &= 1 - p, \\ \Pr(\Delta \log C_t = \mu_h | \Delta \log C_{t-1} = \mu_l) &= 1 - q, \\ \Pr(\Delta \log C_t = \mu_l | \Delta \log C_{t-1} = \mu_l) &= q. \end{aligned}$$

We assume the “stock” pays a dividend equal to consumption, forever. We will write this setup for a general Markov chain with S states, here $S = 2$. Hence, there is a transition matrix $Q(s, s') = \Pr(\Delta \log C_t = \mu(s') | \Delta \log C_{t-1} = \mu(s))$ and a vector $\mu(s)$.

(a) Write the equation which determines the risk-free rate. Note that it will depend on the current state s , call it $R^f(s)$.

(b) Write the equation which determines the price-dividend ratio. (Note: this is a recursion.) Call it $q(s)$.

(c) Write the equation which determines the expected return on the stock: $E_t R_{t+1}^e = ER(s)$.

(d) Consider the following parameters:

$$\begin{aligned} S &= 2, \mu_h = 1.02, \mu_l = .98, \\ p &= q = .95, \beta = .98, \gamma = 2. \end{aligned}$$

Compute the expected return and the expected excess return conditional on each state.

(e) Compute the unconditional expected return and excess return on the stock. To do so, you need to use the invariant (or ergodic) distribution to compute the probability that each state is reached. (See my notes on Markov chains if you are lost, <http://people.bu.edu/fgourio/markov.pdf>)

(f) Produce a plot, as a function of γ , of the unconditional excess return on the stock.

(g) Calibration: for a two-state Markov chain with transition

$$\begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

and means (μ_h, μ_l) , with $\mu_h = \mu + \delta$ and $\mu_l = \mu - \delta$, compute the mean, variance, and serial correlaton of $\Delta \log C_t$. We can measure these statistics in U.S. data and then find p, q, δ , to replicate these statistics, so that our implied time series process for $\Delta \log C_t$ is similar to the data. In US data, typical numbers would be for annual consumption growth: $E(\Delta \log C_t) = 2\%$, $var(\Delta \log C_t) = 0.0004$, and $\rho(\Delta \log C_t) = -.1$. What are the implied δ, p, q ?

Problem Set 1 - Answer key (sketch)

Question 1: Asset pricing basics

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = E \sum_{t \geq 0} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows the process

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_c^2}{2} + \sigma_c \varepsilon_{t+1},$$

where ε_{t+1} is iid $N(0, 1)$.

Consider N assets, $i = 1 \dots N$ which pay respectively dividends D_{it} where each D_{it} follows a different process:

$$\Delta \log D_{it+1} = \mu_i - \frac{\lambda_i^2}{2} - \frac{\chi_i^2}{2} + \lambda_i \varepsilon_{t+1} + \chi_i \eta_{i,t+1},$$

with $\eta_{i,t+1}$ iid $N(0, 1)$ and χ_i, λ_i, μ_i characterize this process. $\eta_{i,t+1}$ is uncorrelated with ε_{it+1} at all leads and lags, i.e.

$$E(\varepsilon_{t+1-k} \eta_{i,t+1}) = 0 \text{ for all } k \geq 0, \text{ and all } k \leq 0$$

Note: you will need the log-normal formula, i.e. if X is $N(\mu, \sigma^2)$ then $E(\exp(X)) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$.

(a) Compute the mean of $\frac{C_{t+1}}{C_t}$ and $\frac{D_{it+1}}{D_{it}}$. (So you can see why I added the terms $-\frac{\sigma_c^2}{2}$, $-\frac{\lambda_i^2}{2}$, etc.)

Answer:

$$E\left(\frac{C_{t+1}}{C_t}\right) = \exp(\mu_c)$$

$$E\left(\frac{D_{it+1}}{D_{it}}\right) = \exp(\mu_d)$$

so the terms “minus half variance” are just here to keep the means constant as σ is changed (i.e. to offset the Jensen’s terms).

(b) Compute the risk-free rate.

Answer: using the Euler equation, and integrating,

$$\log R_f = -\log \beta + \gamma \mu - \frac{\gamma(\gamma+1)}{2} \sigma_c^2.$$

Same as in class, except the mean adjustment implies the coefficient of the effect of uncertainty is different.

(c) Compute the price-dividend ratio on asset i . Explain intuitively how it depends on μ_i, λ_i and χ_i .

Answer: denote it q_i . It is constant since both dividend growth and cons. growth are iid. It satisfies:

$$q_i = (1 + q_i) E_t \left(\frac{D_{it+1}}{D_{it}} \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right)$$

$$\begin{aligned}
\frac{q_i}{1+q_i} &= \beta e^{\mu_i - \gamma\mu_c + \gamma\frac{\sigma_c^2}{2} + \frac{(\lambda_i - \gamma\sigma_c)^2}{2}} - \frac{\lambda_i^2}{2} \\
&= \beta e^{\mu_i - \gamma\mu_c + \gamma(\gamma+1)\frac{\sigma_c^2}{2} - \lambda_i\gamma\sigma_c} \\
&= e^{\mu_i - \lambda_i\gamma\sigma_c} R^f
\end{aligned}$$

Which we can rewrite as

$$q_i = \frac{H}{1-H}$$

with

$$H = e^{\mu_i - \lambda_i\gamma\sigma_c} R^f$$

if $\mu_i, \lambda_i\gamma\sigma_c$ are small we have approximately

$$q_i = \frac{1}{\lambda_i\gamma\sigma_c + r_f - \mu_i}$$

This is an extension of the famous Gordon growth formula:¹

$$\frac{P}{D} = \frac{1}{r-g},$$

but there the return is $r = \lambda_i\gamma\sigma_c + r_f$. (this is the expected equity return, see question (d)).

This approximation is useful for the intuition. Looking at H , you see that a higher μ_i leads to a higher P-D ratio since dividends are expected to grow faster in the future, and the price reflects this; a higher λ_i will decrease the P-D ratio, because it means that the asset is more risky (it is more exposed to aggregate shocks) and hence the “discount rate” i.e. the expected return on the asset increases, so the price falls relative to dividends. Last, χ_i has no effect on the price - it does not affect the expected equity return or dividend growth.

(d) Compute the expected return and expected excess return (i.e. return less the risk-free rate) on asset i . Explain intuitively how it depends on μ_i, λ_i and χ_i . Discuss the statement “idiosyncratic risk is not priced.”

Expected return is

$$\begin{aligned}
E_t R_{it+1} &= \frac{q_i + 1}{q_i} E_t \left(\frac{D_{it+1}}{D_{it}} \right) \\
&= \frac{q_i + 1}{q_i} e^{\mu_d} \\
&= \frac{e^{\mu_d}}{\beta e^{\mu_d - \gamma\mu_c + \gamma(\gamma+1)\frac{\sigma_c^2}{2} - \lambda_i\gamma\sigma_c}} \\
&= R_f e^{\lambda_i\gamma\sigma_c}.
\end{aligned}$$

Expected excess return is, in log

$$\log \frac{E_t R_{it+1}}{R_f} = \lambda_i\gamma\sigma_c.$$

i.e. the risk premium on an asset is the “loading” λ_i on the aggregate shock ε , times the effect of this shock on cons. growth (σ_c), times the risk aversion coefficient.

¹which comes from $P_t = \int_t^\infty D_s e^{-r(s-t)} ds$, and $dD_t/dt = gD_t$ for all $t \geq 0$.

This formula is useful for intuition - suppose an asset expected excess return goes up, then we can think, either λ_i , or γ , or σ_c has increased.

χ_i does not show up! idiosyncratic has no effect on expected returns - the only thing that matters is the covariance of the asset return with consumption growth, and cons. growth is not affected by idiosyncratic shocks.

Going back to (c) above, we have

$$q_i \simeq \frac{1}{E(r_i) - E\left(\frac{D_{it+1}}{D_{it}}\right)},$$

i.e. the Gordon growth formula, with (crucially!) the expected return on the asset for r : this is the risk-free rate + the risk premium.

(e) Define the asset i 's "consumption beta" $\beta_{i,c}$ as the slope of the time-series regression of the asset return on consumption growth:

$$R_{t+1}^i = \alpha_i + \beta_{i,c} \Delta \log C_{t+1} + \nu_{i,t+1}.$$

Compute $\beta_{i,c}$. What is the cross-sectional (i.e., across i) relation between $\beta_{i,c}$ and expected returns? [Hint: you can use the following approximation: if (U, V) is jointly normal, and g is a smooth function, then: $Cov(g(U), V) = E(g'(U)) \times Cov(U, V)$. (this is known as Stein's lemma).]

From the def of OLS,

$$\beta_{i,c} = \frac{Cov(\Delta \log C_{t+1}, R_{t+1}^i)}{Var(\Delta \log C_{t+1})}.$$

We now use that

$$R_{t+1}^i = \frac{q_i + 1}{q_i} \frac{D_{it+1}}{D_{it}},$$

and the approximation: $(\Delta \log C_{t+1}, \log R_{t+1}^i)$ is jointly normal (why?), hence

$$\begin{aligned} Cov(\exp(\log R_{t+1}^i), \Delta \log C_{t+1}) &= E(\exp(\log R_{t+1}^i)) Cov(\log R_{t+1}^i, \Delta \log C_{t+1}) \\ &= E(R_i) Cov(\Delta \log D_{it+1}, \Delta \log C_{t+1}) \\ &= E(R_i) \lambda_i \sigma_c \end{aligned}$$

Hence,

$$\beta_{i,c} = E(R_i) \frac{\lambda_i}{\sigma_c} \simeq \frac{\lambda_i}{\sigma_c}$$

since $E(R_i) \simeq 1$ (this is a gross return). Finally, from the previous question,

$$\log \frac{ER_i}{R_f} \simeq ER_i - R_f = \lambda_i \gamma \sigma_c = \gamma \sigma_c^2 \beta_{i,c}.$$

What's interesting about this eqn is that we can measure the LHS (returns) and the RHS ($\beta_{i,c}$ from the regression above), up to a constant $\gamma \sigma_c^2$. So we can test this asset pricing model: plot the $\beta_{i,c}$ (x-axis) against the $E(R_i)$. Of course, we could have done this with GMM without the extraneous assumptions, but this is clearer intuitively and somewhat simpler.

Interpretation of this equation: the excess return is the product of the "quantity of risk" $\beta_{i,c}$ of the asset, times the market price of risk $\gamma \sigma_c^2$.

NOTE: this problem is important because this simple example includes many of the principles of asset pricing - how P/D ratios are related to expected returns, the role of β_i , the fact that idiosyncratic risk

does not matter, etc.; so make sure you are comfortable with this. However, keep in mind that some assumptions are strong, e.g. the iid assumption is restrictive.

Question 2: Endowment economy with Markov chain

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and

$$\begin{aligned} \Pr(\Delta \log C_t = \mu_h | \Delta \log C_{t-1} = \mu_h) &= p, \\ \Pr(\Delta \log C_t = \mu_l | \Delta \log C_{t-1} = \mu_h) &= 1 - p, \\ \Pr(\Delta \log C_t = \mu_h | \Delta \log C_{t-1} = \mu_l) &= 1 - q, \\ \Pr(\Delta \log C_t = \mu_l | \Delta \log C_{t-1} = \mu_l) &= q. \end{aligned}$$

We assume the “stock” pays a dividend equal to consumption, forever. We will write this setup for a general Markov chain with S states, here $S = 2$. Hence, there is a transition matrix $Q(s, s') = \Pr(\Delta \log C_t = \mu(s') | \Delta \log C_{t-1} = \mu(s))$ and a vector $\mu(s)$.

(a) Write the equation which determines the risk-free rate. Note that it will depend on the current state s , call it $R^f(s)$.

We have

$$\begin{aligned} R^f(s)^{-1} &= E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right) \\ R^f(s)^{-1} &= \sum_{s'} \beta Q(s, s') e^{-\gamma \mu(s')} \end{aligned}$$

(b) Write the equation which determines the price-dividend ratio. (Note: this is a recursion.) Call it $q(s)$.

$$\begin{aligned} P_t &= E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} (P_{t+1} + C_{t+1}) \right) \\ \frac{P_t}{C_t} &= E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{P_{t+1}}{C_{t+1}} + 1 \right) \right) \\ q(s_t) &= E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} (q(s_{t+1}) + 1) \right) \\ q(s) &= E_{s'|s} \left((q(s') + 1) \beta e^{(1-\gamma)\mu(s')} \right) \\ q(s) &= \sum_{s'} Q(s, s') (q(s') + 1) \beta e^{(1-\gamma)\mu(s')}. \end{aligned}$$

(c) Write the equation which determines the expected return on the stock: $E_t R_{t+1}^e = ER(s)$.

$$\begin{aligned} E_t(R_{t+1}^e) &= E_t \left(\frac{\frac{P_{t+1}}{C_{t+1}} + 1}{\frac{P_t}{C_t}} \frac{C_{t+1}}{C_t} \right) \\ &= \sum_{s'} Q(s, s') \frac{q(s') + 1}{q(s)} e^{\mu(s')}. \end{aligned}$$

Matlab code:

```

%% %% Mehra-Prescott
clear all ; close all ; clc ;

S = 2 ;
beta = .98 ;
gamma = 2 ;
p = .95 ;
q = .95 ;
muh = 0.02 ;
mul = -.02 ;
Q = zeros(S) ;
Q = [p,1-p;1-q,q] ;
mu = zeros(S,1) ;
mu = [muh,mul]' ;
%% %% Risk-free rate
Rf = zeros(S,1) ;
for s=1:S
Rf(s) = 1 / (beta*Q(s,:)*exp(-gamma*mu)) ;
end
%% %% iterate on P-D to solve:
PD = zeros(S,1) ;
PDnew = zeros(S,1) ;
err = 1 ;
while err>1e-9
PDnew = PD ;
for s=1:S

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PD(s) = beta*Q(s,.)*((PDnew+1).*exp((1-gamma)*mu)) ;
end
err = max(abs(PD-PDnew)) ;
end
%%% expected return on stock
expret = zeros(S,1) ;
for s=1:S
expret(s) = Q(s,.)*((PDnew+1).*exp(mu))/PDnew(s);
end
%%% stationary distribution
A = Q'^1000 ;
invdis = A(:,1) ;
%%% show results:
disp('careful about the units!!!')
disp('average return on stock:')
invdis'*expret
disp('average return on bond:')
invdis'*Rf
disp('average equity premium:')
invdis'*(expret-Rf)
disp('std dev of stock return:')
sqrt(invdis'*expret.^2-(invdis'*expret)^2)
disp('std dev of risk-free rate return:')
sqrt(invdis'*Rf.^2-(invdis'*Rf)^2)

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Problem Set 2 - Due Tuesday, Feb 2 by 6pm in my mailbox

Question 1: Time-varying volatility and asset pricing

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = E \sum_{t \geq 0} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows the process:

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_t^2}{2} + \sigma_t \varepsilon_{t+1},$$

where ε_{t+1} is *iid* $N(0, 1)$, and σ_{t+1} follows a Markov chain with transition probabilities $Q(\sigma_{t+1}|\sigma_t)$. We assume that the two processes $\{\varepsilon_{t+1}\}$ and $\{\sigma_{t+1}\}$ are independent. Note that σ_t is drawn at time t , and ε_{t+1} is drawn at time $t + 1$.

(Note: This problem does not require a lot of computations - write the simple version of the equations, and compute only when necessary.)

(1) Compute the risk-free rate, as a function of σ_t . What is the effect of σ_t on the risk-free rate? Explain briefly the intuition.

(2) Consider an asset with dividends equal to consumption. Give an equation which determines the price-dividend ratio, as a function of σ_t . (You do not need to solve for this equation.)

(3) Using (2), solve for the expected equity return $E_t R_{t+1}^e$, as a function of σ_t , and for the log equity premium $\log \frac{E_t R_{t+1}^e}{R_{t+1}^f}$. What is the effect of σ_t on each of these two objects? Explain briefly the intuition.

(4) Assuming for this question only that σ_t is *iid*, what is the effect of σ_t on the price-dividend ratio? Explain briefly the intuition. [Note: this result can in fact be generalized to non-*iid* σ_t , as long as σ_t is positively serially correlated.]

(5) (a) Does the model generate the time-series predictability evidence, i.e. if you run the regression

$$R_{t+1}^e - R_{t+1}^f = \alpha + \beta \frac{D_t}{P_t} + \varepsilon_{t+1}$$

in the model, do you get a positive sign? [You do not need a formal proof, and can consider the case σ_t is *iid* only.]

(b) In the data, the regression $R_{t+1}^e = \alpha + \beta \frac{D_t}{P_t} + \varepsilon_{t+1}$ gives the same positive sign as the regression above. Is it the case in this model?

(6) Answer questions (1)-(5) again, but now assume that the process of consumption is

$$\Delta \log C_{t+1} = \mu_t - \frac{\sigma^2}{2} + \sigma \varepsilon_{t+1},$$

i.e. instead of a time-varying risk we have a time-varying growth rate of consumption: μ_t follows a Markov chain with transition $Q(\mu_{t+1}|\mu_t)$.

(7) Answer questions (1)-(5) again, but now assume that the process of consumption is

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_c^2}{2} + \sigma_c \varepsilon_{t+1},$$

while dividends are not equal to consumption anymore, and satisfy

$$\Delta \log D_{t+1} = \mu_d - \frac{\lambda_t^2}{2} + \lambda_t \varepsilon_{t+1},$$

so that there is a time-varying risk of dividends, but not consumption.

(8) Same as (7), but now dividends satisfy

$$\Delta \log D_{t+1} = \mu_t - \frac{\sigma_d^2}{2} + \sigma_d \varepsilon_{t+1},$$

so that there is a time-varying growth rate of dividends, but not consumption.

Question 2 (Stolen from John Campbell)

Consider an economy with a representative agent who has power utility with coefficient of relative risk aversion γ . The agent receives a nonstorable endowment. The process for the log endowment, or equivalently the log of consumption c_t , is $\Delta \log c_{t+1} = \mu(1 - \rho) + \rho \Delta \log c_t + \varepsilon_{t+1}$, where the coefficient ρ may be either positive or negative. If ρ is positive then endowment fluctuations are highly persistent; if it is negative then they have an important transitory component.

(a) Assume that consumption and asset returns are jointly lognormal, with constant variances and covariances.

(i) Use the representative agent's Euler equations to show that the expected log return on any asset is a linear function of the expected growth rate of the endowment. What is the slope coefficient in this relationship?

(ii) Use the representative agent's Euler equations to show that the difference between the expected log return on any asset and the log riskfree interest rate, plus one-half the own variance of the log asset return (call this sum the premium on the asset) is proportional to the conditional covariance of the log asset return with consumption growth. What is the slope coefficient in this relationship?

(b) To a close approximation, the unexpected return on any asset i can be written as

$$r_{i,t+1} - E_t r_{i,t+1} = (E_{t+1} - E_t) \sum_{j=0}^{\infty} \Delta d_{i,t+1+j} - (E_{t+1} - E_t) \sum_{j=1}^{\infty} r_{i,t+1+j}$$

where $d_{i,t}$ is the dividend paid on asset i at time t . [see lecture notes]

(i) Use this expression to calculate the unexpected return on an equity which pays aggregate consumption as its dividend.

(ii) Use this expression to calculate the unexpected return on a real consol bond which has a fixed real dividend each period.

(c) (i) Calculate the equity premium and the consol bond premium.

(ii) Show that the bond premium has the opposite sign to ρ and is proportional to the square of γ . Give an economic interpretation of this result.

(iii) Show that the equity premium is always larger than the bond premium, and the difference between them is proportional to γ . Give an economic interpretation of this result.

(iv) Given the results, discuss which case ($\rho > 0$ or $\rho < 0$) helps explain asset prices.

Problem Set 2 - Answer key

— FIXED —

Question 1: Time-varying volatility and asset pricing

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

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Assume that the endowment follows the process:

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_t^2}{2} + \sigma_t \varepsilon_{t+1},$$

where ε_{t+1} is iid $N(0, 1)$, and σ_{t+1} follows a Markov chain with transition probabilities $Q(\sigma_{t+1}|\sigma_t)$. We assume that the two processes $\{\varepsilon_{t+1}\}$ and $\{\sigma_{t+1}\}$ are independent. Note that σ_t is drawn at time t , and ε_{t+1} is drawn at time $t + 1$.

(Note: this question does not require a huge amount of computations! Just write the simple equations required and compute only when necessary.)

(1) Compute the risk-free rate, as a function of σ_t . What is the effect of σ_t on the risk-free rate? Explain briefly the intuition.

$$R_{t \rightarrow t+1}^f = \frac{1}{\beta E_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right)}$$

$$R^f(\sigma_t) = \frac{1}{\beta} e^{\gamma \mu_c - \gamma(\gamma+1) \frac{\sigma_t^2}{2}}.$$

A higher σ_t leads to more demand for risk-free assets for precautionary savings, to the interest goes down as a result (or: the price of the risk-free bond goes up).

(2) Consider an asset with dividends equal to consumption. Give an equation which determines the price-dividend ratio, as a function of σ_t . (You do not need to solve for this equation.)

Let $q(\sigma_t) = \frac{P_t}{C_t}$. Note that q is a function of σ_t alone since

$$\frac{P_t}{C_t} = E_t \sum_{k=1}^{\infty} \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{1-\gamma},$$

and this conditional expectation depends on the only state variable σ_t . Hence,

$$q(\sigma_t) = E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} (1 + q(\sigma_{t+1})) \right),$$

and we take the expectation over ε_{t+1} and σ_{t+1} , which are independent. Hence:

$$q(\sigma_t) = E_{\sigma_{t+1}|\sigma_t} (q(\sigma_{t+1}) + 1) \times E_{|\sigma_t} \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right),$$

which we can write as

$$\begin{aligned} q(\sigma) &= E_{\sigma'|\sigma} (1 + q(\sigma')) \times \beta e^{(1-\gamma)(\mu_c - \frac{\sigma^2}{2}) + \frac{\sigma^2}{2}(1-\gamma)^2} \\ &= E_{\sigma'|\sigma} (1 + q(\sigma')) \times \beta e^{(1-\gamma)\mu_c - \gamma(1-\gamma)\frac{\sigma^2}{2}}. \end{aligned}$$

(3) Using (2), solve for the expected equity return $E_t R_{t+1}^e$, as a function of σ_t , and for the log equity premium $\log \frac{E_t R_{t+1}^e}{R_{t+1}^f}$. What is the effect of σ_t on each of these two objects? Explain briefly the intuition.

$$\begin{aligned} E_t R_{t+1}^e &= \frac{E_t(q(\sigma_{t+1}) + 1)}{q(\sigma_t)} E_t \left(\frac{C_{t+1}}{C_t} \right) \\ &= \frac{1}{\beta} e^{\gamma\mu_c + \gamma(1-\gamma)\frac{\sigma_t^2}{2}} \end{aligned}$$

and the log equity premium is

$$\begin{aligned} \log \frac{E_t R_{t+1}^e}{R_{t+1}^f} &= \log \frac{1}{\beta} e^{\gamma\mu_c + \gamma(1-\gamma)\frac{\sigma_t^2}{2}} - \log \frac{1}{\beta} e^{\gamma\mu_c - \gamma(\gamma+1)\frac{\sigma_t^2}{2}} \\ &= \gamma\sigma_t^2. \end{aligned}$$

When σ_t rises, the risk rises, and the risk premium $\frac{E_t R_{t+1}^e}{R_{t+1}^f}$ rises. The effect on the equity return is ambiguous: if $\gamma < 1$, then an increase in σ_t^2 raises the expected equity return, and if $\gamma > 1$, it falls. Intuitively, you have offsetting effects because the risk premium goes up, and the riskless interest rate goes down. The effects are weighted by $\gamma(\gamma + 1)$ and γ respectively, so the total effect is $\gamma(\gamma - 1)$.

(4) Assuming for this question only that σ_t is iid, what is the effect of σ_t on the price-dividend ratio? Explain briefly the intuition. [Note: this result can in fact be generalized to non-iid σ_t , as long as σ_t is positively serially correlated.]

If σ is iid, then $E_{\sigma'|\sigma} (1 + q(\sigma'))$ does not depend on σ . Hence,

$$q(\sigma) = \text{constant} \times \beta e^{(1-\gamma)\mu_c - \gamma(1-\gamma)\frac{\sigma^2}{2}},$$

and we can see that q will be increasing in σ if and only if $\gamma > 1$. Again (as in ps1), the intuition can be seen from the Gordon growth formula, $P/D = 1/(r - g)$: an increase in the expected return will decrease the price-dividend ratio. The effect of σ on the expected equity return comes from the previous question.

Note: if σ is not iid but is persistent, then, you can prove using a standard contraction mapping argument that q is increasing if σ is positively serially correlated. (For detailed proofs, you can read a short paper I wrote, "Time series predictability in the disaster model", available from my web page.)

(5) (a) Does the model generate the time-series predictability evidence, i.e. if you run the regression

$$R_{t+1}^e - R_{t+1}^f = \alpha + \beta \frac{D_t}{P_t} + \varepsilon_{t+1}$$

in the model, do you get a positive sign? [An approximate analytical answer in the case where σ_t is iid is fine.]

(b) In the data, the regression $R_{t+1}^e = \alpha + \beta \frac{D_t}{P_t} + \varepsilon_{t+1}$ gives the same sign as the regression above. Is it the case in this model?

(a and b) If σ is iid, we have seen that there are two cases:

(i) $\gamma < 1$: the D-P ratio is increasing in σ . Expected excess returns and returns are both increasing in σ , so the model generates the right signs for both regressions.

(ii) $\gamma > 1$: the D-P ratio is decreasing in σ . Expected excess returns are increasing in σ , but expected equity returns are decreasing in σ . So β is negative in the first regression, but positive in the second regression.

So this model will not match the evidence on predictability if $\gamma > 1$. We need $\gamma < 1$. Of course this will make the calibration difficult since we want a high risk aversion to match the equity premium. [Here Epstein-Zin utility can help \rightarrow the γ in this predictability evidence is (partly) the IES.]

(6) Answer questions (1)-(5) again, but now assume that the process of consumption is

$$\Delta \log C_{t+1} = \mu_t - \frac{\sigma^2}{2} + \sigma \varepsilon_{t+1},$$

i.e. instead of a time-varying risk we have a time-varying growth rate of consumption: μ_t follows a Markov chain with transition $Q(\mu_{t+1}|\mu_t)$.

Risk-free rate:

$$R^f(\mu) = \frac{1}{\beta} e^{\gamma\mu - \gamma(\gamma+1)\frac{\sigma^2}{2}}.$$

PD ratio equation:

$$q(\mu) = E_{\mu'|\mu} (1 + q(\mu')) \times \beta e^{(1-\gamma)\mu - \gamma(1-\gamma)\frac{\sigma^2}{2}}.$$

Expected equity return:

$$ER^e(\mu) = \frac{1}{\beta} e^{\gamma\mu + \gamma(1-\gamma)\frac{\sigma^2}{2}}$$

Expected excess return (i.e. equity premium)

$$\log \frac{ER^e(\mu)}{R^f(\mu)} = \gamma\sigma^2.$$

Discussion: an increase in μ affects the risk-free rate for intertemporal substitution reasons, but does not affect the equity premium. Hence, the expected equity return increases. The effect on the P-D ratio depends on γ : if $\gamma > 1$, a higher μ reduces P/D , and if $\gamma < 1$, it is the opposite. That's because both future cash flows increase, and the interest rate increases.

In terms of the regressions: the regression with excess returns will have a R^2 of 0 (in population). The regression with the equity return will have a positive slope (high D/P forecasts high returns) if $\gamma < 1$, and a negative slope if $\gamma > 1$. So this model will not match the evidence on predictability.

(7) Answer questions (1)-(5) again, but now assume that the process of consumption is

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_c^2}{2} + \sigma_c \varepsilon_{t+1},$$

while dividends are not equal to consumption anymore, and satisfy

$$\Delta \log D_{t+1} = \mu_d - \frac{\lambda_t^2}{2} + \lambda_t \varepsilon_{t+1},$$

so that there is a time-varying risk of dividends, but not consumption.

Risk-free rate is constant now:

$$R^f = \frac{1}{\beta} e^{\gamma\mu_c - \gamma(\gamma+1)\frac{\sigma_c^2}{2}}.$$

The P-D ratio satisfies:

$$\begin{aligned} \frac{P_t}{D_t} &= E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{D_{t+1}}{D_t} \left(1 + \frac{P_{t+1}}{D_{t+1}} \right) \right) \\ q(\lambda) &= E_{\lambda'|\lambda} (1 + q(\lambda')) \times \beta e^{\mu_d - \gamma\mu_c + \frac{\gamma(\gamma+1)\sigma_c^2}{2} - \lambda_t \gamma \sigma_c} \end{aligned}$$

The expected return:

$$\begin{aligned} ER^e(\lambda) &= \frac{E_{\lambda'|\lambda} (1 + q(\lambda'))}{q(\lambda)} E_t \left(\frac{D_{t+1}}{D_t} \right) \\ &= \frac{1}{\beta e^{-\gamma\mu_c + \frac{\gamma(\gamma+1)\sigma_c^2}{2} - \lambda_t \gamma \sigma_c}} \\ &= R^f e^{\lambda_t \gamma \sigma_c} \end{aligned}$$

Equity premium (in log):

$$\log \frac{ER^e(\lambda)}{R^f} = \lambda \gamma \sigma.$$

Intuition: The consumption risk is constant, but equities sometimes become more risky. When they do, the risk premium rises, and the P-D ratio falls. Predictability regressions: an increase in λ leads P/D to fall, and expected returns and expected excess returns to rise. Hence, the model is consistent with both coefficients being positive. The only problem with this model is that there is no link between consumption and dividends, but economic intuition suggests that dividends finance consumption (at least to some extent). Moreover, as we'll see later, there are risk premia not only on stocks but also on other assets, and these risk premia seem to rise sometimes all at the same time – which makes sense if consumption risks increases, but does not make sense if it's dividend risk – you need to assume that all these assets become more risky at the same time.

(8) Same as (7), but now dividends satisfy

$$\Delta \log D_{t+1} = \mu_t - \frac{\sigma_d^2}{2} + \sigma_d \varepsilon_{t+1},$$

so that there is a time-varying growth rate of dividends, but not consumption.

Risk-free rate is still constant, equal to $R^f = \frac{1}{\beta} e^{\gamma\mu_c - \gamma(\gamma+1)\frac{\sigma_c^2}{2}}$.

The P-D ratio satisfies:

$$q(\mu) = E_{\mu'|\mu} (1 + q(\mu')) \times \beta e^{\mu - \gamma\mu_c + \frac{\gamma(\gamma+1)\sigma_c^2}{2} - \gamma\sigma_d \sigma_c}$$

The expected return:

$$ER^e(\mu) = \frac{1}{\beta} e^{\gamma\mu_c - \frac{\gamma(\gamma+1)\sigma_c^2}{2} + \gamma\sigma_c \sigma_d},$$

which is independent of μ .

Equity premium (in log):

$$\log \frac{ER^e(\mu)}{R^f} = \gamma\sigma_c \sigma_d.$$

Intuition: risk is constant. Higher μ leads to higher future cash flows so higher price today. But no change in the risk premium. Hence expected returns and risk premia are constant, and there is no time-series predictability. This is a “pure cash flow” model. This does not the predictability evidence! (And would also imply that price variation is explained by future cash flow growth, which is not true in reality.)

Question 2 (stolen from John Campbell’s web site)

Consider an economy with a representative agent who has power utility with coefficient of relative risk aversion γ . The agent receives a nonstorable endowment. The process for the log endowment, or equivalently the log of consumption c_t , is $\Delta \log c_{t+1} = \mu + \delta \Delta \log c_t + \varepsilon_{t+1}$, where the coefficient δ may be either positive or negative. If δ is positive then endowment fluctuations are highly persistent; if it is negative then they have an important transitory component.

(a) Assume that consumption and asset returns are jointly lognormal, with constant variances and covariances.

(i) Use the representative agent’s Euler equations to show that the expected log return on any asset is a linear function of the expected growth rate of the endowment. What is the slope coefficient in this relationship?

From the EE, as in the lecture notes:

$$0 = \log \beta - \gamma E_t \Delta \log c_{t+1} + E_t \log R_{t+1} + \frac{\gamma^2}{2} \text{Var}_t \Delta \log c_{t+1} + \frac{1}{2} \text{Var}_t \log R_{t+1} - \gamma \text{Cov}_t (\Delta \log c_{t+1}, \log R_{t+1})$$

Hence:

$$E_t \log R_{t+1} = \gamma E_t \Delta \log c_{t+1} - \log \beta - \frac{\gamma^2}{2} \text{Var}_t \Delta \log c_{t+1} - \frac{1}{2} \text{Var}_t \log R_{t+1} + \gamma \text{Cov}_t (\Delta \log c_{t+1}, \log R_{t+1})$$

Slope = $\gamma = 1/IES$.

(ii) Use the representative agent’s Euler equations to show that the difference between the expected log return on any asset and the log riskfree interest rate, plus one-half the own variance of the log asset return (call this sum the premium on the asset) is proportional to the conditional covariance of the log asset return with consumption growth. What is the slope coefficient in this relationship?

As in the lecture notes: Euler eqn + joint lognormality implies that

$$E_t \log \left(\frac{R_{t+1}^i}{R_{t+1}^f} \right) + \frac{1}{2} \text{Var}_t \log R_{t+1}^i = \log E_t \left(\frac{R_{t+1}^i}{R_{t+1}^f} \right) = \gamma \text{Cov} (\Delta \log c_{t+1}, \log R_{t+1}^i),$$

i.e. the slope is $\gamma = RA$.

(b) To a close approximation, the unexpected return on any asset i can be written as

$$r_{i,t+1} - E_t r_{i,t+1} = (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta d_{i,t+1+j} - (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{i,t+1+j}$$

where $d_{i,t}$ is the dividend paid on asset i at time t . [see lecture notes]

(i) Use this expression to calculate the unexpected return on an equity which pays aggregate consumption as its dividend.

If var and covariances are constant, then

$$\begin{aligned} E_t \log R_{t+1} &= \gamma E_t \Delta \log c_{t+1} - \log \beta - \frac{\gamma^2}{2} \text{Var}_t \Delta \log c_{t+1} - \frac{1}{2} \text{Var}_t \log R_{t+1} + \gamma \text{Cov}_t (\Delta \log c_{t+1}, \log R_{t+1}) \\ &= \text{constant} + \gamma E_t \Delta \log c_{t+1} \end{aligned}$$

and hence if $d_t = c_t$, $E_{t+1} r_{t+j+1} = E_{t+1} E_{t+j} r_{t+j+1} = c + \gamma E_{t+1} \Delta \log c_{t+j}$, and $E_t r_{t+j+1} = c + \gamma E_t \Delta \log c_{t+j}$, hence

$$\begin{aligned} (E_{t+1} - E_t) r_{t+1+j} &= \gamma (E_{t+1} - E_t) \Delta c_{t+1+j} \\ &= \gamma \delta^j \varepsilon_{t+1}, \end{aligned}$$

and

$$\begin{aligned} r_{i,t+1} - E_t r_{i,t+1} &= \sum_{j=0}^{\infty} \rho^j \delta^j \varepsilon_{t+1} - \gamma \sum_{j=1}^{\infty} \rho^j \delta^j \varepsilon_{t+1} \\ &= \left(\frac{1}{1 - \rho \delta} - \frac{\gamma \rho \delta}{1 - \rho \delta} \right) \varepsilon_{t+1} \\ &= \left(\frac{1 - \gamma \delta \rho}{1 - \rho \delta} \right) \varepsilon_{t+1}. \end{aligned}$$

(ii) Use this expression to calculate the unexpected return on a real consol bond which has a fixed real dividend each period.

If $d_t = 1$, then $\Delta d_t = 0$ and only the second term remains:

$$r_{t+1} - E_t r_{t+1} = -\frac{\gamma \rho \delta}{1 - \rho \delta} \varepsilon_{t+1}.$$

(c) (i) Calculate the equity premium and the consol bond premium.

The equity premium is:

$$\begin{aligned} \gamma \text{Cov}_t (\Delta \log c_{t+1}, r_{t+1}) &= \gamma \text{Cov}_t \left(\varepsilon_{t+1}, \frac{1 - \gamma \rho \delta}{1 - \rho \delta} \varepsilon_{t+1} \right) \\ &= \frac{\gamma (1 - \gamma \rho \delta)}{1 - \rho \delta} \sigma_\varepsilon^2. \end{aligned}$$

The consol bond premium is

$$\gamma \text{Cov}_t (\Delta \log c_{t+1}, r_{t+1}) = -\frac{\gamma^2 \rho \delta}{1 - \rho \delta} \sigma_\varepsilon^2.$$

(ii) Show that the bond premium has the opposite sign to δ and is proportional to the square of γ . Give an economic interpretation of this result.

We'll discuss this in more details later, but the point is that if $\delta > 0$, long-term bonds have a negative risk premium, since good news (positive ε_{t+1}) indicate high growth for a while, and hence interest rates go up, and bond prices go down. If $\delta < 0$ it is the opposite. Hence δ determines the mean return on long-term bonds (and the slope of the real yield curve).

Proportional to γ^2 : there are 2 gammas: the first one is the risk aversion and determines the risk premium; the second one is the 1/IES and determines the response of interest rate to a shock.

(iii) Show that the equity premium is always larger than the bond premium, and the difference between them is proportional to γ . Give an economic interpretation of this result.

$$\frac{\gamma(1-\gamma\rho\delta)}{1-\rho\delta}\sigma_\varepsilon^2 > -\frac{\gamma^2\rho\delta}{1-\rho\delta}\sigma_\varepsilon^2$$

since the difference is $\frac{\gamma}{1-\rho\delta}\sigma_\varepsilon^2$. The equity premium is a premium for two things: (a) riskiness of cash flows, and (b) duration of cash flows. The first term is $\gamma\frac{\sigma_\varepsilon^2}{1-\rho\delta}$, and the second term is $-\frac{\gamma^2\rho\delta}{1-\rho\delta}\sigma_\varepsilon^2$.

(iv) Given the results, discuss which case ($\delta > 0$ or $\delta < 0$) helps explain asset prices.

(a) $\delta > 0$ leads to a negative long-term bond premium, which seems strange (but might be real, see later in class).

(b) The equity risk premium is $\frac{\gamma(1-\gamma\rho\delta)}{1-\rho\delta}$. If $\gamma < 1$, this is increasing in δ , but if $\gamma > 1$, it is decreasing in δ . Hence, a high δ will not work if, as usual, $\gamma > 1$. (And that's why having Epstein-Zin preferences helps \rightarrow we can separate out the two γ .)

Problem Set 3 - Due Thursday, Feb 18 by 6pm in my mailbox

This problem set is empirical. You will have to download an excel data file from the courseinfo website. This excel file has series for annual consumption (a real index), an index of US stock excess returns (RMRF: market return less risk-free rate), a risk-free rate series, and six returns on portfolios of stocks, sorted by size (small vs. large) and book-to-market (low b/m, medium b/m, and high b/m).

Question 0: Preliminaries and cross-section of stocks

(1) Compute the consumption growth: C_t/C_{t-1} . Note that returns are from Jan 1 to Dec 31. You should match the consumption growth between 1929 and 1930 with the 1929 stock market return. (Some researchers match it with the 1930 stock market return, but I want everyone in class to do the same thing for these results.) Similarly compute inflation, P_t/P_{t-1} . Compute the market return by adding the risk-free rate to RMRF. Subtract inflation from the risk-free rate, the market return, and the 6 portfolio returns, to obtain real returns (Match the inflation between 1929 and 1930 with the 1929 stock return, just like for consumption). (Note: the risk-free rate = short-term US government bonds.)

(2) Report the mean, standard deviation, and autocorrelation of consumption growth, the risk-free rate, the market real return, and the 6 real portfolio returns.

(3) Report the mean and standard deviation of the log returns as well. (These may be more relevant for long-run investments, since the long-horizon stock return is $R_{t \rightarrow t+T} = \left(\prod_{s=0}^{T-1} R_{t+s, t+s+1} \right)^{\frac{1}{T}}$.)

(4) Compute the β of each portfolio return on the market return. Produce a plot with, on the x-axis, the market β , and the y-axis, the mean excess return. What is the prediction of the CAPM? What does the data say?

(5) Compute the β of each portfolio return on consumption growth, as well as the consumption β of the stock market. Produce a plot with, on the x-axis, the consumption β , and the y-axis, the mean excess return. What is the prediction of the Consumption CAPM? What does the data say?

Question 1: Hansen-Jagannathan bounds

In class I concentrated on the simple H-J inequality with one excess return:

$$\frac{\sigma(M)}{E(M)} \geq \frac{E(R^e - R^f)}{\sigma(R^e - R^f)}.$$

However we can in practice find tighter bounds if we use the model for multiple returns R^i . The question is, what is the minimal $\sigma(M)$ given $E(M)$ and given that $E(MR^i) = 1$, for $i = 1 \dots N$. Here is one derivation of the bounds (for more [not required], see the Hansen-Jagannathan JPE 1991 paper).

Write a linear regression of M on demeaned returns:

$$M = a + \sum_{i=1}^N b_i (R_i - E(R_i)) + \varepsilon,$$

i.e.

$$M - E(M) = R'b + \varepsilon,$$

where R is the vector of stacked, demeaned returns. We have

$$b = \text{Var}(R)^{-1} \text{Cov}(M, R).$$

It follows that

$$\text{Var}(M) \geq b' \text{Var}(R) b = \text{Cov}(M, R)' \text{Var}(R)^{-1} \text{Cov}(M, R).$$

From the asset pricing equation,

$$E(MR^i) = 1 \text{ for all } i,$$

hence

$$\text{Cov}(M, R) = E(MR) - E(M)E(R) = \mathbf{1} - E(M)E(R)$$

where again R is the vector of returns.

Conclusion: (this is the only thing you need!)

$$\text{Var}(M) = \sigma(M)^2 \geq (\mathbf{1} - E(M)E(R))' \text{Var}(R)^{-1} (\mathbf{1} - E(M)E(R)).$$

Note that $\mathbf{1}$ is a vector of 1.

Questions:

(1) Create a figure showing, as a function of $E(M)$, the required $\sigma(M)$ to make the inequality hold with equality. First use two assets, the real risk-free rate and the real market return. Then, use 8 assets (adding the 6 portfolios). Plot the two frontiers on the same picture. x-axis= $E(M)$, y-axis= $\sigma(M)$. Choose the scale!

(2) Using the H-J bounds. Consider the standard CRRA utility function

$$E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

Compute the stochastic discount factor. Given the consumption data, compute $E(M)$ and $\sigma(M)$ for various values of $\gamma = .5, 1, 2, 4, 6, 8, 10, 15, 20, 30, 40, \dots, 200$. Plot the resulting pairs $E(M), \sigma(M)$ on the figure of question 1. Is there a value of γ such that the SDF satisfies the H-J bounds?

(3) Now consider the following utility function, with an external habit:

$$E \sum_{t=0}^{\infty} \beta^t \frac{(c_t - \theta h_t)^{1-\gamma}}{1-\gamma},$$

$$h_t = (1 - \delta)h_{t-1} + \delta c_{t-1}.$$

Redo the exercise of question (2), trying different values for δ (e.g. .1, .01), and for θ (e.g. .9, .95) as well as $\gamma = 2, 4$. Are there parameter values such that the SDF satisfies the H-J bounds?

Question 2: GMM estimation

This question asks you to estimate by GMM the standard consumption-based model:

$$E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} \right) = 1.$$

Set $\beta = .998$, so you only have to estimate γ . For each question below, produce a table with the first stage and second stage estimates, the J test of overidentification (if possible) and the associated p-value. (Use Cochrane's book, chapters 10 and 11, for reference on GMM.)

Overall, we want to know

(a) What is the right value of γ ?

(b) Does this model work well or not? With this in mind, make sense of the estimates, standard errors and J-statistic. Try to explain intuitively the results.

Advice: You can use Matlab's minimizer (fminsearch) to solve the minimum of the objective function. It may be useful to plot as a function of γ , the objective function that you try to minimize. It may be useful to report the average error in the Euler equations, at the estimated parameter value, i.e.

$\frac{1}{T} \sum_{t=1}^T .998 \left(\frac{C_{t+1}}{C_t} \right)^{-\hat{\gamma}} R_{t+1} - 1$. This is an error in annual returns, e.g. .02 means that the model misses the mean returns by 200 basis points per year. The second stage of GMM picks an optimal combination of moments to set equal to zero. It may be useful to look which combination GMM is picking (i.e. the weighting matrix W) since in some cases, GMM "discards" some moments because they are not very informative. Please format your results carefully.

Questions:

(a) Estimate the model using only the excess return on the stock market (market return less risk free rate). 1 moment, 1 parameter!

(b) Estimate the model using only the real return on the stock market.

(c) Estimate the model using only the real risk-free return.

(d) Estimate the model using the real risk-free return and the real market return.

(e) Estimate the model using the real risk-free return and the real excess market return.

(f) Estimate the model using the 8 assets.

Problem Set 4 - Due Thursday, March 18 by 6pm in my mailbox

The economy lasts two periods. There are two types of agents. There is a measure 1 of “poor” agents who can only save or borrow in a bond (risk-free asset). These agents all have the same income y^P today. Tomorrow they have the income ηy^P , where η is a random variable which is iid across agents (i.e. this is uninsurable labor income risk). The agent’s problem is, given the gross interest rate $R = 1 + r$:

$$\max_b \{u(y^P - b) + \beta E_\eta u(y^P \eta + Rb)\},$$

where b is the amount borrowed. Denote the solution b_P^* . (Note that we must have $y^P \eta + Rb_P^* > 0$, for all possible realizations of η , so that agents can effectively repay their loan. We will solve the model without checking this condition, and we can verify at the end that it is OK.)

There is a measure 1 of “rich” agents who can buy stocks or bonds. The stock is a claim to a dividend d which is paid in period 2, and is random and distributed according to the cdf F . These agents also have some income y^R today, and they initially hold the stock. They will receive a (certain, same) income y^R tomorrow. Hence, their problem is

$$\max_{b,x} \{v(y^R + P - b - xP) + \beta E_d v(y^R + Rb + xd)\},$$

with solution b_R^* and x^* .

Questions:

- (1) Write the FOC of the poor agents (assuming an interior solution).
- (2) Write the FOCs of the rich agents (assuming an interior solution).
- (3) Write the market-clearing conditions.
- (4) Write a Matlab program to solve for the equilibrium prices and quantities. You can assume for now the following parameter values:

$$\begin{aligned} u(x) &= \frac{x^{1-\gamma_P}}{1-\gamma_P} \text{ with } \gamma_P = 2, \quad v(x) = \frac{x^{1-\gamma_R}}{1-\gamma_R} \text{ with } \gamma_R = 2 \\ \beta &= .95, y^P = 1, y^R = 2, \sigma_d = .1, \sigma_\eta = .2 \end{aligned}$$

and η is log-normally distributed with mean 1 and variance σ_η^2 , and d is log-normally distributed with mean $\bar{d} = 2$ and variance σ_d^2 . You can approximate each of these (independent) log-normal distributions with finite distributions as follows (this is a bit rough but essentially fine): let $N = 10$ points, and let $x = (x_i)_{i=1}^{10}$ be equally spaced between -3 and $+3$, and $\pi(x_i) = \frac{\phi(x_i)}{\sum_{j=1}^{10} \phi(x_j)}$ where $\phi =$ standard normal p.d.f., then η is approximated with the values $e^{x_i \sigma_\eta - \frac{\sigma_\eta^2}{2}}$ with prob $\pi(x_i)$, and d with the values $\bar{d} e^{x_i \sigma_d - \frac{\sigma_d^2}{2}}$.

- (5) Plot the equilibrium risk-free rate (RFR) and equity premium (EP), as a function of σ_η . Explain intuitively your results.
- (6) Plot the RFR and EP, as a function of y_R . Explain intuitively.
- (7) Plot the RFR and EP, as a function of σ_d . Explain intuitively.
- (8) Compare numerically the equilibrium risk-free rate and equity premium with the same economy but with complete markets (i.e. a representative consumer with the utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma = \gamma_p = \gamma_r$.) Explain intuitively.

Sample short questions

Short questions: no formal proof is required, but a clear explanation (perhaps with one or two equations or citing some work) is enough.

- (1) Explain what is the equity premium puzzle in two sentences.
- (2) What is the effect of an increase in expected future growth on the price of a nominal bond? On the price of a real bond (TIPS)?
- (3) In the US, starting around 1985, the volatility of consumption growth fell (the “Great Moderation”). What would a standard asset pricing model predict for the risk-free rate and the equity premium, if people realize immediately in 1985 the decrease in volatility? (You may assume that the IES is > 1 .)
- (4) Answer (3) again if people learn progressively that consumption volatility is falling. (Plot the risk-free rate, the realized equity return, and the P-D ratio.)
- (5) The real return on gold is high when inflation expectations are high. As a result, Gold returns are _____ correlated with long term bond returns. (Fill the blank.)
- (6) Suppose a CEO can pick its firm’s consumption β costlessly. What would he do?
- (7) In the Campbell-Cochrane model, low asset prices (e.g. low P/D ratio) forecasts high future economic growth. T/F/U.
- (8) In the XIXth century, because of the gold standard, the price-level was stable, so an increase in inflation today would lead to lower inflation in the future. What does this imply for the slope of the nominal yield curve? (You may assume that real consumption growth is iid.)
- (9) A journalist tells you, “isn’t it obvious that there are bubbles in the stock and housing markets?”. What do you answer (4 sentences max).
- (10) During the financial crisis, bank stock prices fell by a large amount, even when they were not holding bad assets. Hence, their prices fell substantially more than their expected future earnings. Why could that be, and is there something you could look at which would test your explanation?

Sample short questions - Answer key

Short questions: no formal proof is required, but a clear explanation (perhaps with one or two equations or citing some work) is enough.

(1) Explain what is the equity premium puzzle in two sentences.

Different ways to do it – here’s my take: the covariance between equity returns and cons. growth in the data is low, so we need a high risk aversion to make it consistent with the CRRA-repr. agent Euler equation:

$$E(R^e - R^f) \simeq \gamma \text{Cov}(R^e, \Delta \log C).$$

(2) What is the effect of an increase in expected future growth on the price of a nominal bond? On the price of a real bond (TIPS)?

Higher expected future growth leads interest rates to go up (think Euler equation), so long-term real bond prices fall. Holding expected future inflation constant, long-term nominal bond prices fall too.

(3) In the US, starting around 1985, the volatility of consumption growth fell (the “Great Moderation”). What would a standard asset pricing model predict for the risk-free rate and the equity premium, if people realize immediately in 1985 the decrease in volatility? (You may assume that the IES is > 1 .)

Risk-free rate rises since less precautionary savings. Equity premium falls. No role for IES here!! P-D ratio would rise if $\text{IES} > 1$.

(4) Answer (3) again if people learn progressively that consumption volatility is falling. (Plot the risk-free rate, the realized equity return, and the P-D ratio.)

Risk-free rate would rise progressively, and EXPECTED eq. premium would fall progressively. P-D would rise progressively. Key point: the realized return would go UP as the expected returns go down – a reduction in the risk is good today. (Add plot in class.)

(5) The real return on gold is high when inflation expectations are high. As a result, Gold returns are ----- correlated with long term bond returns. (Fill the blank.)

Negatively – higher inflation expectations lowers the price of a long-term bond but increases the price of gold.

(6) Suppose a CEO can pick its firm’s consumption β costlessly. What would he do?

Set β as low as possible! The stock price would go up... (so it better be costly to reduce β .)

(7) In the Campbell-Cochrane model, low asset prices (e.g. low P/D ratio) forecasts high future economic growth. T/F/U

In the CC model, cons. growth is iid: *nothing* can forecast future economic growth.

(8) In the XIXth century, because of the gold standard, the price-level was stable, so an increase in inflation today would lead to lower inflation in the future. What does this imply for the slope of the nominal yield curve? (You may assume that real consumption growth is iid.)

You have to remember that the slope of the yield curve depends on the average return on a short-term vs. long-term bond. If inflation goes up today, short-term bonds will lose value, but long-term bonds will not lose value much if we expect inflation to be lower in the future (math: the long-term bond price is $Q_t = \beta^k E_t \left(\left(\frac{C_{t+k}}{C_t} \right)^{-\gamma} \frac{1}{P_{t+k}} \right)$, so if P_{t+k} is known in advance there is only real risk (risk to consumption)

as opposed to nominal risk. Hence, long-term bonds are safer assets than short-term bonds, and the yield curve would slope downward. (I think that actually was the case back then.)

(9) *A journalist tells you, "isn't it obvious that there are bubbles in the stock and housing markets?". What do you answer (4 sentences max).*

What do we mean by bubble? Empirically: when prices are too high relative to "fundamentals", then they fall a lot. There's some evidence for this, in the form of the predictability regressions, $R_{t+1} = \alpha + \beta \frac{D_t}{P_t} + \varepsilon_{t+1}$. But (a) this evidence is not super-strong (i.e. it is really hard in real time to tell a bubble apart from a permanent rise in stock prices!), (b) bubbles can be rational, e.g. because of time-varying risk.

(10) *During the financial crisis, bank stock prices fell by a large amount, even when they were not holding bad assets. Hence, their prices fell substantially more than their expected future earnings. Why could that be, and is there something you could look at which would test your explanation?*

Bank stocks β went up like crazy - around 3 or even more. Not only their earnings were low but they were perceived as super-risky because the economy would go down if the banks were going down. You can actually see this using high frequency (daily) correlations.

Exam

Instructions:

- (1) Write your name above
- (2) **Budget your time carefully (1h35mn and 95pts so 1mn = 1pt)**
- (3) If you run out of space write on the back of the page.

Short questions (each question is 5 points)

For the short questions, no formal proof is required, but a clear explanation (perhaps with one or two equations or a citation of some work) is enough.

- (1) State briefly two facts regarding the the nominal yield curve.
- (2) Firms such as airlines or banks have highly cyclical profits. What does this imply for their consumption β and for the expected return on these stocks?
- (3) What is the effect of an increase in expected future inflation on the price of a nominal bond? On the price of a real bond (TIPS)?
- (4) A decrease in the supply of oil (e.g. because of OPEC) typically reduces world output and increases inflation. Suppose that there are no oil demand shocks. What does this imply for the average return on holding oil?
- (5) An increase in world output typically increases world oil demand. Suppose there are no oil supply shocks. What does this imply for the average return on holding oil?
- (6) Plot, for $t = 1 \dots 10$ the path of realized returns for a stock which has an unexpected increase in its systematic risk at time $t = 5$. Plot the path of expected returns on the same graph. Also plot the P-D ratio of the stock. (Make whatever assumptions you feel are necessary/reasonable and state them.)
- (7) According to theory, countries with more volatile consumption and dividends should have more volatile stock prices. T/F/U.
- (8) An increase in the volatility of inflation would increase the average slope of the nominal yield curve. T/F/U.
- (9) A journalist tells you, “Markets are inefficient, since prices often rise then crash without changes in dividends”. What do you answer (4 sentences max).
- (10) Consider the Bansal-Yaron “long-run risk model”, with only the long-run shock (i.e. without the volatility shock). Which facts would that model match, and which would it have difficulty matching?
- (11) More volatile stocks are more risky and should have higher average returns. T/F/U

Long question 1 (20 pts)

Consider a standard representative agent model, with power utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Let $X_{t+1} = \frac{C_{t+1}}{C_t}$. We assume that X_{t+1} is *iid* over time, but its distribution is not log-normal.

- (1) Write the Euler equation for any gross return R_{t+1} .
- (2) Write an equation giving the price Q_t of a one-period risk-free bond.
- (3) Write an equation giving the price P_t of a claim to consumption tomorrow (and tomorrow only, i.e. $D_{t+1} = C_{t+1}$ and $D_{t+k} = 0$ for $k \geq 2$). (Call this the “equity” asset.)
- (4) Describe the real yield curve in this model. Does the expectation hypothesis hold?
- (5) Suppose that $\gamma = 3$ and that $X > 0$, $E(X) < \infty$, $E(X^{-2}) < \infty$, but $E(X^{-3}) = +\infty$. What is the risk-free rate and what is the equity premium in this model? Explain.

Long question 2 (20 pts)

Consider a standard representative agent model, with power utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

We assume that the joint distribution of consumption growth and inflation is iid and normally distributed, i.e.

$$(\Delta \log C_{t+1}, \Delta \log P_{t+1}) \text{ is } N(\mu, \Sigma),$$

where P_t is the price level (1 good = P_t \$), μ is a 2x1 vector and Σ is a 2x2 covariance matrix:

$$\mu = \begin{pmatrix} \mu_c \\ \mu_p \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \sigma_{cc} & \sigma_{cp} \\ \sigma_{cp} & \sigma_{pp} \end{pmatrix}.$$

- (1) Compute the price N_t of a one-period risk-free real bond.
- (2) Compute the price Q_t of a one-period risk-free nominal bond, i.e. a claim to 1\$ tomorrow.
- (3) Compare the expected returns on the nominal bond and on the real bond. Explain the intuition briefly.

Exam Answer Key

Short questions (each question is 5 points)

For the short questions, no formal proof is required, but a clear explanation (perhaps with one or two equations or citing some work) is enough.

(1) *State briefly two facts regarding the the nominal yield curve.*

For instance, the yield curve is upward sloping on average, and the expectation hypothesis fails, i.e. bond returns are predictable. Or: the yield curve slope predicts the business cycle. Or: bond yields tend to move together.

(2) *Firms such as airlines or banks have highly cyclical profits. What does this imply for their consumption β and for the expected return on these stocks?*

* They should have high β and high expected returns since they are very procyclical.

[Note: this is PS1.]

(3) *What is the effect of an increase in expected future inflation on the price of a nominal bond? On the price of a real bond (TIPS)?*

Higher expected inflation reduces the value of a long-term nominal bond. It has no effect on a real bond (holding real consumption constant). [Some people said, perhaps real rates will rise (a “liquidity effect”) e.g. if prices are sticky; in this case the price of the real bond would fall as well.]

[Note: many people gave incorrect answers here, but the basic point is that a nominal bond is a promise to 1\$ and if there will be inflation the real value of 1\$ in the future falls.]

(4) *A decrease in the supply of oil (e.g. because of OPEC) typically reduces world output and increases inflation. Suppose that there are no oil demand shocks. What does this imply for the average return on holding oil?*

[Note on questions 4 and 5 - few people answered correctly. The basic point was to look for the covariance with consumption growth. I could have been slightly more precise in my wording, and said: suppose there are supply shocks from time to time and no demand shocks, what is the average return on holding oil.]

This implies that oil prices go up when output goes down. Hence, oil is a hedge against fluctuations in output (and thus consumption): it is not risky and has a low average return (lower than the risk-free rate).

(5) *An increase in world output typically increases world oil demand. Suppose there are no oil supply shocks. What does this imply for the average return on holding oil?*

This implies that oil prices go up when output goes up. Hence, oil is a risky asset, it has a high average return (higher than the risk-free rate).

(6) *Plot, for $t = 1 \dots 10$ the path of realized returns for a stock which has an unexpected increase in its systematic risk at time $t = 5$. Plot the path of expected returns on the same graph. Also plot the P-D ratio of the stock. (Make whatever assumptions you feel are necessary/reasonable and state them.)*

* This is the plot we did in class: at $t = 5$, the return is low, then the return is high. Expected returns are low then high. The P-D ratio falls on impact.

(7) According to theory, countries with more volatile consumption and dividends should have more volatile stock prices. T/F/U.

Uncertain. This is (by choice) a bit vague and you could talk about different things. In simple models, P/D ratio is constant and vol of stock return = vol of dividends (e.g. the log-normal iid model). Hence the vol of P-D ratio is actually 0, no matter what is the vol. of consumption. The vol of P = the vol of D so this is true in this simple model.

[More elaborate: in more complicated model, and in the data, variation in P/D ratios stock prices is mostly driven by variation in discount rates not dividends. The vol. of cons/dividends itself may be a poor proxy for the variation of discount rates. (e.g. think of the B-Y model: the vol of P-D is largely due to shocks to shocks to uncertainty, not to shocks to cons. growth - the iid shock to cons. growth has little impact on stock prices.)]

(8) An increase in the volatility of inflation would increase the average slope of the nominal yield curve. T/F/U.

Uncertain/False in general: The average slope of the yield curve depends on the covariance between inflation and cons. growth. If it is positive, then increase in inflation occur in good times, and so long-term bonds lose value in good times, so yield curve is downward sloping on average. Then, adding to the vol of inflation would make this covariance stronger, and make the yield curve more downward sloping. But if the covariance is >0 the opposite is true. (Adding pure noise in inflation, i.e. unrelated to cons. growth, would not affect the average slope...)

(9) A journalist tells you, "Markets are inefficient, since prices often rise then crash without changes in dividends". What do you answer (4 sentences max).

Different possible answers... Empirically the journalist is right, see Campbell-Shiller and return predictability. [Almost nobody mentioned the empirical evidence!!] The standard explanation is that it is hard to tell irrationality apart from variation in risk (or more generally, variation in discount rates). With variation in risk, stock prices can fall without changes in expected earnings/dividends.

(10) Consider the Bansal-Yaron "long-run risk model", with only the long-run shock (i.e. without the volatility shock). Which facts would that model match, and which would it have difficulty matching?

The model would get a large equity premium, and volatile stock prices, however it would imply that high P-D ratios forecast high future dividend growth.

[Many of you seem unaware of the paper. You must read the Campbell-Cochrane and Bansal-Yaron papers.]

(11) More volatile stocks are more risky and should have higher average returns. T/F/U.

* False. It's the covariance that matters! An asset which returns is negatively correlated with consumption growth have a lower mean return than the risk-free asset.

Long question 1 (20 pts)

Consider a standard representative agent model, with power utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Let $X_{t+1} = \frac{C_{t+1}}{C_t}$. We assume that X_{t+1} is iid over time, but its distribution is not log-normal.

* (1) Write the Euler equation for any gross return R_{t+1} .

$$E_t (\beta X_{t+1}^{-\gamma} R_{t+1}) = 1.$$

* (2) Write an equation giving the price Q_t of a one-period risk-free bond.

$$Q_t = \beta E_t (X_{t+1}^{-\gamma}).$$

[Many people were confused about this. We've solved many times for this in the class...]

* (3) Write an equation giving the price P_t of a claim to consumption tomorrow (and tomorrow only, i.e. $D_{t+1} = C_{t+1}$ and $D_{t+k} = 0$ for $k \geq 2$). (Call this the "equity" asset.)

$$\begin{aligned} P_t &= E_t (\beta X_{t+1}^{-\gamma} C_{t+1}) \\ &= C_t \beta E_t (X_{t+1}^{1-\gamma}). \end{aligned}$$

(4) Describe the real yield curve in this model. Does the expectation hypothesis hold?

Consumption growth is iid so the real interest rate is constant. Hence the yield curve is flat.

(5) Suppose that $\gamma = 3$ and that $X > 0$, $E(X) < \infty$, $E(X^{-2}) < \infty$, but $E(X^{-3}) = +\infty$. What is the risk-free rate and what is the equity premium in this model? Explain.

$$\begin{aligned} Q_t &= \beta E(X^{-3}) = +\infty, \\ P_t &= C_t \times \beta E(X^{-2}). \end{aligned}$$

The price of a bond is infinite! because of the risk in the left tail of cons. growth. Hence the risk-free rate is 0 and the equity premium is infinite.

Long question 2 (20 pts)

Consider a standard representative agent model, with power utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

We assume that the joint distribution of consumption growth and inflation is iid and normally distributed, i.e.

$$(\Delta \log C_{t+1}, \Delta \log P_{t+1}) \text{ is } N(\mu, \Sigma),$$

where P_t is the price level (1 good = P_t \$), μ is a 2x1 vector and Σ is a 2x2 covariance matrix.

(1) Compute the price N_t of a one-period risk-free real bond.

$$\begin{aligned} N_t &= E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right) \\ &= \beta E_t \exp(-\gamma \Delta \log C_{t+1}) \\ N_t &= N = \beta \exp \left(-\gamma \mu_c + \frac{\gamma^2 \Sigma_{cc}}{2} \right). \end{aligned}$$

(2) Compute the price Q_t of a one-period risk-free nominal bond, i.e. a claim to 1\$ tomorrow.

$$\begin{aligned} Q_t &= E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{1}{P_{t+1}} \right) \\ &= \frac{1}{P_t} E_t \left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{P_t}{P_{t+1}} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{P_t} \beta E_t \exp(-\gamma \Delta \log C_{t+1} - \Delta \log P_{t+1}) \\
&= \frac{1}{P_t} \beta E_t \exp \left(\begin{array}{l} -\gamma E_t \Delta \log C_{t+1} - E_t \Delta \log P_{t+1} + \frac{1}{2} \gamma^2 \text{Var}_t(\Delta \log C_{t+1}) \\ + \frac{1}{2} \text{Var}_t(\Delta \log P_{t+1}) + \gamma \text{Cov}_t(\Delta \log C_{t+1}, \Delta \log P_{t+1}) \end{array} \right) \\
Q_t &= \frac{1}{P_t} \beta \exp \left(-\mu_p - \gamma \mu_c + \frac{\gamma^2}{2} \Sigma_{cc} + \frac{1}{2} \Sigma_{pp} + \gamma \Sigma_{cp} \right)
\end{aligned}$$

(3) Compare the expected returns on the nominal bond and on the real bond. Explain the intuition briefly.

These expected returns are constant since everything is iid, and the expected return on the risk-free real bond is

$$R^f = \frac{1}{N} = \frac{1}{\beta \exp(-\gamma \mu_c + \frac{\gamma^2}{2} \Sigma_{cc})},$$

while on the nominal bond it is

$$\begin{aligned}
R^n &= \frac{1}{Q_t} E_t \left(\frac{1}{P_{t+1}} \right) \\
&= \frac{1}{\beta \exp(-\mu_p - \gamma \mu_c + \frac{\gamma^2}{2} \Sigma_{cc} + \frac{1}{2} \Sigma_{pp} + \gamma \Sigma_{cp})} E_t \left(\frac{P_t}{P_{t+1}} \right) \\
&= \frac{1}{\beta \exp(-\mu_p - \gamma \mu_c + \frac{\gamma^2}{2} \Sigma_{cc} + \frac{1}{2} \Sigma_{pp} + \gamma \Sigma_{cp})} \exp \left(-\mu_p + \frac{1}{2} \Sigma_{pp} \right) \\
&= \frac{1}{\beta \exp(-\mu_p - \gamma \mu_c + \frac{\gamma^2}{2} \Sigma_{cc} + \gamma \Sigma_{cp})}
\end{aligned}$$

hence the difference of expected returns is

$$\frac{R^n}{R^f} = \frac{\beta \exp(-\gamma \mu_c + \frac{\gamma^2}{2} \Sigma_{cc})}{\beta \exp(-\mu_p - \gamma \mu_c + \frac{\gamma^2}{2} \Sigma_{cc} + \gamma \Sigma_{cp})} = \exp(-\gamma \Sigma_{cp}).$$

If $\Sigma_{cp} > 0$, then $R^n < R^f$, while if $\Sigma_{cp} < 0$, $R^n > R^f$.

Intuition: if inflation is procyclical ($\Sigma_{cp} > 0$), the return on the nominal bond is lower in booms, when marginal utility is low, so nominal assets are hedges. Hence, they have a negative risk premium ($R^n < R^f$). If inflation is countercyclical, that's the opposite.