Weak Ex Ante Collusion and Design of Supervisory Institutions[1]

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Abstract

A Principal seeks to design a mechanism for an agent (privately informed regarding production cost with a continuous distribution) and a supervisor/intermediary (with a noisy signal of the agent's cost) that collude *ex ante*, i.e., on *both* participation and reporting decisions. Collusion is 'weak' in the sense that neither colluding party can commit to how they would behave if they fail to mutually agree to a side-contract. We provide conditions under which the Principal's problem reduces to selecting weak collusion-proof (WCP) allocations. We characterize WCP allocations, and use this to show that it is always valuable to employ the supervisor if the good is divisible. Delegation is optimal, but only if supplemented by an appeal/dispute settlement mechanism mediated by the Principal, which serves as an outside option for coalitional bargaining. Changes in bargaining power within the coalition have no effect, while altruism of the supervisor towards the agent makes the Principal worse off.

KEYWORDS: mechanism design, intermediation, supervision, collusion, delegation

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1 Introduction

The potential for collusion is widely acknowledged to be a serious problem for a Principal who relies on information provided by an expert intermediary or supervisor to design a contract for a productive agent. Examples of such contexts abound: an investor that relies on an investment bank or rating agency for information necessary to decide on financing an entrepreneur; shareholders that rely on outside directors of a company to supervise its CEO; an owner that relies on a manager for information needed to set production targets and compensation for workers or suppliers; a government that relies on a regulator to advise on rates for a public utility, or on an auditor to evaluate the eligibility of a private firm for an investment tax credit. In these settings the supervisor is typically better informed about the agent's productivity or cost than the Principal, but less informed than the agent. Eliciting information becomes problematic when the supervisor is willing to misreport information in exchange for suitable side payments with the agent. Evidence for such collusion has recently been forthcoming in many areas, e.g., between outside Directors and CEOs (Hallock (1997), Hwang and Kim (2009), Fracassi and Tate (2012), Kramarz and Thesmar (2013), Schmidt (2015)), between management and workers (Bertrand and Mullainathan (1999, 2003), Atanassov and Kim (2009), Cronqvist et al (2009)), 'revolving doors' betwen credit-rating agencies and firms (de Haan et al (2015), Cornaggia et al (2016)) and between auditors and their clients (Lennox (2005), Lennox and Park (2007), Firth et al (2012)).

Early literature on the mechanism design problem with collusion (e.g., Tirole (1986), Laffont and Tirole (1993), Kofman and Lawarree (1993)) was based on the assumption of hard information (where the supervisor cannot lie, and can only withhold information), and exogenous transaction costs of collusion. Subsequent literature has considered contexts where the collusion problem is harder to control, owing to soft information (which allows the supervisor to report anything) and absence of exogenous transaction costs of collusion (e.g., Laffont and Martimort (1997, 2000), Baliga and Sjostrom (1998), Faure-Grimaud, Laffont and Martimort (2003), Che and Kim (2006), Celik (2009)). Most of the literature, however, considers only the possibility of *interim* collusion, where supervisor and agent can collude over reporting decisions, but not whether to participate in the mechanism.

The interim collusion formulation [[in the adverse selection setting]] is subject to two significant drawbacks. The first is that some results turn out to depend sensitively on fine details of the information structure, for reasons that are not particularly transparent or intuitive. One of the key questions is whether delegation to the supervisor (denoted S hereafter) is an optimal response of the Principal (denoted P) to collusion. In their seminal papers, Faure-Grimaud, Laffont and Martimort (2003, denoted FLM hereafter) and Celik (2009) have studied this and obtain contrasting results in a setting where the agent (denoted A) has either two or three possible types, depending on whether S's information is represented by full support or a partition (with shifting support).[5]

The context of interim collusion also requires some implicit, unmodeled restrictions on communication possibilities between the players at the participation stage (where S and A commit non-cooperatively to participate in the mechanism proposed by P). In the absence of any restriction of message spaces, it is trivially possible for P to overcome collusion completely by requiring S or A to communicate their private information aside from their participation decision at this stage. For instance, P may offer each a menu of contracts, from which S and A are required to respectively select at the same time that they communicate their participation decision. This obviates the need for any subsequent communication after both have accepted and selected their respective contracts. Since by assumption S and A cannot collude at the participation stage, such a design would completely eliminate any scope for collusion that undermines P's payoff. The analysis of interim collusion therefore requires an exogenous restriction to binary message spaces at the participation stage. While it may be possible to imagine environments where such a formulation is relevant[6], such formulations of the contracting environment are non-standard and invoke additional frictions beyond private information and collusion *per se*. This suggests the need to consider an environment of ex ante collusion, where S and A collude **before** agreeing to participate. In an ex ante collusion setting, no *ad hoc* restrictions on message spaces need be imposed.

[5]Celik also finds delegation to be optimal in some settings involving interim collusion and a partition information structure, if optimal output schedules are non-monotone in type.

[6][[For instance, interim collusion would be relevant in a formulation where (i) there are many potential supervisors and agents who will be matched by P, each S and A are not informed who they will be matched with while committing to participate, and it is too costly for specific S-A pairs to enter into collusion contracts ex ante, conditional on being matched; (ii) although there are many potential supervisors and agents, S is still somehow informed about every A, even before being matched with one of them; (iii) S is indispensable for production; (iv) P's communication possibilities are restricted at the participation stage because it is too costly to write menus of contracts at that stage or alternatively certain features of the project are not well defined at that stage rendering the contract incomplete.]]

This motivates the current paper: we examine a setting of ex ante collusion, with a continuum type space for A and monitoring structures which accommodate both full support and partition cases. We study optimal mechanisms without imposing any exogenous constraints on message spaces, and obtain general results that do not depend sensitively on details of the information structure, and are relatively easy to explain and interpret. We examine the extent to which the results in the interim collusion context obtained by FLM and Celik extend to this setting.

Further details of the context we study are the following. P has a benefit function which is strictly increasing and concave in an output supplied by A. For most part we consider the 'divisible output' case where the benefit function is strictly concave and satisfies Inada conditions, so that optimal production allocations are always interior. However we also consider an extension to an 'indivisible output' case where P's decision is limited to whether or not to buy a fixed quantity. A is privately informed regarding its (constant unit) cost of production whose realization is drawn from a continuous distribution over a compact support. S costlessly obtains a signal with finite support which is informative regarding the realization of A's cost; the realization of this signal is also observed by A. Side contracts which coordinate participation, reports to P and side-payments between S and A, are assumed to be costlessly enforceable. This side contract is negotiated between S and A; the key friction in collusion is the existence of one-sided asymmetric information between S and A regarding the latter's cost realization. We restrict attention to contexts involving weak collusion, where neither S nor A can commit to how they would behave in the event that they fail to agree on a collusive side-contract (whereupon they play the contract designed by P noncooperatively). The solution concept we employ requires that an allocation designed by the Principal should (besides meeting interim participation constraints) leave no room for design of a non-null side contract (and selection of a weak Perfect Bayesian Equilibrium of the resulting continuation game) which is Pareto improving for the colluding parties, and generates a strict improvement for the designer of the side contract.[7]

[7]The Appendix shows that the Principal would not benefit from allowing collusion to occur on the equilibrium path. Moreover, in the formulation of the side contracting problem there is no loss of generality in restricting attention to side contracts that are always accepted by the colluding parties, thereby addressing a problem highlighted by Celik and Peters (2011). [[The latter authors construct an example of collusion among oligopolistic firms where some payoffs can be sustained only via side contracts that are accepted by some types but not others. Our results generalize the traditional approach based on assumptions of 'passive

Our first main finding is that delegation of contracting to S is **never** an optimal strategy for P; instead it is dominated by a setting where P does not employ a supervisor at all. This is driven directly by the feature of ex ante collusion wherein participation decisions can be coordinated between S and A, giving rise to a 'limited liability' constraint for S in the pure delegation setting. Ex ante collusion thus generates the classic phenomenon of double marginalization of rents (DMR): S is privately informed regarding the rents that are paid to A within the coalition, resulting in a cascading of rents. Such rent cascades can be avoided if P were not to appoint S at all, and contract directly with A. Hence, results concerning optimality of delegation as a response to interim collusion for some specific information structures fails to extend to the ex ante collusion context in a very general and robust manner.

Nevertheless, a form of modified delegation always turns out to be optimal, where A does not contract directly with P on the equilibrium path, but has the option to do so off the equilibrium path. This provides A with an outside option while negotiating the side-contract with S, which varies with A's true type. Such a type-dependent outside option allows P to manipulate the side-contract by providing 'countervailing incentives' to A which reduces the production and welfare loss from collusion.[8] We show that ex ante collusion is optimally countered by providing A with the opportunity to trigger an 'appeals' mechanism whereupon S's authority to contract with A is revoked, and both S and A are required to submit independent reports to P.

We thereafter examine whether such countervailing incentives can control the DMR problem sufficiently so as to rationalize the appointment of S. This turns out to depend on whether the good in question is indivisible or divisible. In the divisible good case, hiring S is shown to be *always* valuable. For certain ranges of A's cost, P raises the latter's outside option sufficiently in order to force S to supply higher output relative to a setting where S is not appointed. Over other ranges, P lowers A's outside option, thereby inducing output to contract. This enlarges S's rent over the latter range, while shrinking it over the former range. As S is privately informed regarding the realization of the signal of A's cost, she

beliefs' employed in previous papers (e.g., Laffont and Martimort (1997, 2000) or FLM), in a manner that addresses the Celik-Peters problem.]] Celik and Peters (2013) develop an alternative approach to this problem which involves 'reciprocal' side-contract offers by colluding side parties.

[8] The effectiveness of such countervailing incentives as a method of deterring collusion have previously been highlighted by Mookherjee and Tsumagari (2004) and Celik (2009).

evaluates the relative probability of the two ranges differently from the way that P does. This enables the construction of a 'modified delegation' contract which realizes 'gains from trade', benefitting P without inducing S to quit. In the indivisible good case, the same turns out to be true for most but not all possible parameter values. Intuitively, this is because [[the value of S's information is more restricted in the indivisible case:]] the decision space is restricted to 'buy-not buy', with no corresponding variation on the intensive margin regarding the 'quantity to buy'.

Does collusion create a welfare loss for P? For the divisible good case, we provide sufficient conditions where this is the case. This result contrasts with the general results obtained by Che and Kim (2006) or Motta (2009) for interim collusion settings where the second-best welfare can be achieved by P. In the case of an indivisible good, the second-best can be achieved for some parameter values when S's signal is binary, but this turns out to be generically impossible when S's signal takes three or more possible values.

Our final set of results concerns the effect of altruism or variations in bargaining power between S and A. We show that altruism of S with respect to A hurts P in general. This result is intuitive, as altruism limits the frictions created by asymmetric information within the coalition. On the other hand, changes in bargaining power between S and A over the side-contract turn out not to matter. This is a consequence of the assumption of weak collusion, wherein coordinated deviations occur only if there is room for a feasible (interim) Pareto improvement relative to the given outside options defined by P's appeal mechanism. Consequently, if collusion is deterred for any specific set of bargaining weights, it is also deterred for any other. These results happen to be common to both settings of interim and ex ante collusion.

In the indivisible good case, we examine the nature of optimal mechanisms in more detail when S's signal has two possible realizations, and are able to numerically compute optimal solutions for the case of uniformly distributed cost. In this setting, we examine the relationship between the solutions in interim and ex ante collusion. We verify that the contrasting results of FLM and Celik concerning (sub-)optimality of delegation for interim collusion in the discrete type case extend respectively to non-degenerate parameter subsets in the continuum type case. Moreover, the solution to the interim collusion problem turns out to coincide with the solution to the ex ante collusion in exactly those cases (such as a partition information structure) where delegation is suboptimal under interim collusion.

Some general implications of our results for hierarchical organizational design are as follows. Our theory rationalizes the widespread prevalence of supervisors, despite the potential for collusion. Moreover, collusion is typically costly for the Principal, in contrast to interim collusion which can be overcome via mechanisms of the sort constructed by Motta (2009). Our theory does not rationalize unconditional delegation of authority to the supervisor; instead, delegation needs to be supplemented by scope for agents to 'appeal' and trigger direct communications with the Principal. Such appeals do not arise in equilibrium. But the scope for such appeals indirectly promote the agent's bargaining power with the supervisor (by altering outside options in coalitional bargaining), which reduces the severity of the double-marginalization-of-rents (DMR) problem sufficiently to result in a net benefit to the Principal. Within firms, it explains the role of worker rights to appeal the evaluations reported by their managers to higher level managers or an ombudsman appointed for this purpose. This echoes Williamson's (1975) view of such dispute settlement procedures as an advantage of hierarchies over market relationships. [[It is also resembles Hirschman's (1970) depiction of the value of 'voice' within organizations over and above exit options.]]

Our result concerning the irrelevance of allocation of bargaining power within the coalition implies that collusion costs are unaffected by alternative mechanisms for matching supervisors and agents, e.g., whether an agent should be allowed to select an auditor on a competitive market, or whether the Principal should appoint the auditor instead. This result depends on the assumption of 'weak' collusion, which does not allow either colluding partner to make commitments regarding how it will behave following a breakdown in collusion negotiations. When such commitments are possible, the notion of weak collusion needs to be replaced by a suitable notion of 'strong' collusion, the implications of which we are currently exploring in a subsequent paper. The result concerning effects of altruism of S towards A implies that the Principal ought to appoint 'outside' self-interested supervisors rather than 'insiders' likely to be altruistic towards the agent. In the context of corporate governance, for instance, this is an argument in favor of appointing 'outsiders' rather than 'insiders' to a company's Board of Directors.[9] In the context of regulation, it confirms the normal intuition in favor of preventing any direct conflict of interest for the supervisor (e.g., who should not have a financial stake in the agent's fortunes, nor have any social or

[9] See Harris and Raviv (2008) for a model based on limited commitment by P where this result may not hold in some settings.

personal connections with the agent). This result is not entirely obvious as altruism has some benefits for the Principal: it limits the inclination of the supervisor to extract rents from the agent that is the source of the DMR problem.

The paper is organized as follows. Section 2 describes relation to existing literature in more detail. Section 3 introduces the model, followed by definition and characterization of WCP allocations. To simplify the exposition, we focus initially on the divisible goods case where optimal allocations are always interior, and the supervisor has all the bargaining power within the coalition. The main results concerning properties of optimal weak-collusion-proof mechanisms are presented in Section 4. Section 5 then considers a number of extensions: (a) alternative allocations of bargaining power within the coalition, wherein side contracts are designed and offered by a third party maximizing a weighted sum of supervisor and agent's payoffs; and (b) the supervisor may exhibit altruism towards the agent. Section 6 considers how the preceding results are modified in the indivisible goods case, and the relation between solutions to *interim* and *ex ante collusion.* Finally, Section 7 concludes with a summary, and a discussion of extensions of the model in various other directions.

2 Relation to Existing Literature

The literature on mechanism design with collusion and 'soft' information can be classified by the context (auctions, team production or supervision), the nature of collusion (ex ante or interim, weak or strong collusion), and whether type spaces are discrete or continuous. Auctions and team production involve multiple privately informed agents and no supervisor. Baliga and Sjostrom (1998) consider a team setting with two productive agents that collude, involving moral hazard and limited liability rather than adverse selection. For auctions, Dequiedt (2007) considers strong ex ante collusion with binary agent types and shows that efficient collusion is possible, implying that the second-best cannot be implemented. In contrast, Pavlov (2008) considers a model with continuous types where the second-best can be implemented with weak ex ante collusion, and Che and Kim (2009) find the same result with either weak or strong ex ante collusion with continuous types.

Team production with binary types is studied by Laffont and Martimort (1997), who show the second best can be implemented with weak interim collusion; this analysis is extended to a public goods context in Laffont and Martimort (2000) where the role of correlation of types is explored. Che and Kim (2006) show how second-best allocations can be implemented in a team production context with continuous types in the presence of weak interim collusion. Quesada (2004) on the other hand shows strong ex ante collusion is costly in a team production model with binary types. Mookherjee and Tsumagari (2004) show delegation to one of the agents is worse than centralized contracting in the presence of weak ex ante collusion. The logic of this result is similar to that underlying our result that delegation to the supervisor is worse than not appointing a supervisor. Their paper also considers delegation to a supervisor who is perfectly informed about the costs of each agent, and show that its value relative to centralized contracting depends on complementarity or substitutability between inputs supplied by different agents. The current paper differs insofar as there is only one agent, and there is asymmetric information within the supervisor-agent coalition owing to the supervisor receiving a noisy signal of the agent's cost. This friction in coalitional bargaining plays a key role in the current paper.

In the context of collusion between a supervisor and agent, existing models (with the exception of Mookherjee-Tsumagari (2004)) have explored *interim* collusion only. Faure-Grimaud, Laffont and Martimort (2003) consider a model with binary types and signals (with full support for conditional distributions), a risk-averse supervisor where collusion is costly, where (unconditional) delegation turns out to be an optimal response to collusion. Celik (2009) considers a model with three types and two signals (where the support of conditional distributions depends on the signal), and risk neutral supervisor and agent, in which unconditional delegation is dominated by no supervision, which in turn is dominated strictly by centralized contracting with supervision. Celik's results are similar to ours, but he considers interim rather than ex ante collusion. Our results can be viewed as finding that the results he derived in the context of interim collusion with a special information structure happen to obtain quite generally with ex ante collusion and continuous types. The need to examine ex ante rather than interim collusion is highlighted by Motta (2009) who shows that collusion can be rendered costless in models with discrete type and signal spaces and interim collusion, by using mechanisms where the Principal offers a menu of contracts to the agent which the latter must respond to before colluding with the supervisor.

3 Model

3.1 Environment

We consider an organization composed of a principal (P), an agent (A) and a supervisor (S). P can hire A who delivers an output $q \geq 0$ at a personal cost of θq. P's return from q is $V(q)$ where $V(q)$ is twice continuously differentiable, increasing and strictly concave satisfying $\lim_{q\to 0} V^{'}(q) = +\infty$, $\lim_{q\to+\infty} V^{'}(q) = 0$ and $V(0) = 0$. These conditions imply that $q^*(\theta) \equiv \arg_q \max V(q) - \theta q$ is continuously differentiable, positive on $\theta \in [0, \infty)$ and strictly decreasing. In Section 6 we shall describe how the results are modified when P procures an indivisible good.

We use θ to denote a random variable whose realization is privately observed by A. It is common knowledge that everybody shares a common prior $F(\theta)$ over θ on the interval $\Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \Re_+$. F has a density function $f(\theta)$ which is continuously differentiable and everywhere positive on its support. The 'virtual cost' $H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$ is assumed to be strictly increasing in θ; this assumption is inessential and is made in order to simplify the proofs.

The supervisor S plays no role in production, and costlessly acquires an informative signal η about A's cost θ.[10] The set of possible realizations of η is Π, a finite set with $\#\Pi \geq 2$. The finiteness of this set is assumed for technical convenience, and is relatively inessential as long as S's information regarding θ is not perfect. It is common knowledge that the realization of η is observed by both S and A. $a(\eta \mid \theta) \in [0, 1]$ denotes the likelihood function of η conditional on θ, which is common knowledge among all agents. $a(\eta \mid \theta)$ is continuously differentiable and positive on $\Theta(\eta)$, where $\Theta(\eta)$ denotes the set of values of θ for which signal η can arise with positive probability. We assume $\Theta(\eta)$ is an interval, for every $\eta \in \Pi$. Define $\underline{\theta}(\eta) \equiv \inf \Theta(\eta)$ and $\bar{\theta}(\eta) \equiv \sup \Theta(\eta)$. We assume that for any $\eta \in \Pi$, $a(\eta \mid \theta)$ is not a constant function on Θ, and there are some [[subsets]] of θ with positive measure such that $a(\eta \mid \theta) \neq a(\eta^{'} \mid \theta)$ for any $\eta, \eta^{'} \in \Pi$. In this sense each possible signal realization conveys information about the agent's cost. The information conveyed is partial, since Π is finite.

[10] If signal acquisition involves a fixed cost, P will need to reimburse S for this cost. Hence it will have to be subtracted from P's payoff when S is appointed. The extension of our results to this context is straightforward.

The conditional density function and the conditional distribution function are respectively denoted by $f(\theta \mid \eta) \equiv f(\theta)a(\eta \mid \theta)/p(\eta)$ (where $p(\eta) \equiv \int_{\underline{\theta}(\eta)}^{\bar{\theta}(\eta)} f(\tilde{\theta})a(\eta \mid \tilde{\theta})d\tilde{\theta}$) and $F(\theta \mid \eta) \equiv \int_{\underline{\theta}(\eta)}^{\theta} f(\tilde{\theta} \mid \eta)d\tilde{\theta}$. The 'virtual' cost conditional on the signal η is $h(\theta \mid \eta) \equiv \theta + \frac{F(\theta|\eta)}{f(\theta|\eta)}$. We do not impose any monotonicity assumption for $h(\theta \mid \eta)$. Let $\hat{h}(\theta \mid \eta)$ be constructed from $h(\theta \mid \eta)$ and $F(\theta \mid \eta)$ by the ironing procedure introduced by Myerson (1981).

All players are risk neutral. P's objective is to maximize the expected value of $V(q)$, less expected payment to A and S, represented by X_A and X_S respectively. S's objective is to maximize expected transfers $X_S - t$ where t is a transfer from S to A. A seeks to maximize expected transfers received, less expected production costs, $X_A + t - \theta q$. Both A and S have outside options equal to 0.

In this environment, a feasible (deterministic) allocation is represented by $(u_A, u_S, q) = \{(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta)) \in \Re^2 \times \Re_+ \mid (\theta, \eta) \in K\}$ where $K \equiv \{(\theta, \eta) \mid \eta \in \Pi, \theta \in \Theta(\eta)\}$, u_S, u_A denotes S and A's payoff respectively, and q represents the production level. P's payoff equals $u_P = V(q) - u_S - u_A - \theta q$. These payoffs relate to transfers and productions as follows: $u_A \equiv X_A + t - \theta q; u_S \equiv X_S - t; u_P \equiv V(q) - X_S - X_A$.

3.2 Mechanism in the Absence of Collusion

Consider as a benchmark the case where A and S do not collude, and P designs contracts for both. We call this organization NC (no collusion). Owing to risk-neutrality of all parties, [[concavity of V and linearity of A's payoff in q,]] P can restrict attention to a deterministic grand contract:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S); M_A, M_S)$$

where M_A (resp. M_S) is a message set for A (resp. S). This mechanism assigns a deterministic allocation, i.e. transfers X_S, X_A and output q, for any message $(m_A, m_S) \in M_A \times M_S$. M_A includes A's exit option $e_A \in M_A$, with the property that $m_A = e_A$ implies $X_A = q = 0$ for any $m_S \in M_S$. Similarly M_S includes S's exit option $e_S \in M_S$, where $m_S = e_S$ implies $X_S = 0$ for any $m_A \in M_A$. The set of all possible deterministic grand contracts is denoted by $\mathcal{GC}$.

A grand contract induces a Bayesian game of incomplete information between A and S. To set up notation which will be useful in later sections, we use $p(\eta)$ to denote [[an arbitrary set of]] beliefs held by S regarding θ, in a state where signal η has been realized. [[In the absence of collusion, the relevant set of beliefs will be $p_\emptyset(\eta)$, the posterior beliefs of S based on Bayesian updating of prior beliefs on the basis of observation of η alone.]]

Definition 1 *A Bayesian equilibrium of the game played by A and S in state η relative to beliefs $p(\eta)$ is [[represented by a set of functions $c \equiv (m_A(\theta,\eta); m_S(\eta))$ where m_A maps K into M_A, while m_S maps Π into M_S,]] such that the following conditions are satisfied for all $\theta \in [\underline{\theta}(\eta), \bar{\theta}(\eta)]$:*

$$m_A(\theta,\eta) \in \arg\max_{m_A \in M_A} [X_A(m_A, m_S(\eta)) - \theta q(m_A, m_S(\eta))] \tag{1}$$

$$m_S(\eta) \in \arg\max_{m_S \in M_S} E_{p(\eta)}[X_S(m_A(\theta,\eta), m_S)] \tag{2}$$

where $E_{p(\eta)}$ denotes expectation taken with respect to beliefs $p(\eta)$. $C(p(\eta);\eta)$ denotes the set of Bayesian equilibria corresponding to the beliefs $p(\eta)$ in state η.

The timing of events in NC is as follows.

(NC1) A observes θ and η, S observes η.

(NC2) P offers the grand contract $GC \in \mathcal{GC}$, and for any $\eta \in \Pi$ recommends a Bayesian equilibrium $c(p_\emptyset(\eta);\eta)$ relative to posterior beliefs $p_\emptyset(\eta)$ based on Bayesian updating by S on the basis of observation of η alone.

(NC3) A and S play the recommended Bayesian equilibrium.

The order of the timing between (NC1) and (NC2) can be interchanged without altering any of the results. If P offers a null contract to S (defined by the property that M_S is the empty set and $X_S = 0$), this is an organization without a supervisor, which we will denote by NS. Such an organization obviously leaves no scope for collusion between A and S.

It is well-known that in NC the Principal can restrict attention to direct revelation games, where M_A and M_S reduce to reports of private information, besides participation decisions. Define the *second-best allocation* $(u_A^{SB}, u_S^{SB}, q^{SB})$ as follows:

$$u_A^{SB}(\theta,\eta) = \int_\theta^{\bar{\theta}(\eta)} q^{SB}(y,\eta)dy,$$

$$E[u_S^{SB}(\theta,\eta) \mid \eta] = 0$$

and

$$q^{SB}(\theta,\eta) \equiv q^*(\hat{h}(\theta \mid \eta)) = \arg\max_q [V(q) - \hat{h}(\theta \mid \eta)q]$$

where $\hat{h}(\theta \mid \eta)$ is constructed from $h(\theta \mid \eta)$ and $F(\theta \mid \eta)$ by the ironing procedure. It is well-known [[(e.g., extending arguments in Baron and Myerson (1982))]] that this is the optimal allocation in NC, where P observes η directly. It turns out that in NC it is possible for the second-best to be implemented as a unique Bayesian equilibrium.[11]

3.3 Mechanism with Weak Ex Ante Collusion

Now we describe the game played with *weak ex ante* collusion. The 'ex ante' feature refers to the assumption that collusion takes the form of communication and side-contracting between A and S, which takes place before they respond to P's offer of the grand contract (including participation decisions). This is distinguished from (interim) collusion where they do not collude on their participation decisions, but collude on the reports they send to P [[while exchanging]] side payments in the event of joint participation. The 'weak' adjective additionally refers to the lack of commitment power of either colluding partner with respect to how they would behave (i.e., play the grand contract) in the event that they fail to agree on the side contract. In this event they would play the grand contract noncooperatively (relative to beliefs formed subsequent to the breakdown of the side contract).

The game with weak ex ante collusion is different from the game without collusion following stage NC2. At that point, A and S can enter into a side-contract in which A sends a message to S following which they jointly decide on participation, reporting and side-payments. The side-contract is unobserved by P. As in existing literature, we assume the side-contract is costlessly enforceable. Moreover we assume S has all the bargaining power *vis-a-vis* A: S can make a take-it-or-leave-it offer of a side-contract. This assumption turns out to be inessential: Section 5.1 explains how the same results obtain with side contracts offered by an uninformed third party that assigns arbitrary welfare weights to the supervisor and agent. After S offers the side contract, A retains the option of rejecting it; given that A's true cost is not known to S, this still enables A to earn some rents. This information friction within the coalition plays a key role in our analysis.

The game replaces (NC3) above (while (NC1) and (NC2) are unchanged) by the following three-stage subgame (conditional on any $\eta \in \Pi$):

(i) S offers a side-contract SC which determines for any $\tilde{\theta} \in \Theta(\eta)$ to be privately reported by A to S, a probability distribution over joint messages $(m_A, m_S) \in M_A \times M_S$, and

[11] A proof [[is provided in the online Appendix.]]

a side payment from S to A.[12] Formally, it is a pair of functions $\{\tilde{m}(\tilde{\theta},\eta), t(\tilde{\theta},\eta)\}$ where $\tilde{m}(\theta,\eta) : \Theta(\eta) \times \{\eta\} \longrightarrow \Delta(M_A \times M_S)$, the set of probability measures over $M_A \times M_S$, and $t : \Theta(\eta) \times \{\eta\} \longrightarrow \Re$. The case where S does not offer a side contract is represented by a null side-contract (NSC) with zero side payments ($t(\theta,\eta) \equiv 0$), and (deterministic) messages $(m_A(\theta,\eta); m_S(\eta))$ the same as those in the Bayesian equilibrium of the grand contract recommended by the Principal. We abuse terminology slightly and refer to the situation where no side contract is offered as one where NSC is offered.

(ii) A either accepts or rejects the SC offered, and the game continues as follows.

(iii) If A accepts the offered SC, he sends a private report $\theta' \in \Theta(\eta)$ to S, following which the SC is executed.[13] If A rejects SC, S updates his beliefs to $p(SC;\eta)$ which is restricted to be $p_\emptyset(\eta)$ if NSC was offered in stage (i) above.[14] A and S then play a Bayesian equilibrium c of the grand contract relative to beliefs $p(SC;\eta)$.

[[

3.4 Suboptimality of Delegated Supervision

Before proceeding further, we consider the special case of *Delegated Supervision (DS)* where P delegates authority to S over contracting with A. Here the GC designed by P involves a null contract for A: the latter submits no report to P directly, receives no production instructions or payments from P. P contracts entirely with S, requiring the latter to submit a report to P which determines a production target and a payment to the (S,A) coalition. Following receipt of this contract, S designs a side contract for A which selects a production target and payment for the latter as a function of a report submitted by A to S, provided A accepts the side contract. After receiving a report from A (conditional on A agreeing to participate), S submits a participation and report to send to P. Note that in contrast

[12] The option of randomizing over possible messages is useful for technical reasons. Owing to quasilinearity of payoffs, there is no need to randomize over side transfers.

[13] [[Standard arguments show that the restriction to direct revelation mechanisms for the side contract entails no loss of generality.]]

[14] This ensures that it is immaterial whether or not NSC was accepted or rejected, since in either case they play the grand contract non-cooperatively with prior beliefs.

to the setting of interim collusion, S can postpone submission of the participation decision *after* receiving a report from A.

Our first main result is that delegation is *never* optimal in ex ante collusion, as it is strictly dominated by the case where S is not appointed at all. which we refer to as *No Supervision (NS).*

Proposition 1 *Delegated Supervision is worse for the Principal compared to No Supervision.*

The FLM result concerning optimality of delegation therefore does not extend to the setting of our model with ex ante collusion, risk neutrality and continuous types. The underlying argument is simple and very general (e.g., it also applies to the indivisible good procurement case). P contracts for delivery of the good with S, so the problem reduces to contracting with a single agent S. In order to deliver the good to P, S needs to procure it in turn from A. The cost that S expects to incur equals A's virtual cost function $h(\theta|\eta)$ corresponding to the signal observed by S. This is unambiguously higher than the delivery cost θ of A if P were to contract directly with A, in the absence of any supervision. This is the well-known problem of double marginalization of rents (DMR), arising due to exercise of monopsony power by S in side-contracting with A. Unlike the context of interim collusion, S can postpone her own participation decision *after* receiving A's report. This effectively translates into a kind of 'limited liability' constraint for A, which constitutes the source of the DMR problem.[15]

Given this result, we hereafter focus on centralized contracting with supervision, where P offers a non-null contract to both S and A in GC.

3.5 Centralized Contracting and Weak Collusion Proofness

We now introduce the notion of weak collusion proofness in the context of centralized contracts.]] A justification for this solution concept is provided in Section 3.7 below.

[15]While it is relatively easy to show that DS cannot dominate NS, the proof establishes the stronger result that DS is **strictly** dominated by NS. The proof of strict domination is also straightforward in the case that $h(\theta|\eta)$ is continuous and nondecreasing in θ over a common support $[\underline{\theta}, \bar{\theta}]$ for every η. In that case an argument based on Proposition 1 in Mookherjee and Tsumagari (2004) can be applied. In the general case there are a number of additional technical complications, but the result still goes through.

Informally, an allocation is weakly collusion proof if the supervisor cannot benefit from offering a non-null side contract when the Principal selects a grand contract based on the associated direct revelation mechanism (i.e., when agent and supervisor make consistent reports about the state, the allocation corresponding to that state is chosen). This requires the null side contract to be the optimal side contract for S, when the outside option of A corresponds to his payoff resulting from the allocation.

Before proceeding to the formal definition, note that a *deterministic allocation* can be represented by payoff functions $(u_A(\theta,\eta), u_S(\theta,\eta))$ of the true state (θ,η) combined with the output function $q(\theta,\eta)$, as these determine the Principal's payoff function $u_P(\theta,\eta) \equiv V(q(\theta,\eta)) - u_S(\theta,\eta) - u_A(\theta,\eta) - \theta q(\theta,\eta)$, and the aggregate net transfers of S (equals $u_S(\theta,\eta)$) and A (equals $u_A(\theta,\eta) + \theta q(\theta,\eta)$). For technical convenience we consider randomized allocations, though it will turn out they will never actually need to be used on the equilibrium path.[16] In a randomized allocation, $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ denotes the expected payoffs of A, S and the expected output, conditional on the state (θ,η). For (conditional expected) allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$, define functions $(\hat{X}(m), \hat{q}(m))$ on domain $m \in \hat{M} \equiv K \cup \{e\}$ (where $K \equiv \{(\theta,\eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$) as follows:

$$(\hat{X}(\theta,\eta), \hat{q}(\theta,\eta)) = (u_A(\theta,\eta) + \theta q(\theta,\eta) + u_S(\theta,\eta), q(\theta,\eta))$$

$$(\hat{X}(e), \hat{q}(e)) = (0,0)$$

$(\hat{X}(\theta,\eta), \hat{q}(\theta,\eta))$ denote corresponding expected values of the sum of payments $X_S + X_A$ made by the principal, and the output delivered, in state (θ,η). Also, let $\Delta(\hat{M})$ denote the set of the probability measures on $\hat{M}$, and use $\tilde{m} \in \Delta(\hat{M})$ to denote a randomized message submitted by the coalition to P. With a slight abuse of notation, we shall denote the corresponding conditional expected allocation by $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$, which is defined on $\Delta(\hat{M})$. $\tilde{m} = (\theta,\eta)$ or e will be used to denote the probability measure concentrated at (θ,η) or e respectively.

S's choice of an optimal (randomized) side-contract can be formally posed as follows. Given a grand contract and a noncooperative equilibrium recommended by P, let the corresponding conditional expected allocation as defined above be denoted by $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ and $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$. For any $\eta \in \Pi$, the associated side-contracting problem $P(\eta)$ is to select

[16] This owes to the assumption that A's payoff is linear in the output produced.

$(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta))$ to maximize S's expected payoff

$$E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta \hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to $\tilde{m}(\theta \mid \eta) \in \Delta(\hat{M})$,

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}(\tilde{m}(\theta' \mid \eta))$$

for any $\theta, \theta' \in \Theta(\eta)$, and

$$\tilde{u}_A(\theta, \eta) \geq u_A(\theta, \eta)$$

for all $\theta \in \Theta(\eta)$. The first constraint states truthful revelation of the agent's true cost to S is consistent with the agent's incentives, and the second constraint requires A to attain a payoff at least as large as what he would expect to attain by playing the grand contract noncooperatively.

Let the maximum payoff of S in the side contracting problem in state η be denoted by $W(\eta)$.

Definition 2 *The (conditional expected) allocation* $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta)) : K \rightarrow \Re^2 \times \Re_+$ *is weakly collusion proof (WCP) if for every* $\eta \in \Pi$*:* $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta)) = ((\theta, \eta), u_A(\theta, \eta))$ *solves problem* $P(\eta)$ *in which S achieves a maximum payoff of* $W(\eta) = E[u_S(\theta, \eta) \mid \eta]$*.*

3.6 Characterization of WCP Allocations

We now characterize WCP allocations. This requires us to define a family of 'modified' virtual cost functions, representing the effective cost incurred by the coalition in delivering a unit of output to P, following selection of an optimal side-contract.

Definition 3 *For any* $\eta \in \Pi$*,* $Y(\eta)$ *is a collection of* **coalitional shadow cost (CSC)** *functions* $\pi(\cdot \mid \eta) : \Theta(\eta) \rightarrow \Re$ *which satisfy the following property. For any function in this collection, there exists a real-valued function* $\Lambda(\theta|\eta)$ *which is non-decreasing in* $\theta \in \Theta(\eta)$ *with* $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ *and* $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$*, such that*

$$\pi(\theta|\eta) \equiv \theta + \frac{F(\theta \mid \eta) - \Lambda(\theta \mid \eta)}{f(\theta \mid \eta)} \tag{3}$$

Equation (3) modifies the usual expression for virtual cost $h(\theta|\eta) \equiv \theta + \frac{F(\theta|\eta)}{f(\theta|\eta)}$ by subtracting from it the non-negative term $\frac{\Lambda(\theta|\eta)}{f(\theta|\eta)}$. [[In order to overcome the DMR problem in

Delegated Supervision, in the centralized regime]] P contracts with both S and A, thereby providing A an outside option (of $u_A(\theta, \eta)$) that effectively raises his bargaining power *vis-a-vis* S while negotiating the side contract. Meeting a larger outside option for A effectively induces S to deliver a higher output to P: this is what paying a higher rent to A necessitates. The extent of DMR is then curbed: the shadow cost for the coalition in delivering a unit of output to P is lowered. This lowering of the virtual cost is represented by the subtraction of the term $\frac{\Lambda(\theta|\eta)}{f(\theta|\eta)}$ from what it would have been ($h(\theta|\eta)$) under Delegated Supervision. [[As Jullien (2000) describes it in the analogous context of contracting with a single agent with type dependent outside options, $\Lambda(\theta \mid \eta)$ represents the shadow value of a uniform reduction in A's outside option for all types below θ. Clearly, the $\Lambda(\theta \mid \eta)$ function must be non-decreasing.]]

However, $\pi(\theta|\eta)$ is not the correct expression for the shadow cost of output for the coalition, if it is non-monotone in θ. In that case, it has to be replaced by its 'ironed' version (Myerson (1981)), using the distribution function $F(\theta|\eta)$. Let the corresponding ironed version of $\pi(\theta|\eta)$ be denoted by $z(\theta|\eta)$: we call this a *coalitional virtual cost function.*

Definition 4 *For any $\eta \in \Pi$, the set of* **coalitional virtual cost (CVC)** *functions is the set*

$$Z(\eta) \equiv \{z(\cdot \mid \eta) \quad \textit{is the ironed version of some} \quad \pi(\cdot \mid \eta) \in Y(\eta)\}$$

of functions obtained by applying the ironing procedure to the set $Y(\eta)$ of CSC functions.[17] *Denote by $\Theta(\pi(\cdot \mid \eta), \eta)$ the corresponding pooling region of θ where $\pi(\cdot|\eta)$ is flattened by the ironing procedure.*

As the next result shows, every WCP allocation satisfies coalitional participation and incentive constraints corresponding to some coalitional virtual cost function z. Combined with an individual incentive compatibility constraint for A, and a constraint that output must be constant over regions where the ironing procedure flattens the underlying CSC function, these coalitional constraints characterize WCP allocations.[18]

Proposition 2 *The allocation (u_A, u_S, q) is WCP if and only if the following conditions hold for every η. There exists a CVC function $z(\cdot|\eta) \in Z(\eta)$ such that*

[17] The ironing procedure ensures these functions are continuous and non-decreasing.

[18] [[See Mookherjee and Tsumagari (2004), Celik (2008) and Pavlov (2009) for similar characterizations of weakly collusion proof mechanisms.]]

(i) For every $(\theta,\eta),(\theta^{'},\eta^{'}) \in K \equiv \{(\theta,\eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$,

$$X(\theta,\eta) - z(\theta \mid \eta)q(\theta,\eta) \geq X(\theta^{'},\eta^{'}) - z(\theta \mid \eta)q(\theta^{'},\eta^{'})$$

$$X(\theta,\eta) - z(\theta \mid \eta)q(\theta,\eta) \geq 0$$

where

$$X(\theta,\eta) \equiv u_A(\theta,\eta) + u_S(\theta,\eta) + \theta q(\theta,\eta)$$

(ii) For any $\theta,\theta^{'} \in \Theta(\eta)$,

$$u_A(\theta,\eta) \geq u_A(\theta^{'},\eta) + (\theta^{'} - \theta)q(\theta^{'},\eta)$$

(iii) $q(\theta,\eta)$ *is constant on any interval of* θ *which is a subset of the corresponding pooling region of the CVC function* z.

Condition (i) represents the coalitional incentive and participation constraints corresponding to contracting with a single agent with a unit cost of z. Condition (ii) is the individual incentive compatibility constraint for A. Condition (iii) states that the output must be constant over every interval in the pooling region.

3.7 Justification for WCP Allocations

In this section, we provide a justification for focusing attention on WCP allocations.

We use the notion of Weak Perfect Bayesian Equilibrium (WPBE) of the game with collusion.[19] [[The same results apply if we use the slightly stronger notion of Perfect Bayesian Equilibrium but the proofs are somewhat more complicated, so we examine WPBE.[20]]] As there are typically multiple WPBEs of the continuation game following any given GC offer, we need to specify how these might be selected.

If the mechanism design problem is stated as selection of an allocation by the Principal subject to the constraint that it can be implemented as the outcome of some WPBE following a choice of a grand contract, it is presumed that the Principal is free to select continuation beliefs and strategies for noncooperative play of the grand contract following off-equilibrium path rejections of offered side contracts by S to A. It can be shown that in such a setting the problem of collusion can be completely overcome by the Principal, with

[19]For definition of WPBE, see Mas-Colell, Whinston and Green (1995, p.285).

[20]The online Appendix explains how to modify the proofs for the PBE solution concept.

appropriate selection of off-equilibrium-path continuations. [[This is formally shown in the online Appendix.]] A heuristic description of how the second-best payoff can be achieved by the Principal as a WPBE is as follows. P selects a grand contract and recommends a noncooperative equilibrium of this contract in which (i) conditional on participation by S, noncooperative play results in the second-best allocation; (ii) S is paid nothing; and (iii) if S does not participate, P offers A a 'gilded' contract providing the latter a high payoff in all states. On the equilibrium path S always offers a null side contract. If A rejects any offer of a non-null side-contract, they mutually believe that subsequently S will not participate in the grand contract, and A will receive the gilded contract. This forms a WPBE as rejection of any non-null side contract is sequentially rational for A given A's belief that S will exit following any rejection. And exit is sequentially rational for S given his belief that A will reject the side contract and they will subsequently play the grand contract noncooperatively where S will be paid nothing.

Collusion is overcome by the Principal here by exploiting a lack of coordination among A and S over continuation beliefs and play of the side contracting game. This denies the essence of collusive activity, which involves coordination by the colluding parties 'behind the Principal's back'. It is therefore reasonable to insist that S and A can collectively coordinate on the choice of side-contracting equilibria that are Pareto-undominated (for the coalition). Specifically, this rules out WPBE outcomes for which (following some realization of η) there exists some side-contract offer and a PBE of the subsequent continuation game played by S and A which generates a higher expected payoff for S, without lowering the expected payoff of any type of A.

Definition 5 *Following the selection of a grand contract by P, a WPBE(wc) is a Weak Perfect Bayesian Equilibrium (WPBE) of the subsequent game with the following property. There does not exist some signal realization η, and some deviating side-contract offer SC(η) for which there is a [[Weak Perfect Bayesian Equilibrium (WPBE)]] of the subsequent continuation game in which (conditional on η) S's payoff is strictly higher and A's payoff not lower for any type.*

Definition 6 *An allocation (u_A, u_S, q) is implementable in the weak collusion game if there exists a grand contract and a WPBE(wc) of the subsequent game which results in this allocation.*

We now show that the WPBE(wc) refinement corresponds to WCP allocations that satisfy interim participation constraints. Note that the WPBE(wc) notion allows for collusion to occur (i.e., a non-null side contract to be offered and accepted by some types of A), and also for side-contract offers to be rejected by some types of A. Hence the WCP notion does not rest on any arbitrary restrictions on side contract outcomes, e.g., which rule out the possibility of equilibrium-path rejections by A of the side contract offered by S. The problem discussed by Celik and Peters (2011) therefore does not apply to this setting. Moreover, the restriction to WCP allocations which correspond to equilibrium outcomes in which collusion does not occur on the equilibrium path, is also without loss of generality.

Proposition 3 *An allocation (u_A, u_S, q) is implementable in the weak collusion game, if and only if it is a WCP allocation satisfying interim participation constraints*

$$E[u_S(\theta,\eta)|\eta] \geq 0 \textit{ for all } \eta \tag{4}$$

$$u_A(\theta,\eta) \geq 0 \textit{ for all } (\theta,\eta) \tag{5}$$

4 Main Results

We are now in a position to present our main results. In this section we will compare the following organizational alternatives, besides No Supervision (NS) the context where P contracts directly with A in the absence of S.

(a) *Centralized Supervision (CS):* This is the unrestricted version of centralized model, where P offers a grand contract involving both S and A. A has an outside option of rejecting the side contract offered by S and participating in the grand contract noncooperatively. We shall denote the resulting profit of P by Π_{CS}.

(b) *Conditional Delegation (CD):* This is a hybrid of the delegation and centralized arrangements, where P delegates to (i.e., contracts and communicates only with) S on the equilibrium path. However, S and A are both given the option to switch from the 'normal' delegation mode to a centralized mode in which both S and A send messages to P, followed by decisions made by P. The resulting profit of P is denoted Π_{CD}.

We will also assess these relative to the benchmark of no collusion, which is associated with the second-best allocation defined previously. The associated profit will be denoted

Π_{SB}. Since S has access to information about A's cost that is valuable in contracting with A, it is obvious that $\Pi_{NS} < \Pi_{SB}$, i.e., hiring S is valuable if there is no collusion.

Proposition 4 $\Pi_{NS} < \Pi_{CS}$*: the Principal is strictly better off hiring S and contracting directly with both S and A, compared to hiring no supervisor.*

This states that P always benefits from hiring S despite the presence of ex ante collusion between S and A. Combining with the previous result, it follows that S is valuable only provided P does not delegate authority to S: it is essential that P contracts simultaneously with A as well, thus providing A an outside option which raises A's bargaining power within the coalition. This limits the DMR problem by countervailing S's tendency to behave monopsonistically with respect to A. By raising A's outside option, the coalitional virtual cost z is reduced, allowing an increase in output delivered, and raising P's expected payoff.

This helps explain how contracting directly with both S and A helps reduce the DMR problem inherent in DS which rendered it inferior to NS. However, it does not help explain why it manages to do so sufficiently that CS ends up being superior to NS. The explanation for this is more subtle, arising from P's ability to profitably utilize S's superior information concerning the agent's cost with a simple mechanism. This arises ultimately from the discrepancy between relative likelihoods of different cost states by P and S, which they use to weight different states in computing their respective payoffs.

It may help to outline the WCP allocation that can be used by P. Starting with the optimal allocation in NS (which corresponds to the special case of CS where $\Lambda(\theta \mid \eta)$ is chosen equal to $F(\theta \mid \eta)$, ensuring that the CSC and CVC functions both reduce to the identity function ($\pi(\theta|\eta) = z(\theta|\eta) = \theta$)), P can construct a small variation in the CVC function z in some state η^*, raising it above θ for some interval Θ_H and lowering it for some other interval Θ_L, both of which have positive probability given η^*. The corresponding quantity procured $q(\theta, \eta^*)$ is set equal to $q^{NS}(z(\theta|\eta^*))$, the quantity procured in NS when the agent reported a cost of $z(\theta|\eta^*)$. This corresponds to raising the quantity procured from the coalition over Θ_L and lowering it over Θ_H. Payments to the coalition are set analogously at $X^{NS}(z(\theta|\eta^*))$, what the agent would have been paid in NS following such a cost report.[21] The agent is offered the associated rent: $u_A(\theta, \eta^*) = \int_{\theta}^{\bar{\theta}} q^{NS}(z(y|\eta^*))dy$. By

[21] Specifically, $X^{NS}(z(\theta|\eta)) = z(\theta|\eta)q^{NS}(z(\theta|\eta)) + \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(y)dy$.

construction, this allocation satisfies the agent's incentive and participation constraints, as well as the coalitional incentive constraint.[22]

Proposition 2 ensures such an allocation is WCP, provided S's interim participation constraint is satisfied. The variation over Θ_L lowers rents earned by S, and over Θ_H raises them. Since S does not earn any rents to start with (i.e, in NS), it is necessary to construct the variation such that S's expected rents in state η^* do not go down. The rate at which S's rents vary locally in state θ with the quantity procured equals $\frac{F(\theta|\eta^*)}{f(\theta|\eta^*)}$.[23] Intuitively this is the saving that can be pocketed by S when procuring one less unit of the good from A. Maintaining S's expected rent therefore implies a marginal rate of substitution between output variations over Θ_L and Θ_H that equals the ratio of the (average) conditional inverse hazard rates $\frac{F(\theta|\eta^*)}{f(\theta|\eta^*)}$ over these two intervals respectively.

On the other hand, P's benefit from a small expansion in output delivered in state θ equals $V'(q^{NS}(\theta))-\theta$, where $q^{NS}(\theta)$ denotes the optimal allocation in NS.[24] This allocation satisfies $V'(q^{NS}(\theta)) = H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$, the virtual cost of procurement without conditioning on information regarding η. Hence P's marginal benefit from output expansion in state θ equals the unconditional inverse hazard rate $\frac{F(\theta)}{f(\theta)}$. This implies that P's marginal rate of substitution between output variations over Θ_L and Θ_H equals the ratio of the (average) unconditional inverse hazard rates $\frac{F(\theta)}{f(\theta)}$ over these two intervals. The informativeness of S's signals implies that P's marginal rate of substitution differs from S's in some state η^* over some pair of intervals Θ_L, Θ_H. Hence there exist variations of the type described above which raise P's expected payoff, while preserving the expected payoff of S.

One may wonder whether the gains achieved by the Principal from hiring S are marginal rather than substantial. Section 6 shows that the second-best payoff is achievable for some cases in the context [[of procurement of an indivisible good.]] This is consistent with results

[22] This requires checking that there exists a CSC function $\pi(\theta|\eta)$ corresponding to some function $\Lambda(\cdot \mid \eta)$ on $[\underline{\theta}(\eta), \bar{\theta}(\eta)]$ satisfying the requirements in the definition of a CSC function, such that $z(\theta \mid \eta)$ is the ironed version of $\pi(\theta \mid \eta)$. This is true, since we can select $\Lambda(\theta \mid \eta) = (\theta - z(\theta \mid \eta))f(\theta \mid \eta) + F(\theta \mid \eta)$, which is strictly increasing over Θ_L and Θ_H for a sufficiently small variation of z from the identity function. Then $\Lambda(\cdot \mid \eta)$ is a function which satisfies the required properties and generates $\pi(\theta|\eta) = z(\theta \mid \eta)$, since $z(\theta \mid \eta)$ is a non-decreasing function.

[23] S's interim rent in state η equals the expected value conditional on η of $X^{NS}(z(\theta|\eta)) - u_A(z(\theta|\eta)) - \theta q^{NS}(z(\theta|\eta))$, i.e., equals $E[\{z(\theta|\eta) - h(\theta|\eta)\}q^{NS}(z(\theta|\eta)) - \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(z)dz|\eta]$.

[24] This follows from the fact that $\frac{\partial X^{NS}(z)}{\partial z} = zq^{NS'}(z)$, implying that the marginal increase in payment evaluated at $z = \theta$ equals θ times the marginal output change.

of Pavlov (2008) and Che and Kim (2009) in the case of auctions. [[In the current setting of procurement of a divisible good,]] the following result shows that the second-best is not achievable provided the benefit function exhibits sufficient curvature (besides some standard restrictions on the information structure).

Proposition 5 $\Pi_{CS} < \Pi_{SB}$*: P cannot attain the second-best payoff in CS if the following conditions hold:*

(i) The support of θ *does not vary with the signal:* $\Theta(\eta) = \Theta$ *for any* $\eta \in \Pi$*;*

(ii) there exists $\eta^* \in \Pi$ *such that* $f(\theta|\eta^*)$ *and* $\frac{f(\theta|\eta^*)}{f(\theta|\eta)}$ *are both strictly decreasing in* θ *for any* $\eta \neq \eta^*$*; and*

(iii) $V'''(q) \leq \frac{(V''(q))^2}{V'(q)}$ *for any* $q \geq 0$*.*

Condition (i) states that the support of θ does not vary with η, while (ii) is a form of a monotone likelihood [ratio property: there is a signal realization η^* which is 'better' news about θ than any other realization, in the sense of shifting weight in favor of low realizations of θ. It additionally requires that the conditional density $f(\theta|\eta^*)$ is strictly decreasing in θ, i.e., higher realizations of θ are less likely than low realizations when $\eta = \eta^*$. (ii) is satisfied for instance when θ has a uniform prior and there are just two possible signal values satisfying the standard monotone likelihood ratio property. Condition (iii) is satisfied if V is exponential ($V = 1 - \exp(-rq), r > 0$). It corresponds to the assumption of 'non-increasing absolute risk aversion' of the Principal's benefit function.

The proof develops necessary conditions for implementation of the second best given the distributional properties (i) and (ii). If the outputs are second-best, they must be a monotone decreasing function of the (ironed) virtual cost $\hat{h}(\theta \mid \eta)$ in the second-best setting. If they also satisfy the coalitional incentive constraints, they must be monotone in CVC $z(\theta \mid \eta)$. These conditions imply the existence of a monotone transformation from $\hat{h}$ to z, and enable S's ex post rent to be expressed as a function of $\hat{h}$ alone. Condition (iii) is used to show that this rent function is strictly convex which in turn is used to show that the expected rents of S must be strictly higher in state η^* than any other state.

Our final result in this Section shows that the optimal allocation in CS can be implemented by a modified form of delegation, where P communicates and transacts only with S on the equilibrium path. In this arrangement, S is 'normally' expected to contract on behalf of the coalition $\{S, A\}$ with P, sending a joint participation decision and report of the state (θ, η) to P after having entered into a side contract with A. However A has the option of circumventing this 'normal' procedure and asking P to activate a grand contract in which A and S will send independent reports and participation decisions to P. The presence of this option ensures that A has sufficient bargaining power within the coalition; it does not have to be 'actually' used, i.e., on the equilibrium path. This mechanism can implement any implementable allocation as a WPBE(wc) outcome.

The argument is as follows; [[the formal proof is provided in the online Appendix]]. Take any WCP allocation $(u_S(\theta, \eta), u_A(\theta, \eta), q(\theta, \eta))$ defined on K which satisfies interim participation constraints, and let aggregate payments to the coalition be $X(\theta, \eta) = u_A(\theta, \eta) + u_S(\theta, \eta) + \theta q(\theta, \eta)$. Let the associated grand contract be denoted as follows. The message spaces are $\tilde{M}_S, \tilde{M}_A$, where $\tilde{M}_S = \Pi \cup \{e_S\}$ and $\tilde{M}_A = K \cup \{e_A\}$. Both S and A report η, and A additionally reports θ. P cross-checks the two η reports, and conditional on these agreeing with one another, transfers are set in the obvious way corresponding to the allocation $(u_S(\theta, \eta), u_A(\theta, \eta), q(\theta, \eta))$, e.g., when neither party exits, both report η and A reports θ, $\tilde{X}_S(\theta, \eta) = u_S(\theta, \eta), \tilde{X}_A(\theta, \eta) = u_A(\theta, \eta) + \theta q(\theta, \eta), \tilde{q}(\theta, \eta) = q(\theta, \eta)$, otherwise these are all zero.

This 'original' grand contract can be augmented as follows. A is offered a message space $M_A = \tilde{M}_A \cup \{\emptyset\}$, while S is offered $M_S = \tilde{M}_S \cup K \cup \{e\}$. The interpretation is that if $m_A = \emptyset$, A decides not to communicate directly with P. And if $m_S \in K \cup \{e\}$, S decides to submit a joint report (θ, η) (or else communicates a joint shutdown decision e) to P on behalf of the coalition. The choice of $m_A = \emptyset, m_S \in K \cup \{e\}$ will correspond to the 'normal' delegation mode.

When the normal delegation mode is in operation, i.e., $m_A = \emptyset, m_S \in K \cup \{e\}$, P will communicate and transact with S alone. Hence transfers and output assignments in the augmented mechanism are defined as follows: (X_S, X_A, q) equals $(\tilde{X}_S, \tilde{X}_A, \tilde{q})$ on $\tilde{M}_S \times \tilde{M}_A$, $(0, X(m_S), q(m_S))$ if $m_A = \emptyset, m_S \in K \cup \{e\}$, and $(-T, -T, 0)$ otherwise where T is a large positive number. The last feature ensures that A and S will always coordinate on either the normal delegation mode, or the grand contract.

It is easy to check that this augmented mechanism has a WPBE(wc) where both S and A opt for the normal delegation mode, S offers A a side contract with $m_S(\theta, \eta) = (\theta, \eta) \in K$ and $u_A^*(\theta, \eta) = u_A(\theta, \eta)$ for all (θ, η), which A accepts. To see this note first that if S and

A play this augmented grand contract noncooperatively, A will never select $m_A = \emptyset$, since this results in a negative payoff for A no matter what S does. If $m_A = \emptyset, m_S \in K \cup \{e\}$, A is committed to producing a positive quantity while not getting paid anything, while $m_A = \emptyset, m_S \in \tilde{M}_S$ implies $X_A = -T, q = 0$. And given that A does not select $m_A = \emptyset$, neither will S select m_S in $K \cup \{e\}$, owing to the large penalty T for mis-coordination. Rejection of a side contract will effectively result in noncooperative play of the original grand contract.

Hence A has an outside option of earning $u_A(\theta, \eta)$ by rejecting any side contract offered by S. This (along with the fact that the allocation is WCP) implies that the side contract offered by S in equilibrium is optimal for S. The reason is that the outcome of any feasible side contract in the normal delegation mode was also attainable as the outcome of some feasible side contract in the original mechanism.

Proposition 6 $\Pi_{CD} = \Pi_{CS}$: *any implementable allocation with weak collusion can be implemented as a WPBE(wc) outcome of the modified delegation mechanism described above, where P communicates and transacts with S alone on the equilibrium path.*

The reverse pattern of modified delegation, where P communicates only with A on the equilibrium path, also happens to be an alternative way of implementing an optimal WCP allocation [[(the proof of this is provided in the online Appendix)]]. It implies that the model does not provide any argument for superiority of either form of modified delegation over the other. In the context of legal procedures, this suggests the equivalence of plea bargaining arrangements (where the judge seeks a report from accused party and reserves the right to go to trial should a 'not-guilty' plea be made) with the reverse system where the judge seeks a report from a public prosecutor initially and then decides whether or not to go to trial based on this report. If we were to extend our model to include fixed costs of communication of the Principal with either the supervisor or the agent (but not both), it would provide a way of discriminating between the two alternatives. If for instance communication with S is costless while with A is costly, modified delegation to S will be optimal and will dominate modified delegation to A.

5 Extensions

5.1 Side Contracts Designed by a Third Party, and Alternative Allocations of Bargaining Power

We now explain how the preceding results extend when the side contract is designed not by S, but instead by a third-party that manages the coalition and assigns arbitrary welfare weights to the payoffs of S and A respectively. Such a formulation has been used by a number of authors to model collusion, such as Laffont and Martimort (1997, 2000), Dequiedt (2007) and Celik and Peters (2011). An advantage of this approach is that it allows an evaluation of the effects of varying the allocation of bargaining power between colluding partners.

Our results extend to such a setting, under the following formulation of side contracts designed by a third party. We assume the third-party's objective is to maximize a weighted sum of S and A's interim payoffs. The third party designs the side contract after learning the realization of η.[25] Both S and A have the option to reject the side contract, in which case they play the grand contract noncooperatively.

The notion of WCP allocations is extended as follows. Letting $\alpha \in [0, 1]$ denote the welfare weight assigned by the third-party to A's payoff, the side contract design problem reduces to selecting randomized message $\tilde{m}(\theta \mid \eta)$ and A's payoff $\tilde{u}_A(\theta, \eta)$ to (using the same notation for the formulation $P(\eta)$ of side contracts in Section 3.3):

$$\max E[(1-\alpha)\{\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta)\} + \alpha\tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to $\tilde{m}(\theta \mid \eta) \in \Delta(\hat{M})$,

$$\tilde{u}_A(\theta, \eta) \geq u_A(\theta, \eta)$$

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}(\tilde{m}(\theta' \mid \eta))$$

$$E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta] \geq E[u_S(\theta, \eta) \mid \eta].$$

Besides modifying the objective function, this formulation adds a participation constraint for S. We refer to this as problem $TP(\eta; \alpha)$. The definition of WCP can be extended to WCP(α) by requiring the null side contract to be optimal in $TP(\eta; \alpha)$ for every η.

[25] This assumption can be dropped without affecting the results, since it can be shown the third-party can use cross-reporting of η by S and A to learn its true value.

In Appendix A, we explain how WCP(α) allocations can continue to be justified by a suitable extension of the WPBE(wc) concept to this setting. In order to address the Celik-Peters (2011) problem, side contracts consist of two stages: an initial collusion-participation stage, followed by a reporting or execution stage in the event of both parties agreeing to participate at the first stage. The collusion-participation stage enlarges a dichotomous (exit-participate) message set for each party to a larger message set which includes auxiliary messages for A. At the end of the first stage, S and A observe their respective first stage messages; conditional on both agreeing to participate, they communicate type reports to P at the second stage. The auxiliary first-stage messages enable A to communicate more information to S than is possible with a dichotomous participation decision, and replicate outcomes achievable when side contract offers are rejected by some types of A. This enables attention to be restricted to side contracts which are always accepted on the equilibrium path.[26]

In this setting, the WPBE(wc) notion is extended in the obvious manner: it should never be possible for the third party to deviate to some alternative side-contract whose subsequent continuation game has a PBE which generates a higher payoff for the third-party, without lowering the payoff of S or any type of A. In Appendix A we show that allocations implementable as WPBE(wc) outcomes coincide with the set of WCP(α) allocations.

We now claim that the set of WCP(α) allocations is independent of α. This implies that all our preceding results extend to side contracts designed by a third party.[27]

Proposition 7 *The set of WCP(α) allocations is independent of* $\alpha \in [0, 1]$.

The reasoning is straightforward, [[so the formal proof is relegated to the online Appendix.]] The WCP criterion amounts to the absence of incentive compatible deviations that are Pareto improving for the coalition: this property does not vary with the precise welfare weights. Consider any $\alpha \in (0, 1)$. A given allocation is WCP(α) if and only if there is no other allocation attainable by some non-null side contract which satisfies the incentive constraint for A, and which Pareto-dominates it (for A and S) with at least one of them strictly better off. The same characterization applies to any interior $\alpha' \in (0, 1)$, implying that the set of WCP(α) allocations is independent of $\alpha \in (0, 1)$. The transferability of util-

[26][[Celik and Peters (2013) provide an alternative approach to address this problem.]]

[27]FLM provide an analogous result for the case of interim collusion.

ity can then be used to show that the set of WCP allocations for interior welfare weights are also the same at the boundary.[28]

5.2 Altruistic Supervisors

Now consider a different variant, where S offers a side-contract to A, but S is altruistic towards A rather than just concerned with his own income. Suppose S's payoff is $u_S = X_S + t + \alpha[X_A - t - \theta q]$, where $\alpha \in [0,1]$ is the weight he places on A's payoff. A on the other hand is concerned with only his own income: $u_A = X_A - t - \theta q$.

Our analysis extends as follows. It is easy to check that the expression for coalitional shadow cost is now modified to

$$\pi_\alpha(\theta|\eta) \equiv \theta + (1-\alpha)\frac{F(\theta \mid \eta) - \Lambda(\theta \mid \eta)}{f(\theta \mid \eta)}$$

instead of $\pi(\theta|\eta)$ in Definition 3. In DS, the corresponding expression for the cost of procuring one unit from S is modified from $h(\theta \mid \eta)$ to $h_\alpha(\theta \mid \eta) = \theta + (1-\alpha)\frac{F(\theta|\eta)}{f(\theta|\eta)}$. As long as $\alpha < 1$, this is strictly higher than θ, so DS will still continue to result in a lower profit than NS. The proof that CS dominates NS also goes through *in toto*.

It is interesting to examine the effect of changes in the degree of altruism on P's payoffs. An increase in α lowers S's shadow cost of output in DS $h_\alpha(\theta \mid \eta)$, which benefits P. This is intuitive: the DMR problem becomes less acute with a more altruistic supervisor. Note that with perfect altruism $\alpha = 1$, and the DMR problem disappears: DS then becomes equivalent to NS.

On the other hand, an increase in altruism cannot benefit P in CS. The set of WCP allocations can be shown to be non-increasing in α. Take any WCP allocation corresponding to α: the following argument shows that it is a WCP allocation corresponding to any $\alpha' < \alpha$. Let $z(\theta \mid \eta)$ be the CVC function that is associated with the allocation at α, i.e., it is the ironed version of $\pi_\alpha(\theta|\eta)$ corresponding to some function $\Lambda_\alpha(\cdot|\eta)$ satisfying the stipulated requirements in the definition of CSC functions on $[\underline{\theta}(\eta), \bar{\theta}(\eta)]$. We can then select

$$\Lambda_{\alpha'}(\theta \mid \eta) = \frac{\alpha - \alpha'}{1-\alpha'}F(\theta \mid \eta) + \frac{1-\alpha}{1-\alpha'}\Lambda_\alpha(\theta \mid \eta)$$

[28] If an allocation is WCP(1) but not WCP(α) for some interior α, there must exist a non-null side contract SC^* which allows S to attain a strictly higher payoff, which leaves A's payoff unchanged. Then there exists another feasible non-null side-contract which gives A a slightly higher payoff in all states, which meets S's participation constraint. Hence it is possible to design a feasible side contract that raises A's expected payoff, so the original allocation could not have been WCP(1).

when the altruism parameter is $\alpha^{'}$, which satisfies the stipulated requirements since $\alpha > \alpha^{'}$. This ensures that the same CSC and CVC function is available when the altruism parameter is $\alpha^{'}$, since by construction $\pi_\alpha(\theta|\eta) = \pi_{\alpha^{'}}(\theta|\eta)$. Hence the allocation satisfies the sufficient condition for WCP when the altruism parameter is $\alpha^{'}$.

Finally, if $\alpha = 1$, the CSC function π_α coincides with the identity function θ, the cost of the agent in NS. We thus obtain

Proposition 8 *In CS, P's optimal payoff is non-increasing in α. In DS, P's optimal payoff is increasing in α. When $\alpha = 1$, P's optimal payoffs in DS, NS and CS coincide.*

6 The Indivisible Good Case

We now consider the case where P procures an indivisible good. Moreover, S has access to a signal which takes two possible values. This simple context helps provide better understanding of the nature of the mechanism design problem and how it can be solved. We present numerical computation of third-best allocations when costs are uniformly distributed, which helps assess the magnitude of benefits from hiring a supervisor despite the presence of collusion. The final subsection provides analytical results in this setting concerning the solution to the case of interim collusion and how this relates to the solution with ex ante collusion.

6.1 Characterization of Optimal WCP Allocations

Let $q \in \{0, 1\}$ denote the decision of whether or not to procure the indivisible good, which delivers a gross benefit of V to P. S receives a binary signal $\eta \in \{\eta_1, \eta_2\}$ where $\eta = \eta_1$ represents information that cost is 'low'. We continue to assume the density $f(\theta)$ is well-defined, continuous and positive everywhere on $[0, 1]$. $a(\theta \mid \eta_i)$ $(i = 1, 2)$ denotes the likelihood function of θ conditional on η_i. In addition to continuously differentiability property of $a(\theta \mid \eta_i)$[[on $[\underline{\theta}_i, \bar{\theta}_i] \equiv [\underline{\theta}(\eta_i), \bar{\theta}(\eta_i)]$, we assume the following monotone likelihood ratio property]]:

Assumption 1 $0 = \underline{\theta}_1 \leq \underline{\theta}_2 \leq \bar{\theta}_1 \leq \bar{\theta}_2 = 1$ *and* $\frac{a(\theta|\eta_1)}{a(\theta|\eta_2)}$ *is decreasing in* θ *on* $(\underline{\theta}_2, \bar{\theta}_1)$.

Define $h_i(\theta) \equiv \theta + \frac{F(\theta|\eta_i)}{f(\theta|\eta_i)}$ and $l_i(\theta) \equiv \theta + \frac{F(\theta|\eta_i)-1}{f(\theta|\eta_i)}$ for $i \in \{1, 2\}$. [[These are upper and lower bounds for coalitional virtual costs, corresponding to the lowest and highest possible values of the shadow value $\Lambda(\theta|\eta_i)$.]]

With Assumption 1, $F(\theta \mid \eta_1) > F(\theta \mid \eta_2)$ and $h_1(\theta) > h_2(\theta)$ on $(\underline{\theta}_2, \bar{\theta}_1)$. Our focus is often provided to two examples of information structure.

(1) Partition Case

The information structure partitions the type space into two subintervals. The signal $\eta = \eta_1$ is received when the true θ lies in the interval $[\underline{\theta}_1, \bar{\theta}_1] = [0, c]$ for some $c \in (0, 1)$. And $\eta = \eta_2$ reveals that cost is 'high': that it lies in $[c, 1]$. Then the conditional distribution functions are $F(\theta \mid \eta_1) = \frac{F(\theta)}{F(c)}$ on $[0, c]$ and $F(\theta \mid \eta_2) = \frac{F(\theta)-F(c)}{1-F(c)}$ on $[c, 1]$. Then $h_1(\theta) = \theta + \frac{F(\theta)}{f(\theta)}$, $l_1(\theta) = \theta + \frac{F(\theta)-F(c)}{f(\theta)}$, $h_2(\theta) = \theta + \frac{F(\theta)-F(c)}{f(\theta)}$ and $l_2(\theta) = \theta + \frac{F(\theta)-1}{f(\theta)}$.

(2) Full Support Case

$a(\eta_i \mid \theta) \in (0, 1)$ on $[0, 1]$ for $i = 1, 2$. For example this property is satisfied with linear likelihood function $a(\eta_1 \mid \theta) = d - (2d - 1)\theta$ for $\theta \in \Theta_1 = [\underline{\theta}_1, \bar{\theta}_1] \equiv [0, 1]$ and $a(\eta_2 \mid \theta) = 1 - d + (2d - 1)\theta$ for $\theta \in \Theta_2 = [\underline{\theta}_2, \bar{\theta}_2] \equiv [0, 1]$ with $d \in (1/2, 1)$.

To avoid technical problems associated with the need to iron the coalitional virtual cost functions, we impose the following assumption.

Assumption 2 $H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$ *is increasing in* θ *on* $[0, 1]$*, and* $h_i(\theta)$ *and* $l_i(\theta)$ *are strictly increasing in* θ *on* $[\underline{\theta}_i, \bar{\theta}_i]$ *for any* $i \in \{1, 2\}$.

This assumption is automatically satisfied in both the partition case and the full support case with linear likelihood function, if θ is uniformly distributed.

We also confine attention to mechanisms not involving any randomization.[29]

Using the general characterization of feasible mechanisms established earlier in the paper, it is easy to show that the Principal's choice reduces to selecting: (i) a total payment X_0 to the coalition in the event that the good is not delivered; (ii) an additional bonus b when it is delivered; and (iii) cost thresholds $\theta_i, i = 1, 2$ where $\theta_1 \in [\underline{\theta}_1, \bar{\theta}_1]$ and $\theta_2 \in [\underline{\theta}_2, \bar{\theta}_2]$ where the agent delivers the good in state η_i if and only if $\theta < \theta_i$. Let p_i denote $\int_{\underline{\theta}_i}^{\bar{\theta}_i} a(\eta_i \mid \theta) f(\theta) d\theta$. P's maximization problem reduces to

[29]In the case where V is strictly concave, this assumption entails no loss of generality. We are not sure whether the same is true in this context as well.

$$\max[V - b][p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)] - X_0$$

subject to

$$X_0 \geq F(\theta_i \mid \eta_i)[\theta_i - b] \quad \text{for } i \in \{1, 2\} \tag{6}$$

$$X_0 \geq 0 \tag{7}$$

and (θ_1, θ_2, b) satisfies

$$\text{If} \quad \theta_i \in (\underline{\theta}_i, \bar{\theta}_i), l_i(\theta_i) \leq b \leq h_i(\theta_i) \tag{8}$$

$$\text{If} \quad \theta_i = \underline{\theta}_i, b \leq \underline{\theta}_i \tag{9}$$

$$\text{If} \quad \theta_i = \bar{\theta}_i, b \geq \bar{\theta}_i. \tag{10}$$

The cost threshold θ_i ends up being the 'price' that S offers to A for supplying the good, following signal η_i. Hence (14) represents S's participation constraint in this state, requiring that the fixed payment X_0 must be sufficient to cover the expected 'net' cost of paying A (after taking into account the bonus received from P for delivering the good). Condition (7) represents the constraint that collusion is *ex ante*. If it were not satisfied, the coalition would choose to exit in the event that A reported a cost above the offered price θ_i. In the case of *interim* collusion, this condition would not be imposed: S would have to commit to participating before hearing a cost report from A, whence (14) would suffice to ensure S's participation. Hence *ex ante* collusion represents a kind of 'limited liability' constraint.

The remaining three conditions (8, 9, 10) represent coalitional incentive constraints: it must be in S's interest to offer the price θ_i upon observing η_i.[30] All that is needed (for an 'interior' price) is that the bonus b lie somewhere in-between the upper and lower bounds on coalitional virtual cost (modified in an obvious way for non-interior prices). As shown previously, any price offer lying within these bounds can be induced by P by offering suitable outside options to A.

[30] We use here the fact that coalition incentive compatibility requires that the good will be delivered in state η_i if and only if the bonus b exceeds $z(\theta|\eta_i)$, where $z(\theta|\eta_i) = \theta + \frac{F(\theta|\eta_i) - \Lambda(\theta|\eta_i)}{f(\theta|\eta_i)}$ is the coalitional virtual cost function, where $\Lambda(\theta \mid \eta_i)$ is a non-decreasing function taking value 0 and 1 at the endpoints $\underline{\theta}_i$ and $\bar{\theta}_i$ respectively. Hence conditions (8, 9, 10) are necessary. Conversely, given these three conditions, we can find a coalitional virtual cost function satisfying coalition incentive compatibility.

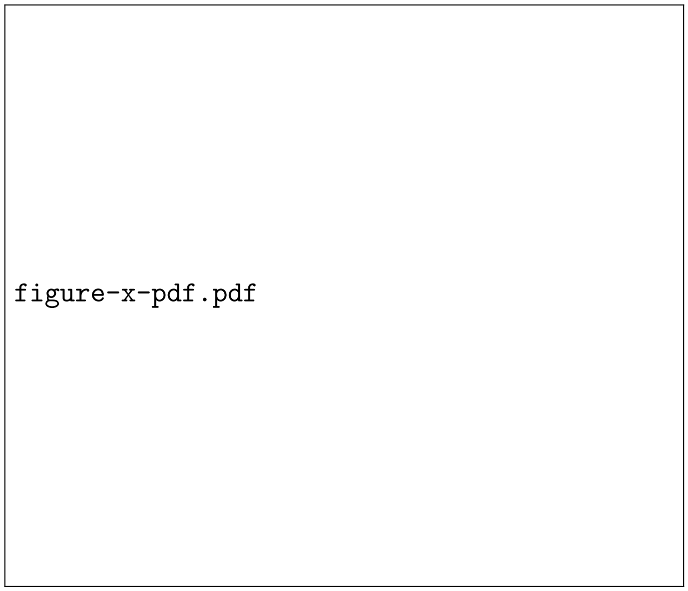

Figure 1: An illustration of a feasible mechanism.

6.2 Value of Supervisor with Partition Information Structure

With the restriction to partition case of information structure with $\theta_1 \in [0, c]$ and $\theta_2 \in [c, 1]$, we examine some properties in optimal WCP allocation.[31] The case of unconditional delegation corresponds to constraining b to equal the upper bound $h_i(\theta_i)$. This is dominated by P contracting with A in the absence of the supervisor, whence b is constrained to equal θ_i. When P contracts with both S and A, b can be lowered further, up to $l_i(\theta_i)$. Figure 1 provides an illustration of a feasible mechanism where $h_i(\theta_i) > \theta_i > b$ for $i = 1, 2$, in which a given bonus b allows a higher probability of supply in each state than would result with unconditional delegation, or if P were to not hire S. Of course, raising θ_i above b comes at a cost: a positive fixed payment X_0 has to be made to ensure S's participation constraint (14).

We first examine in this context whether hiring a supervisor is strictly valuable. It makes sense to exclude cases where $V \leq c$, where hiring S is not valuable even in the absence of collusion.[32] Hence we focus on the case where $V > c$, where hiring S is strictly

[31] We can provide a similar analysis for the full support case.

[32] If $V \leq c$, the second-best with a honest supervisor involves zero probability of procurement in state η_2, and offering A a price of θ_1^{SB} which satisfies $V = H(\theta_1^{SB}) = h_1(\theta_1^{SB})$ in state η_1. This can be implemented by P offering A a price of θ_1^{SB} irrespective of η_i; hence S is not needed.

valuable in the absence of collusion.[33]

Proposition 9 *Suppose $V > c$, so hiring S is strictly valuable in the second-best situation. In the presence of weak ex ante collusion, there exists an interval (V_1, V_2) with $V_1 \geq c$, such that hiring S is strictly valuable if and only if $V \in (V_1, V_2)$. $V_1 > c$ if and only if $H(\max\{0, l_2(c)\}) > c$, while $V_2 \geq H(1)$.*

This result shows that in contrast to previous Sections with divisible quantities and a strictly concave benefit function, collusion may destroy the value of supervision in some circumstances.[34] This can happen for instance when P's benefit from the good V is very large, so she ends up procuring in both states in the third-best outcome. This is only possible if P offers to pay the maximum cost of 1 for delivery in either state η_i.[35] Owing to collusion, it is no longer possible to offer a lower price in state η_1 and still guarantee delivery.

For lower values of V where the good will not always be delivered, the result is less obvious. Proposition 9 states that the condition $H(\max\{0, l_2(c)\}) \leq c$ is sufficient to ensure hiring S is strictly valuable for all values of V slightly above c. This can be explained as follows. In the absence of S, P would offer a price θ^{NS} below c, if V lies between c and $H(c)$. Then P would not procure the good in state η_2. This corresponds to the allocation $\theta_1 = \theta^{NS} = b, \theta_2 = c, X_0 = 0$. Upon hiring S, P can offer the following allocation which would generate a strict improvement. θ_1 could be left unchanged at θ^{NS}, while $\theta_2^{'}$ could be raised slightly above c. See Figure 2. This enables the delivery probability to be increased in state η_2 and left unchanged in state η_1. For $\theta_2^{'}$ close enough to c, it is true that $F(\theta_2^{'} \mid \eta_2) < F(\theta^{NS} \mid \eta_1)$. Hence a contract $(X_0^{'}, b^{'})$ can be chosen to satisfy

$$X_0^{'} = F(\theta^{NS} \mid \eta_1)(\theta^{NS} - b^{'}) = F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - b^{'}).$$

where the bonus $b^{'}$ is now slightly lower than before, satisfying the following condition, $\max\{l_1(\theta^{NS}), l_2(\theta_2^{'})\} \leq b^{'} < \theta^{NS}$. Given that $H(max\{0, l_2(c)\}) \leq c < V(= H(\theta^{NS}))$

[33]With $V > c$, P will procure with positive probability in state η_2 and the second-best price offered to A will necessarily differ between the two states η_1, η_2, since the price offered in state η_1 will not exceed c while it will exceed it in state η_2. Hence S is valuable in the second-best situation.

[34]It can be shown that if the participation constraint (14) for S is strengthened to hold *ex post* rather than *interim*, then supervision ceases to be valuable. This is in contrast to the case where the benefit function is strictly concave, whence it may be possible in some circumstances to hire a supervisor even with *ex post* participation constraints.

[35]If $\theta_i = \bar{\theta}_i$, condition (10) requires $b \geq \bar{\theta}_i$. Hence $b = 1$.

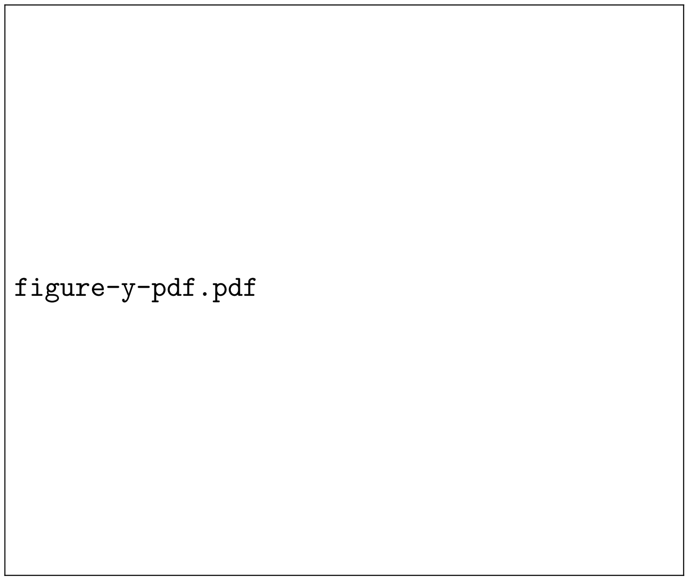

Figure 2: Value of Supervisor.

implies $l_2(c) < \theta^{NS}$ then for θ_2' close enough to c, $l_2(\theta_2') < \theta^{NS}$, making this choice of b' possible. Then P benefits as S continues to earn zero rent in either state, while moving the allocation closer to the second-best.

The condition $H(\max\{0, l_2(c)\}) \leq c$ turns out to also be necessary to ensure a strict value of supervision for values of V slightly above c. The proof of this is somewhat involved (see the Appendix), but the underlying idea is the following. Suppose $H(\max\{0, l_2(c)\}) > c$, implying $l_2(c) > \theta^{NS}$ for V close enough to c. An improvement over no-supervision would require P to procure with positive probability in state θ_2. This requires raising the bonus b' above $l_2(c)$, which is higher than $b = \theta^{NS}$. Correspondingly, the optimal θ_1 also needs to be raised discontinuously, which lowers profits of P in state η_1. If V is sufficiently close to c, the increased profits in state η_2 are negligible, and cannot outweight the losses in state η_1.

Part of the reason that the value of supervision is lower in the indivisible good case is that the set of controls available to P are limited: e.g., there is no scope for varying the level of provision. On the other hand, there exist a range of parameter values where the benefits of hiring S are substantial: the second-best payoff can be achieved.

Proposition 10 *Suppose that $V > c$. The second-best payoff can be achieved by P in the* *presence of collusion if and only if* $F(\theta_1^{SB} \mid \eta_1) > F(\theta_2^{SB} \mid \eta_2)$ *and*

$$\max\{l_1(\theta_1^{SB}), l_2(\theta_2^{SB})\} \leq \frac{\theta_1^{SB} F(\theta_1^{SB} \mid \eta_1) - \theta_2^{SB} F(\theta_2^{SB} \mid \eta_2)}{F(\theta_1^{SB} \mid \eta_1) - F(\theta_2^{SB} \mid \eta_2)}. \tag{11}$$

where θ_i^{SB} *denotes the second-best solution. In the case of a uniform distribution* $F(\theta) = \theta$ *and* $c = 1/2$*, this condition reduces to* $1/2 < V \leq 3/4$*.*

The underlying argument is straightforward. Implementation of the second-best allocation entails setting $\theta_i = \theta_i^{SB}$, and ensuring that S earns zero rent in each state. This requires existence of X_0, b such that

$$X_0 = F(\theta_1^{SB} \mid \eta_1)[\theta_1^{SB} - b] = F(\theta_2^{SB} \mid \eta_2)[\theta_2^{SB} - b] \geq 0 \tag{12}$$

for which it is necessary that $F(\theta_1^{SB} \mid \eta_1) > F(\theta_2^{SB} \mid \eta_2)$, and b is set equal to the right-hand-side of (11). Since $\theta_i^{SB} \geq b$, this allocation is feasible if condition (11) is satisfied.

This argument indicates, however, that implementation of the second-best will be generically impossible if there are three or more possible signals observed by S. For example, with three signals, in order to ensure S earns zero rent for all η_i, there must exist b such that

$$F(\theta_1^{SB} \mid \eta_1)[\theta_1^{SB} - b] = F(\theta_2^{SB} \mid \eta_2)[\theta_2^{SB} - b] = F(\theta_3^{SB} \mid \eta_3)[\theta_3^{SB} - b] \geq 0.$$

which requires

$$B(\theta_1^{SB}, \theta_2^{SB}) = B(\theta_2^{SB}, \theta_3^{SB})$$

where

$$B(\theta_i, \theta_j) \equiv \frac{\theta_i F(\theta_i \mid \eta_i) - \theta_j F(\theta_j \mid \eta_j)}{F(\theta_i \mid \eta_i) - F(\theta_j \mid \eta_j)}.$$

This condition will not hold generically.

In the case of strictly concave $V(q)$, our result concerning the impossibility of the second best allocation under suitable conditions was based on a different kind of argument, relying on the continuity of the second best output schedule. When the good is indivisible, such arguments do not apply as the second best output schedule q^{SB} jumps discontinuously from 1 to 0 at certain points.

It is interesting to note an implication of Proposition 10: second-best implementation requires the good not be procured with positive probability in states η_1 and η_2, which in turn requires V to not be too large. This is similar to the result of Pavlov (2008) and Che

2D.pdf

Figure 3: Comparing Optimal Collusion-Proof, Second-Best and No Supervision Allocations, with uniformly distributed cost and $c = \frac{1}{2}$

and Kim (2009) in the context of auctions, whence second-best implementation requires trade to not occur with positive probability.

With a uniform distribution and $c = 1/2$, we can numerically compute optimal allocations under the second-best, third-best and no-supervision respectively. The results are shown in Figure 3. As shown above, the second best allocation can be implemented in the case $1/2 < V \leq 3/4$. Hiring S is valuable if V is between 3/4 and 2. Compared to the second-best, we see that for some intervals of V between 3/4 and 2 the probability of procurement decreases, especially in state η_2.

6.3 Relating Solutions to Ex Ante and Interim Collusion

Finally, we derive the solution to the interim collusion problem, and show how it relates to the ex ante collusion solution. This helps relate our work to that of FLM and Celik (2009). The P's maximization problem in interim collusion differs from ex-ante one only in that we can drop the coalitional participation constraint (7). Let $(\theta_1^I, \theta_2^I, b^I, X_0^I)$ be the solution for the interim collusion problem and $(\theta_1^E, \theta_2^E, b^E, X_0^E)$ be the solution for the ex-ante collusion

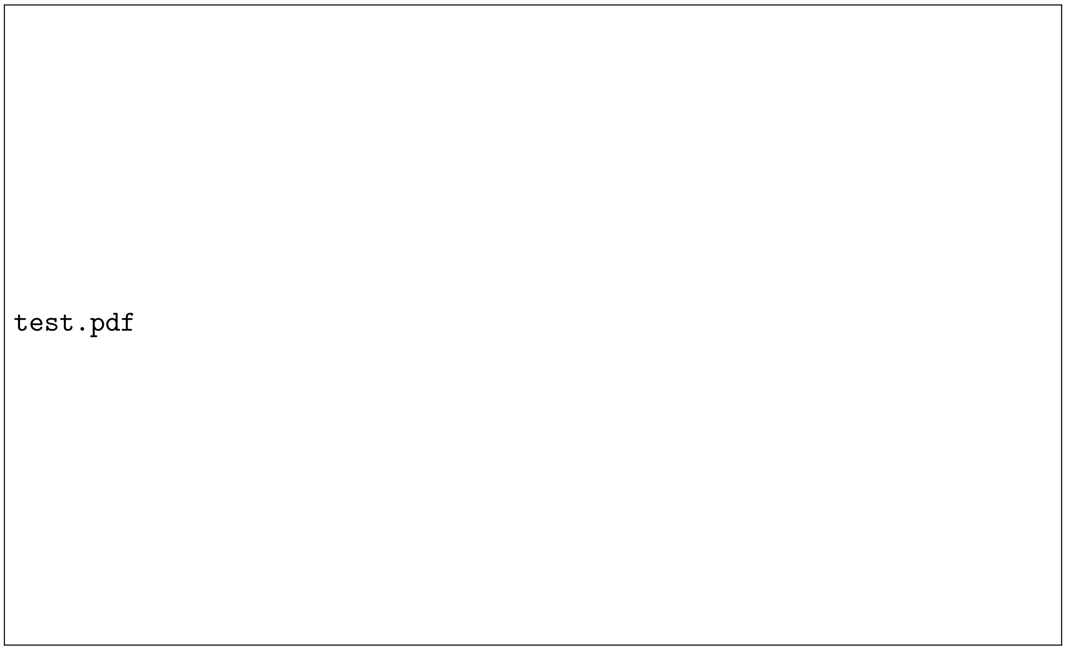

Figure 4: Optimality of Pure Delegation in Interim Collusion

problem. If $X_0^I \geq 0$ in the solution of the interim collusion problem, it is evident that the interim collusion and the ex-ante collusion have the same solution.

Our main result for this subsection can now be stated.

Proposition 11 *Consider the indivisible good case where Assumptions 1 and 2 hold.*

(i) The solutions to interim and ex-ante collusion differ if and only if the solution to interim collusion can be attained via pure delegation to S.

(ii) If S's information has a partition structure with $c \in (0, 2/3)$ and θ is uniformly distributed, the solution to interim collusion cannot be attained via pure delegation to S.

(iii) If S's information has the full-support structure, there exists a non-degenerate set of values of V for which the solution to interim collusion can be attained via pure delegation to S. For other values of V, the solution to interim collusion cannot be attained via pure delegation to S.

Parts (ii) and (iii) show how the results of Celik (2009) and FLM concerning optimality of pure delegation extend to the continuum of cost types context. For the full support case with uniformly distributed θ and linear likelihood function $a(\eta_1 \mid \theta) = d - (2d-1)\theta$, Figure 4 shows the curve in (V, d) space dividing the two regions where delegation is and is not an optimal solution in interim collusion: the region above the line is where delegation is optimal. Part (i) in the preceding Proposition relates the solution to interim and ex ante collusion, which diverge if and only if pure delegation is optimal in interim collusion. Our earlier result concerning the suboptimality of pure delegation in ex ante collusion implies the 'if' part, which clearly holds very generally. The 'only if' part is the novel result in this setting — stating that whenever delegation is suboptimal in interim collusion (e.g., a partition information structure), the solution to interim collusion is feasible (and hence optimal) in ex ante collusion. The intuition for this is that in such cases, it is optimal in the second-best as also in interim collusion for P to procure the good with higher probability when S's signal is that the agent's cost is low. The solution to interim collusion then involves a low-powered incentive contract, involving a low bonus b and a non-negative fixed payment X_0, which renders it feasible under ex ante collusion.[36]

7 Concluding Comments

We have analyzed implications of weak ex ante collusion between a supervisor and agent, where collusion arises with regard to both participation and reporting decisions, and outside option payoffs in coalitional bargaining are determined by noncooperative equilibria of a grand contract designed by the Principal. We showed in such settings that the Principal can still benefit from employing the supervisor. This requires the Principal to design a grand contract involving both the supervisor and the agent, rather than delegating authority over contracting with the agent to the supervisor in an unconditional manner. It is essential for the Principal to give both parties suitable outside option payoffs by designing such a grand contract judiciously. The presence of such a centralized safeguard as an option then allows optimal outcomes to be implemented by delegating authority to the supervisor. These results are consistent with the widespread prevalence of delegation to information intermediaries, and highlight the importance of centralized oversight mechanisms that are needed to mitigate their 'abuse of power'. While the commonsense justification for such

[36]Specifically, extraction of S's interim rents under interim collusion requires solving for b and X_0 in the two equations $F(\theta_i|\eta_i)(b-\theta_i) + X_0 = 0, i = 1, 2$. If $F(\theta_1|\eta_1) \geq F(\theta_2|\eta_2)$ [[and $\theta_1 \leq \theta_2$,]] the solution satisfies $X_0 \geq 0$.

mechanism is typically based on considerations of fair treatment of agents, our analysis shows how such mechanisms are essential to prevent inefficient output contractions and loss of profits of the Principal owing to monopsonistic behavior by intermediaries to whom authority is delegated.

We now describe briefly how our results are modified if our model is extended in various directions:

1. If A cannot observe S's signal, the side contract will be subject to bilateral asymmetric information within the coalition. This extension is considered in Tsumagari (2016a), for the case where S's signal is binary. All the key results of this paper are shown to extend to that setting, excepting the result concerning suboptimality of pure delegation which has been verified only for some special cases.

2. Tsumagari (2016b) considers the case where A's type space is discrete rather than continuous. Provision of countervailing incentives by P then becomes more costly. Intuitively, countervailing incentives provided [[via outside option payoffs of A that are decreasing]] in the latter's cost cause incentive constraints to [[bind in the downward direction, i.e., they tempt high cost types to mimic low cost types.]] The cost of this to P becomes larger when the gap between successive cost types becomes larger. As a consequence, it may not be valuable for P to appoint S in some cases involving a divisible good. All other results of this paper, however, do extend.

3. The allocation of bargaining power between colluding members matters in the case of strong collusion, where one or more of the colluding partners are 'powerful' in the sense of being able to commit how they would behave in the event that others veto a coalitional proposal. We are currently studying this extension. In such contexts, P has less control over outside options of S and A when they bargain over a side contract. If A alone is 'strong', it turns out that the solution is unaffected. However, in other cases, P is worse off, and bargaining welfare weights end up affecting P's welfare in interesting ways. We postpone this issue to a subsequent paper.

Finally, we mention some important qualifications to our analysis. We have ignored the possibility of other coalitions that may co-exist with the S-A coalition. For instance, if P can enter into a side-contract with S that is unobserved by A, the costs of collusion can be lowered. Ortner and Chassang (2015) show this in a setting where P can offer randomized contracts to S that are unobserved by A. This enlarges the extent of asymmetric information within the S-A coalition, which benefits P. Second, we have not modeled the enforcement of side contracts within the coalition. Modeling self-enforcing collusion via a relational contract in a side game between colluding parties seems to be an interesting extension that could be pursued in future research.

References

Atanassov, J. and E. Kim (2009), "Labor and Corporate Governance: International Evidence from Restructuring Decisions," *Journal of Finance*, 64(1), 341-374.

Baliga, S. and T. Sjostrom (1998), "Decentralization and Collusion," *Journal of Economic Theory*, 83(2), 196-232.

Bertrand, M. and S. Mullainathan (1999), "Is There Discretion in Wage Setting? A Test Using Takeover Legislation," *Rand Journal of Economics*, 30(3), 535-554.

Bertrand, M. and S. Mullainathan (2003), "Enjoying the Quiet Life? Corporate Governance and Managerial Preferences," *Journal of Political Economy*, 111(5), 1043-1075.

Celik, G. (2009), "Mechanism Design with Collusive Supervision", *Journal of Economic Theory*, 144(1), 69-95.

Celik, G. and M. Peters (2011), "Equilibrium Rejection of a Mechanism", *Games and Economic Behavior*, 73(2), 375-387.

Celik, G and M. Peters (2013), "Reciprocal Relationships and Mechanism Design", Working Paper, University of British Columbia.

Che, Y. K. and J. Kim (2006), "Robustly Collusion-Proof Implementation", *Econometrica*, 74(4), 1063-1107.

Che, Y. K. and J. Kim (2009), "Optimal Collusion-Proof Auctions", *Journal of Economic Theory*, 144(2), 565-603.

Cornaggia, J., K. J. Cornaggia and H. Xia (2016), "Revolving Doors on Wall Street," *Journal of Financial Economics*, forthcoming.

Cronqvist, H., F. Heyman, M. Nilsson, H. Svaleryd, and J. Vlachos (2009), "Do Entrenched Managers Pay Their Workers More?," *Journal of Finance*, 64(1), 309-339.

deHaan, E., S. Kedia, K. Koh and S. Rajgopal (2015), "The Revolving Door and the SEC's Enforcement Outcomes: Initial Evidence from Civil Litigation," *Journal of Accounting and Economics*, 60(2), 65-96.

Dequiedt, V. (2007), "Efficient Collusion in Optimal Auctions", *Journal of Economic Theory*, 136(1), 302-323.

Faure-Grimaud, A., J. J. Laffont and D. Martimort (2003), "Collusion, Delegation and Supervision with Soft Information", *Review of Economic Studies*, 70(2), 253-279.

Firth, M., O. M. Rui and X. Wu (2012), "How do Various Forms of Auditor Rotation Affect Audit Quality? Evidence from China," *International Journal of Accounting*, 47(1), 109-138.

Fracassi, C., and G. Tate (2012), "External Networking and Internal Firm Governance," *Journal of Finance*, 67(1), 153-194.

Hallock, K. F. (1997), "Reciprocally Interlocking Boards of Directors and Executive Compensation," *Journal of Financial and Quantitative Analysis*, 32(03), 331-344.

Harris M. and A. Raviv (2008), "A Theory of Board Control and Size", *Review of Financial Studies,* 21(4), 1797-1832.

Hirschman, A (1970) "Exit, Voice and Loyalty: Responses to Decline in Firms, Organizations, and States", Harvard University Press, Cambridge/Mass.

Hwang, B. H. and S. Kim (2009), "It Pays to Have Friends," *Journal of Financial Economics*, 93(1), 138-158.

Jullien, B. (2000), "Participation Constraints in Adverse Selection Models," *Journal of Economic Theory*, 93(1), 1-47.

Kofman, F. and J. Lawarree (1993), "Collusion in Hierarchical Agency," *Econometrica*, 61(3), 629-656.

Kramarz, F. and D. Thesmar (2013), "Social Networks in the Boardroom," *Journal of the European Economic Association*, 11(4), 780-807.

Laffont, J. J. and D. Martimort (1997), "Collusion under Asymmetric Information", *Econometrica*, 65(4), 875-911.

Laffont, J. J. and D. Martimort (2000), "Mechanism Design with Collusion and Correlation", *Econometrica*, 68(2), 309-342.

Laffont, J. J. and J. Tirole (1993), "A Theory of Incentives in Procurement and Regulation", MIT Press, Cambridge, MA.

Lennox, C. (2005), "Audit Quality and Executive Officers' Affiliations with CPA Firms," *Journal of Accounting and Economics*, 39(2), 201-231.

Lennox, C. S. and C. W. Park (2007), "Audit Firm Appointments, Audit Firm Alumni, and Audit Committee Independence," *Contemporary Accounting Research*, 24(1), 235-58.

Mas-Colell, A., M. D. Whinston, and J. Green (1995) "Microeconomic Theory", Oxford University Press, New York.

Mookherjee, D. and M. Tsumagari (2004), "The Organization of Supplier Networks: Effects of Delegation and Intermediation", *Econometrica*, 72(4), 1179-1219.

Motta, A. (2009), "Collusion and Selective Supervision", Working Paper.

Myerson, R. (1981), "Optimal Auction Design," *Mathematics of Operations Research* 6, 58–73.

Ortner, J and S. Chassang (2015), "Making Corruption Harder: Asymmetric Information, Collusion, and Crime", Working Paper.

Pavlov, G. (2008), "Auction Design in the presence of Collusion", *Theoretical Economics*, 3(3), 383-429.

Quesada, L. (2004), "Collusion as an Informed Principal Problem", University of Wisconsin.

Schmidt, B. (2015), “Costs and Benefits of Friendly Boards during Mergers and Acquisitions,” *Journal of Financial Economics*, 117(2), 424-447.

Tirole, J. (1986), “Hierarchies and Bureaucracies: On the Role of Collusion in Organizations”, *Journal of Law, Economics, and Organization*, 2, 181-214.

Tsumagari, M. (2016a), “On the Role of Collusive Supervisor: Case of Bilateral Asymmetric Information”, Working Paper.

Tsumagari, M. (2016b), “Weak Ex Ante Collusion and Value of Supervisor: Discrete Type Model”, Working Paper.

Williamson, O. E. (1975) “Markets and Hierarchies: Antitrust Analysis and Implications”, New York: The Free Pres.

Appendix A: Justification for WCP Allocations

Here we explain how WCP allocations can be justified when side contracts are offered by a third party.

To address the problem highlighted by Celik and Peters (2011), the side-contract is modelled as a two stage game played by S and A. The first stage is a 'participation' stage where they communicate their participation decisions in the side contract, in addition to some auxiliary messages in the event of agreeing to participate. The role of these messages is to allow A to signal information about his type while agreeing to participate, which can help replicate whatever information is communicated by side-contract rejection in a setting where communication concerning participation decisions is dichotomous. A and S observe the messages sent by each other at the end of the first stage. At the second stage, A and S submit type reports, conditional on having agreed to participate at the first stage.

Let (D_A^p, D_S^p) denote the message sets of A and S at the participation stage (or p-stage). $e_A \in D_A^p$ and $e_S \in D_S^p$ are the exit options of A and S respectively.

What occurs at the second stage ('execution' or e-stage) depends on $d^p = (d_A^p, d_S^p)$ chosen at the first stage.

- If $d_A^p \neq e_A$ and $d_S^p \neq e_S$, A and S select $(d_A^e, d_S^e) \in D_A^e(d^p) \times D_S^e(d^p)$ respectively. The report to P is selected according to $\tilde{m}(d^p, d^e) \in \Delta(M_A \times M_S)$, associated with the transfers to A and S, $t_A(d^p, d^e)$ and $t_S(d^p, d^e)$ respectively. Owing to wealth constraint of the third party, these are constrained to satisfy $t_A(d^p, d^e) + t_S(d^p, d^e) \leq 0$.
- If either $d_A^p = e_A$ or $d_S^p = e_S$, A and S play GC non-cooperatively.

Given GC and η, the third party decides whether to offer a side-contract $SC(\eta)$ or not (i.e., offer a null side-contract NSC). If a non-null side-contract is offered, A and S play a game denoted by $GC \circ SC(\eta)$ with two stages as described above. On the other hand, if the third party offers a null side-contract NSC at the first stage, A and S play GC non-cooperatively based on prior beliefs $p_0(\eta)$. The third-party's objective is to maximize $E[\alpha u_A(\theta, \eta) + (1 - \alpha) u_S(\theta, \eta) \mid \eta]$ in state η.

The refinement WPBE(wc) introduced in the text for the case where the side contract is offered by S, can now be extended as follows.

Definition 7 *Following the selection of a grand contract by P, a WPBE(wc) is a Weak*

Perfect Bayesian Equilibrium (WPBE) of the subsequent game in which side-contracts are designed by a third party, which has the following property. There does not exist some η, and some deviating side-contract offer $SC(\eta)$ for which there is a Perfect Bayesian Equilibrium (PBE) of the subsequent continuation game in which (conditional on η) the third-party's payoff is strictly higher, without lowering the payoff of S andor any type of A.

Definition 8 *An allocation (u_A, u_S, q) is implementable in the weak collusion game with side contracts designed by a third party assigning welfare weight α to A, if there exists a grand contract and a WPBE(wc) of the subsequent side contract game which results in this allocation.*

Lemma 1 *An allocation (u_A, u_S, q) is implementable in the weak collusion game with side contracts designed by a third party assigning welfare weight α to A, if and only if it is a WCP(α) allocation satisfying the interim participation constraints $u_A(\theta, \eta) \geq 0$ and $E[u_S(\theta, \eta) \mid \eta] \geq 0$.*

Proof of Lemma 1

Proof of Necessity

For some GC, suppose that allocation (u_A, u_S, q) is implemented in the game with weak collusion. Suppose the allocation is achieved as the outcome of a WPBE(wc) in which a non-null side contract $SC^*(\eta)$ is offered on the equilibrium path in some state η, which is rejected by some types of A. We show it can also be achieved as the outcome of a WPBE(wc) in which a non-null side contract is offered in state η and accepted by all types of A. Let Θ_r be the set of types who reject $SC^*(\eta)$. Following A's rejection ($d_A^p = e_A$), suppose that A and S play the grand contract GC based on S's updated belief $p(\cdot \mid \Theta_r, \eta)$. Since we are using the PBE as the solution concept, these beliefs do not depend on S's participation decision. Similarly in the event that A accepts, but S rejects $SC^*(\eta)$, A and S play the grand contract GC based on S's updated belief conditioned on the observation of $d_A^p \neq e_A$. Let $d_A^{p*}(\theta, \eta)$ denote A's decision (on the equilibrium path) at the participation stage. Denoting these beliefs by $p(d_A^p) \equiv p(\cdot \mid d_A^{p*}(\theta, \eta) = d_A^p, \eta)$, S's expected payoff from rejecting $SC^*(\eta)$ is

$$E[u_S(\theta, \eta, c(p(d_A^{p*}(\theta, \eta)))) \mid \eta]$$

where $u_S(\theta,\eta,c) \equiv X_S(m_A(\theta,\eta), m_S(\eta))$ for $c = (m_A(\theta,\eta), m_S(\eta))$.

Now construct a new side-contract $\tilde{SC}(\eta)$ which differs from $SC^*(\eta)$ only in that A's message set at the participation stage is $D_A^p \cup \{\tilde{d}_A^p\}$ instead of D_A^p, and A's choice of $\tilde{d}_A^p$ results in A and S playing of $c(p(e_A))$ in GC without any transfers. It is easily verified that the continuation game $GC \circ \tilde{SC}(\eta)$ has a PBE where no type of A rejects the side-contract, realizing the same allocation (u_A, u_S, q) in an equilibrium. In this equilibrium, type $\theta \in \Theta_r$ reports $d_A^p = \tilde{d}_A^p$ instead of $d_A^p = e_A$. In the off-equilibrium-path event that A rejects $\tilde{SC}(\eta)$, A and S play the grand contract based on the belief $p(\cdot \mid \Theta_r, \eta)$. Since S receives the same information from A's decision about d_A^p, he does not have an incentive to change his decision at the second stage; this in turn implies he has no incentive to change his decision at the participation stage. Since the original equilibrium was a WPBE(wc), so is the newly constructed equilibrium.

Next we show that if allocation (u_A, u_S, q) is realized in a WPBE (wc) in which side contracts are not rejected on the equilibrium path, it must be a WCP(α) allocation. Suppose not: the allocation resulting from some non-null side contract $(\tilde{u}_A^*(\theta,\eta), \tilde{m}^*(\theta,\eta)) \neq (u_A(\theta,\eta), (\theta,\eta))$ solves the problem $TP(\eta;\alpha)$ for some η. Define $\tilde{u}_S^*(\theta,\eta) \equiv \hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}_A^*(\theta,\eta)$. It is evident that

$$E[\alpha\tilde{u}_A^*(\theta,\eta) + (1-\alpha)\tilde{u}_S^*(\theta,\eta) \mid \eta] > E[\alpha u_A(\theta,\eta) + (1-\alpha)u_S(\theta,\eta) \mid \eta],$$

$$\tilde{u}_A^*(\theta,\eta) \geq u_A(\theta,\eta)$$

and

$$E[\tilde{u}_S^*(\theta,\eta) \mid \eta] \geq E[u_S(\theta,\eta) \mid \eta].$$

We can find $m^c(\theta,\eta) \in \Delta(M_A \times M_S)$ for GC such that

$$(X_A(m^c(\theta,\eta)) + X_S(m^c(\theta,\eta)), q(m^c(\theta,\eta))) = (\hat{X}(\tilde{m}^*(\theta \mid \eta)), \hat{q}(\tilde{m}^*(\theta \mid \eta))).$$

Now construct a new side-contract $SC(\eta)$ which realizes

$$(\tilde{u}_A^*(\theta,\eta), \tilde{u}_S^*(\theta,\eta), \hat{q}(\tilde{m}^*(\theta \mid \eta)))$$

as a PBE outcome of the game $GC \circ SC(\eta)$, contradicting the hypothesis that (u_A, u_S, q) is realized in a WPBE (wc), since by construction the interim participation constraints are satisfied. $SC(\eta)$ is specified as follows:

- $D^p \equiv D^{p*}$ where $D^{p*} = (D_A^{p*}, D_S^{p*})$ are A and S's message sets at the participation stage of the original equilibrium side-contract $SC^*(\eta)$.
- $D_A^e = \Theta(\eta)$ and $D_S^e = \{\phi\}$
- A's choice of $d_A^e = \theta \in \Theta(\eta)$ generates the report to P according to $m^c(\theta, \eta)$, associated with the transfers to A and P respectively:

$$t_A(\theta, \eta) = \tilde{u}_A^*(\theta, \eta) - [X_A(m^c(\theta, \eta)) - \theta q(m^c(\theta, \eta))]$$

and

$$t_S(\theta, \eta) = \tilde{u}_S^*(\theta, \eta) - X_S(m^c(\theta, \eta)).$$

For this side-contract $SC(\eta)$, we claim the following is a PBE of the game $GC \circ SC(\eta)$. Given any (d_A^p, d_S^p) with $d_A^p \neq e_A$ and $d_S^p \neq e_S$ at the participation stage, A always selects $d_A^e = \theta$, since $\theta^{'} = \theta$ maximizes

$$X_A(m^c(\theta^{'}, \eta)) - \theta q(m^c(\theta^{'}, \eta)) + t_A(\theta^{'}, \eta) = \tilde{u}_A^*(\theta^{'}, \eta) + (\theta^{'} - \theta)\hat{q}(\tilde{m}^*(\theta \mid \eta)).$$

At the participation stage, A is indifferent among any $d_A^p \in D_A^p \backslash \{e_A\}$ as the optimal response to $d_S^p \neq e_S$, since the same outcome is realized in the continuation for any of these choices. Therefore it is optimal for A to choose the same $d_A^*(\theta, \eta)$ as in the original equilibrium. It implies that S's rejection induces non-cooperative play of GC based on the same updated beliefs as in the original equilibrium. $E[\tilde{u}_S^*(\theta, \eta) \mid \eta] \geq E[u_S(\theta, \eta) \mid \eta]$ guarantees S's participation. On the other hand, specify that A's choice of $d_A^p = e_A$ induces non-cooperative play of GC based on the same beliefs as in the original equilibrium. It guarantees A's participation $d_A^p \neq e_A$. Hence this is a PBE resulting in $(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta))$, completing the argument. This completes the proof of the necessity.

Proof of Sufficiency

Take an allocation which is WCP(α) and satisfies the interim participation constraints. To show that it is implementable as a WPBE(wc) outcome, consider the grand contract GC:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(m_A, m_S) = X_S(m_A, m_S) = q(m_A, m_S) = 0$$

for (m_A, m_S) such that either $m_A = e_A$ or $m_S = e_S$.

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$ for $\eta_A = \eta_S$ and $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (-T, 0)$ for $\eta_A \neq \eta_S$
- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$

where $T > 0$ is sufficiently large. The WCP(α) property implies that $u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$. The interim participation constraints imply that this grand contract has a non-cooperative pure strategy equilibrium

$$(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$$

based on prior beliefs. For this grand contract, we claim there exists a WPBE(wc) resulting in $(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$. This requires us to check that there is no alternative $SC(\eta)$ in any state η with an associated PBE of the continuation game which generates a higher expected payoff for the third party, without making S or any type of A worse off. With sufficiently large $T > 0$, the third party never benefits from a side-contract which instructs the coalition to submit a report to P with $\eta_A \neq \eta_S$. Then the WCP(α) property implies that the third party does not benefit from any manipulation of the report to P, while guaranteeing $E[u_S(\theta, \eta) \mid \eta]$ to S and $u_A(\theta, \eta)$ to A. ■

Appendix B: Proofs of Results in the Text

Proof of Proposition 2: Consider the necessity part. Suppose the allocation (u_A, u_S, q) is WCP. Then the null side contract is optimal for S for every η, so must be feasible in $P(\eta)$. This implies $(u_A(\theta,\eta), q(\theta,\eta))$ satisfies A's incentive compatibility condition. Now consider the problem $P(\eta)$. The incentive constraint

$$\tilde{u}_A(\theta,\eta) \geq \tilde{u}_A(\theta^{'},\eta) + (\theta^{'} - \theta)\hat{q}(\tilde{m}(\theta^{'} \mid \eta))$$

is equivalent to

$$\tilde{u}_A(\theta,\eta) = \tilde{u}_A(\bar{\theta}(\eta),\eta) + \int_{\theta}^{\bar{\theta}(\eta)} \hat{q}(\tilde{m}(y \mid \eta))dy$$

and $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. Then the problem can be rewritten as

$$\max E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta,\eta) \mid \eta]$$

subject to $\tilde{m}(\theta \mid \eta) \in \Delta(\hat{M})$ where $\hat{M} \equiv K \cup \{e\}$,

$$\tilde{u}_A(\theta,\eta) = \tilde{u}_A(\bar{\theta}(\eta),\eta) + \int_{\theta}^{\bar{\theta}(\eta)} \hat{q}(\tilde{m}(y \mid \eta))dy \geq u_A(\theta,\eta)$$

and $\hat{q}(\tilde{m}(\theta \mid \eta))$ non-increasing in θ. Since randomized side contracts can be chosen, the objective function is concave, the feasible set is convex [[and has non-empty interior.]] So the solution maximizes (subject to the constraint $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ) the following Lagrangian expression corresponding to some non-decreasing function $\tilde{\Lambda}(\theta \mid \eta)$:

$$\begin{aligned} \mathcal{L} \equiv\ & E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta,\eta)|\eta] \\ +\ & \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\tilde{u}_A(\theta,\eta) - u_A(\theta,\eta)]d\tilde{\Lambda}(\theta \mid \eta) \end{aligned}$$

where $\hat{X}(\tilde{m}), \hat{q}(\tilde{m})$ denote expected values of $\hat{X}(m), \hat{q}(m)$ taken with respect to probability measure $\tilde{m}$ over $m \in \hat{M}$. Note that without loss of generality, $\tilde{u}_A(\theta,\eta)$ is a deterministic function.

A's incentive constraint implies $\tilde{u}_A(\theta,\eta)$ is continuous on $\Theta(\eta)$. Hence integration by parts yields:

$$\begin{aligned}\int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \tilde{u}_A(\theta,\eta)d\tilde{\Lambda}(\theta \mid \eta) &= \tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta)\tilde{u}_A(\bar{\theta}(\eta),\eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)\tilde{u}_A(\underline{\theta}(\eta),\eta) \\ &+ \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \tilde{\Lambda}(\theta \mid \eta)\hat{q}(\tilde{m}(\theta \mid \eta))d\theta \\ &= [\tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)]\tilde{u}_A(\bar{\theta}(\eta),\eta) \\ &+ \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\tilde{\Lambda}(\theta \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)]\hat{q}(\tilde{m}(\theta \mid \eta))d\theta.\end{aligned}$$

The second equality comes from

$$\tilde{u}_A(\underline{\theta}(\eta),\eta) = \tilde{u}_A(\bar{\theta}(\eta),\eta) + \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \hat{q}(\tilde{m}(y \mid \eta))dy.$$

Next consider the effect of raising uniformly A's outside option function from $u_A(\theta,\eta)$ to $u_A(\theta,\eta) + \Delta$ where Δ is an arbitrary positive scalar. It is evident that the solution is unchanged, except that $\tilde{u}_A(\theta,\eta)$ is raised uniformly by Δ. Hence the maximized payoff of S must fall by Δ, implying that

$$\int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} \Delta d\tilde{\Lambda}(\theta \mid \eta) = [\tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)]\Delta = \Delta,$$

and so $\tilde{\Lambda}(\bar{\theta}(\eta) \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta) = 1$ in the optimal solution. Now define $\Lambda(\theta \mid \eta) \equiv \tilde{\Lambda}(\theta \mid \eta) - \tilde{\Lambda}(\underline{\theta}(\eta) \mid \eta)$. Then $\Lambda(\theta \mid \eta)$ is non-decreasing in θ with $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ and $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$.

This implies

$$\begin{aligned}\mathcal{L} \equiv & \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\hat{X}(\tilde{m}(\theta \mid \eta)) - \pi(\theta \mid \eta)\hat{q}(\tilde{m}(\theta \mid \eta))]dF(\theta \mid \eta) \\ & - \int_{(\underline{\theta}(\eta),\bar{\theta}(\eta)]} u_A(\theta,\eta)d\Lambda(\theta \mid \eta)\end{aligned} \tag{13}$$

where $\pi(\theta \mid \eta) \equiv \theta + \frac{F(\theta|\eta)-\Lambda(\theta|\eta)}{f(\theta|\eta)}$. This has to be maximized subject to the constraint that $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. This reduces to the unconstrained maximization of the corresponding expression where the CSC function $\pi(\cdot \mid \eta)$ is replaced by the corresponding CVC function $z(\cdot \mid \eta)$ using the ironing procedure relative to the cdf $F(\theta \mid \eta)$.

If $\tilde{m}^*(\theta \mid \eta)$ is optimal in problem $P(\eta)$, there exists $\pi(\cdot \mid \eta) \in Y(\eta)$ so that the optimal side contract $\tilde{m} = \tilde{m}^*(\theta \mid \eta)$ maximizes

$$\hat{X}(\tilde{m}(\theta \mid \eta)) - z(\theta \mid \eta)\hat{q}(\tilde{m}(\theta \mid \eta))$$

where $z(\theta \mid \eta)$ [[is the ironed version of $\pi(\cdot \mid \eta), \eta)$.]] Moreover $\hat{q}(\tilde{m}^*(\theta \mid \eta))$ must be non-increasing in θ and flat on any interval of θ which is a subset of $\Theta(\pi(\cdot \mid \eta), \eta)$.

If the optimal side contract is degenerate and concentrated at (θ, η), it must be the case that

$$\hat{X}(\theta,\eta) - z(\theta \mid \eta)\hat{q}(\theta,\eta) \geq \hat{X}(\tilde{m}^{'}) - z(\theta \mid \eta)\hat{q}(\tilde{m}^{'})$$

for any $\tilde{m}^{'} \in \Delta(\hat{M})$. This implies

$$\hat{X}(\theta,\eta) - z(\theta \mid \eta)q(\theta,\eta) \geq \hat{X}(\theta^{'},\eta^{'}) - z(\theta \mid \eta)q(\theta^{'},\eta^{'})$$

$$\hat{X}(\theta,\eta) - z(\theta \mid \eta)q(\theta,\eta) \geq 0$$

for any $(\theta,\eta), (\theta^{'},\eta^{'})$, implying (i) in the proposition. Obviously $q(\theta,\eta)$ must be non-increasing in θ and must be flat on any interval of θ which is a subset of $\Theta(\pi(\cdot \mid \eta), \eta)$ (implying (iii) in the proposition).

Now consider the sufficiency part. Consider any state η. Suppose there is a CSC function $\pi(\cdot \mid \eta) \in Y(\eta)$ which is ironed to yield the CVC function $z(\cdot|\eta)$ such that $(u_S(\theta,\eta), u_A(\theta,\eta), q(\theta,\eta))$ satisfies all the conditions in the proposition. Define $(\hat{X}(m), \hat{q}(m))$ on $\hat{M} \equiv K \cup \{e\}$ such that

$$(\hat{X}(\theta,\eta), \hat{q}(\theta,\eta)) = (u_S(\theta,\eta) + u_A(\theta,\eta) + \theta q(\theta,\eta), q(\theta,\eta))$$

and

$$(\hat{X}(e), \hat{q}(e)) = (0,0).$$

and extend this to $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$ on $\Delta(\hat{M})$ in the obvious manner. Consider the problem $P(\eta)$ as selection of $\tilde{m}(\theta|\eta), \tilde{u}_A(\theta,\eta)$ to maximize

$$E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta,\eta) \mid \eta]$$

subject to

$$\tilde{u}_A(\theta,\eta) \geq u_A(\theta,\eta)$$

for any $\theta \in \Theta(\eta)$,

$$\tilde{u}_A(\theta,\eta) \geq \tilde{u}_A(\theta^{'},\eta) + (\theta^{'} - \theta)\hat{q}(\tilde{m}(\theta^{'} \mid \eta))$$

for any $\theta, \theta^{'} \in \Theta(\eta)$. For $\tilde{u}_A(\theta,\eta)$ which satisfies constraints of the problem, we have

$$\int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\tilde{u}_A(\theta,\eta) - u_A(\theta,\eta)] d\Lambda(\theta \mid \eta) \geq 0.$$

Then

$$\begin{aligned} & E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta,\eta) \mid \eta] \\ \leq \;& E[\hat{X}(\tilde{m}(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}(\theta \mid \eta)) - \tilde{u}_A(\theta,\eta) \mid \eta] \\ + & \int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\tilde{u}_A(\theta,\eta) - u_A(\theta,\eta)] d\Lambda(\theta \mid \eta). \end{aligned}$$

Now consider the problem of maximizing the right hand side of this inequality, subject to the constraint that $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. Using the same steps in the proof of the necessity part, this can be expressed as a problem of selecting $\tilde{m}(\theta|\eta)$ to maximize the Lagrangean (13) subject to the constraint that $\hat{q}(\tilde{m}(\theta \mid \eta))$ is non-increasing in θ. Conditions (i)-(iii) imply that the right-hand-side is maximized at $\tilde{m}(\theta \mid \eta) = (\theta,\eta)$ and $\tilde{u}_A(\theta,\eta) = u_A(\theta,\eta)$. Since

$$\int_{[\underline{\theta}(\eta),\bar{\theta}(\eta)]} [\tilde{u}_A(\theta,\eta) - u_A(\theta,\eta)] d\Lambda(\theta \mid \eta) = 0$$

when $\tilde{u}_A(\theta,\eta) = u_A(\theta,\eta)$, this shows that the left hand side of the above inequality is also maximized at $\tilde{m}(\theta \mid \eta) = (\theta,\eta)$ and $\tilde{u}_A(\theta,\eta) = u_A(\theta,\eta)$. Hence $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$ solves $P(\eta)$. ■

Proof of Proposition 3:

Proof of Necessity

Suppose (u_A, u_S, q) is implementable in the weak collusion game. It is evident that it satisfies interim participation constraints of A and S. Here we show that it is also a WCP allocation. Suppose not. Then there exists $\eta \in \Pi$ such that $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$ does not solve the side-contracting problem $P(\eta)$. Suppose that $(\tilde{m}^*(\theta \mid \eta), \tilde{u}^*_A(\theta,\eta))$ is the solution of $P(\eta)$. Defining

$$\tilde{u}^*_S(\theta,\eta) \equiv \hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}^*_A(\theta,\eta),$$

we have

$$E[\tilde{u}^*_S(\theta,\eta) \mid \eta] > E[u_S(\theta,\eta) \mid \eta]$$

and

$$\tilde{u}^*_A(\theta,\eta) \geq u_A(\theta,\eta)$$

for any $\theta \in \Theta(\eta)$. Since (u_A, u_S, q) is implementable in the weak collusion game, there exists a grand contract GC and an associated WPBE(wc) which results in this allocation. From the property of WPBE(wc), there exists belief $p(\eta)$ and non-cooperative equilibrium $c(\eta)$ of GC based on the belief $p(\eta)$ such that A's payoff is not better than $u_A(\theta, \eta)$ for any $\theta \in \Theta(\eta)$.

For $\tilde{m}^*(\theta \mid \eta) \in \Delta(K \cup e)$, there exists $\tilde{m}^c(\theta, \eta) \in \Delta(M_A \times M_S)$ such that

$$(\hat{X}(\tilde{m}^*(\theta \mid \eta)), \hat{q}(\tilde{m}^*(\theta \mid \eta))) = (X_A(\tilde{m}^c(\theta, \eta)) + X_S(\tilde{m}^c(\theta, \eta)), q(\tilde{m}^c(\theta, \eta))).$$

Given GC and η, suppose that S offers the side-contract $SC^c(\eta)$ such that the report to P is selected according to $\tilde{m}^c(\theta', \eta)$ on the basis of A's report of $\theta' \in \Theta(\eta)$, associated with the transfer to A:

$$t_A^c(\theta', \eta) = \tilde{u}_A^*(\theta', \eta) - [X_A(\tilde{m}^c(\theta', \eta)) - \theta' q(\tilde{m}^c(\theta', \eta)))].$$

Now construct a Perfect Bayesian Equilibrium (PBE) in the game induced by GC and $SC^c(\eta)$, as follows. It is evident that if A accepts this side-contract, it is optimal for him to truthfully report $\theta \in \Theta(\eta)$, generating payoffs $\tilde{u}_A^*(\theta, \eta)$ and $\tilde{u}_S^*(\theta, \eta)$ for A and S respectively. If A rejects the side-contract, A and S play $c(\eta)$ based on the belief $p(\eta)$ specified above. Since $\tilde{u}_A^*(\theta, \eta) \geq u_A(\theta, \eta)$, all types of A participate in the side-contract, given this choice of non-cooperative equilibrium in the event that A rejects the side-contract. The argument shows that $(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta))$ is realized as a PBE outcome. Since S is better off without making any type of A worse off, it contradicts the fact that (u_A, u_S, q) is realized as the outcome of a WPBE(wc).

Proof of Sufficiency

Suppose that (u_A, u_S, q) is a WCP allocation satisfying interim participation constraints of A and S. We show that there exists a grand contract which realizes (u_A, u_S, q) as a WPBE(wc) outcome. Consider the following grand contract, corresponding to $T > 0$ chosen sufficiently large:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(m_A, m_S) = X_S(m_A, m_S) = q(m_A, m_S) = 0$$

for (m_A, m_S) such that either $m_A = e_A$ or $m_S = e_S$.

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$ for $\eta_A = \eta_S$ and $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (-T, 0)$ for $\eta_A \neq \eta_S$

- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$

WCP implies that $u_A(\theta, \eta) \geq u_A(\theta^{'}, \eta) + (\theta^{'} - \theta)q(\theta^{'}, \eta)$ for any $\theta, \theta^{'} \in \Theta(\eta)$. Then together with interim participation constraints of A and S, this grand contract has a non-cooperative truthful equilibrium $(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$ based on prior beliefs. Then there exists a WPBE where S offers the null side-contract on the equilibrium path for any $\eta \in \Pi$. In this WPBE, for any non-null side-contract, A's rejection always induces the truthful equilibrium based on prior beliefs. Then since (u_A, u_S, q) is WCP, S cannot benefit from any non-null side-contract. This equilibrium also satisfies the robustness criterion in WPBE(wc), since there is no room for S to achieve a higher payoff, while leaving a payoff of at least $u_A(\theta, \eta)$ to all types of A. Therefore (u_A, u_S, q) is a WPBE(wc) outcome, given GC. ■

Proof of Proposition 1:

At the first step, note that the optimal side contract problem for S in DS involves an outside option for A which is identically zero. This reduces to a standard problem of contracting with a single agent with adverse selection and an outside option of zero, where the principal has a prior distribution $F(\theta|\eta)$ over the agent's cost θ in state η. The CSC function equals $h(\theta|\eta)$, and the CVC function $z(\theta|\eta)$ reduces to $\hat{h}(\theta|\eta)$ obtained by applying the ironing rule to $h(\theta|\eta)$ and distribution $F(\theta|\eta)$.

Given this, P's contract with S in DS is effectively a contracting problem for P with a single supplier whose unit supply cost is $\hat{h}(\theta|\eta)$. P's prior over this supplier's cost is given by distribution function

$$G(h) \equiv \Pr((\theta, \eta) \mid \hat{h}(\theta \mid \eta) \leq h)$$

for $h \geq \underline{\theta}$ and $G(h) = 0$ for $h < \underline{\theta}$. Let $G(h \mid \eta)$ denote the cumulative distribution function of $h = \hat{h}(\theta \mid \eta)$ conditional on η:

$$G(h \mid \eta) \equiv \Pr(\theta \mid \hat{h}(\theta \mid \eta) \leq h, \eta)$$

for $h \geq \hat{h}(\underline{\theta}(\eta) \mid \eta)(= \underline{\theta}(\eta))$ and $G(h \mid \eta) = 0$ for $h < \underline{\theta}(\eta)$. Then $G(h) = \Sigma_{\eta \in \Pi} p(\eta) G(h \mid \eta)$. Since $\hat{h}(\theta \mid \eta)$ is continuous on $\Theta(\eta)$, $G(h \mid \eta)$ is strictly increasing in h on $[\underline{\theta}(\eta), \hat{h}(\bar{\theta}(\eta) \mid \eta)]$. However, $G(h \mid \eta)$ may fail to be left-continuous.

Hence P's problem in DS reduces to

$$\max E_h[V(q(h)) - X(h)]$$

subject to

$$X(h) - hq(h) \geq X(h^{'}) - hq(h^{'})$$

for any $h, h^{'} \in [\underline{\theta}, \bar{h}]$ and

$$X(h) - hq(h) \geq 0$$

for any $h \in [\underline{\theta}, \bar{h}]$ where the distribution function of h is $G(h)$ and $\bar{h} \equiv \max_{\eta \in \Pi} \hat{h}(\bar{\theta}(\eta) \mid \eta)$. The corresponding problem in NS is

$$\max E_\theta[V(q(\theta)) - X(\theta)]$$

subject to

$$X(\theta) - \theta q(\theta) \geq X(\theta^{'}) - \theta q(\theta^{'})$$

for any $\theta, \theta^{'} \in \Theta$ and

$$X(\theta) - \theta q(\theta) \geq 0$$

for any $\theta \in \Theta$. The two problems differ only in the underlying cost distributions of P: $G(h)$ in the case of DS and $F(\theta)$ in the case of NS. Since $\theta < \hat{h}(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$,

$$G(h \mid \eta) \equiv \Pr(\theta \mid \hat{h}(\theta \mid \eta) \leq h, \eta) < \Pr(\theta \mid \theta \leq h, \eta) = F(h \mid \eta)$$

for $h \in (\underline{\theta}(\eta), \hat{h}(\bar{\theta}(\eta) \mid \eta))$, implying

$$G(h) = \Sigma_{\eta \in \Pi} p(\eta) G(h \mid \eta) < \Sigma_{\eta \in \Pi} p(\eta) F(h \mid \eta) = F(h)$$

for any $h \in (\underline{\theta}, \bar{h})$. Therefore the distribution of h in DS (strictly) dominates that of θ in NS in the first order stochastic sense.

It remains to show that this implies that P must earn a lower profit in DS. We prove the following general statement. Consider two contracting problems with a single supplier which differ only in regard to the cost distributions G_1 and G_2, where $G_1(h) < G_2(h)$ for any $h \in (\underline{h}, \bar{h})$. Let the maximized profit of P with distribution G be denoted $W(G)$. We will show $W(G_1) < W(G_2)$.

Let $q_1(h)$ denote the optimal solution of the problem based on $G_1(h)$.

(i) First we show that $V^{'}(q_1(h)) < h$ does not hold for any h. Suppose otherwise that there exists some interval over which $V^{'}(q_1(h)) < h$. Then we can replace the portion of $q_1(h)$ with $V^{'}(q_1(h)) < h$ by $q^*(h)$ with $V^{'}(q^*(h)) = h$, without violating the constraint that $q(h)$ is non-increasing. It raises the value of the objective function, since $V(q_1(h)) - hq_1(h) < V(q_1^*(h)) - hq_1^*(h)$ for h where $q_1(h)$ is replaced by $q^*(h)$, and $\int_h^{\bar{h}} q(y)dy$ decreases with this replacement. This is a contradiction.

(ii) Next we show that for any $h^{'} \in [\underline{h}, \bar{h})$, there exists a subinterval of $[h^{'}, \bar{h})$ over which $V^{'}(q_1(h)) > h$. Otherwise, there exists $h^{'} \in [\underline{h}, \bar{h})$ such that $q_1(h) = q^*(h)$ almost everywhere on $[h^{'}, \bar{h})$. Then for any $h \in [h^{'}, \bar{h})$,

$$V(q^*(h)) - hq^*(h) - \int_h^{\bar{h}} q^*(y)dy = V(q^*(\bar{h})) - \bar{h}q^*(\bar{h}),$$

since $V(q^*(h)) - hq^*(h) = \int_h^{\bar{h}} q^*(y)dy + V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})$ (which follows from the Envelope Theorem: $d[V(q^*(h)) - hq^*(h)]/dh = -q^*(h)$). Then

$$\begin{aligned} W(G_1) &= (1 - G_1(h^{'}))[V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})] \\ &+ \ G_1(h^{'})E[V(q_1(h)) - hq_1(h) - \int_h^{h^{'}} q_1(y)dy \mid h \leq h^{'}] - G_1(h^{'}) \int_{h^{'}}^{\bar{h}} q^*(y)dy. \end{aligned}$$

Now consider output schedule $q(h)$ such that $q(h) = q_1(h)$ for $h \leq h^{'}$ and $q(h) = q^*(\bar{h})$ for $h > h^{'}$. It is evident that $q(h)$ is non-increasing in h and generates a higher value of the objective function, since $\int_{h^{'}}^{\bar{h}} q^*(y)dy > \int_{h^{'}}^{\bar{h}} q^*(\bar{h})dy$. This is a contradiction.

(iii) We show there does not exist q such that $q_1(h) = q$ almost everywhere. Otherwise, $q_1(h) = q$ almost everywhere for some q. Then

$$V(q) - hq - \int_h^{\bar{h}} qdy = V(q) - \bar{h}q,$$

which is not larger than $V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})$ which equals $\max_{\tilde{q}}[V(\tilde{q}) - \bar{h}\tilde{q}]$. We can show that the value of the objective function is increased by choosing the following output schedule $\tilde{q}(h)$:

$$\tilde{q}(h) = \begin{cases} q^*(\bar{h}) & h \in [h^*, \bar{h}] \\ q^*(\bar{h}) + \epsilon & h \in [\underline{h}, h^*] \end{cases}$$

where h^* is any element of $(\underline{h}, \bar{h})$, and $\epsilon > 0$ is chosen so that $V(q^*(\bar{h})+\epsilon) - V(q^*(\bar{h})) > \epsilon h^*$. This is possible since $\lim_{\epsilon \to 0} \frac{V(q^*(\bar{h})+\epsilon)-V(q^*(\bar{h}))}{\epsilon} = V'(q^*(\bar{h})) = \bar{h}$, implying existence of $\epsilon > 0$ such that $V(q^*(\bar{h}) + \epsilon) - V(q^*(\bar{h})) > \epsilon h^*$ for any $h^* < \bar{h}$.

Then we obtain a contradiction, since

$$\begin{aligned}
& V(q^*(\bar{h})) - \bar{h}q^*(\bar{h}) \\
< \;& (1 - G_1(h^*))[V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})] + G_1(h^*)[V(q^*(\bar{h}) + \epsilon) - \bar{h}q^*(\bar{h}) - \epsilon h^*] \\
= \;& \int_{\underline{h}}^{\bar{h}} [V(\tilde{q}(h)) - h\tilde{q}(h) - \int_h^{\bar{h}} \tilde{q}(y)dy]dG_1(h).
\end{aligned}$$

(iv) Define

$$\Phi(h) \equiv V(q_1(h)) - hq_1(h) - \int_h^{\bar{h}} q_1(y)dy.$$

We claim that $\Phi(h)$ is left-continuous and bounded. First we show that $q_1(h)$ is left-continuous. Otherwise, there exists $h' \in (\underline{h}, \bar{h})$ such that $q_1(h'-) > q_1(h')$. Now consider $\tilde{q}_1(h)$ (which is left-continuous at h') such that $\tilde{q}_1(h') = q_1(h'-)$ and $\tilde{q}_1(h) = q_1(h)$ for any $h \neq h'$. Defining $\tilde{\Phi}(h) \equiv V(\tilde{q}_1(h)) - h\tilde{q}_1(h) - \int_h^{\bar{h}} \tilde{q}_1(y)dy$, observe that $\tilde{\Phi}(h) = \Phi(h)$ for $h \neq h'$ and $\tilde{\Phi}(h) > \Phi(h)$ when $h = h'$. Then

$$\begin{aligned}
& \int_{[\underline{h},\bar{h}]} \tilde{\Phi}(h)dG(h) = \int_{[\underline{h},\bar{h}]\backslash h'} \tilde{\Phi}(h)dG(h) + \tilde{\Phi}(h')[G(h'+) - G(h'-)] \\
\geq \;& \int_{[\underline{h},\bar{h}]\backslash h'} \tilde{\Phi}(h)dG(h) + \Phi(h')[G(h'+) - G(h'-)] = \int_{[\underline{h},\bar{h}]} \Phi(h)dG(h)
\end{aligned}$$

with strict inequality if $G(h)$ is discontinuous at $h = h'$. This is a contradiction. This implies in turn that $\Phi(h)$ is also left-continuous. Moreover, $\Phi(h)$ is bounded, since

$$\Phi(h) \leq \Phi(\underline{h}) \leq V(q_1(\underline{h})) - \underline{h}q_1(\underline{h}) \leq V(q^*(\underline{h})) - \underline{h}q^*(\underline{h}) < \infty$$

because of $\underline{h} > 0$, and

$$\Phi(h) \geq \Phi(\bar{h}) = V(q_1(\bar{h})) - \bar{h}q_1(\bar{h}) \geq 0$$

because of $V'(q) > V'(q_1(\bar{h})) \geq \bar{h}$ for $q < q_1(\bar{h})$ and $V(0) = 0$.

(v) We claim that $\Phi(h)$ is non-increasing in h and is not constant on $(\underline{h}, \bar{h})$. To show the former, note that for any h, we have

$$\begin{aligned}
& \lim_{\epsilon\to 0+}\frac{\Phi(h+\epsilon)-\Phi(h)}{\epsilon} \\
= & \lim_{\epsilon\to 0+}(1/\epsilon)[V(q_1(h+\epsilon))-(h+\epsilon)q_1(h+\epsilon)-\int_{h+\epsilon}^{\bar{h}}q_1(y)dy \\
- & [V(q_1(h))-hq_1(h)-\int_{h}^{\bar{h}}q_1(y)dy]] \\
= & [V^{'}(\hat{q}(h))-h]\lim_{\epsilon\to 0+}\frac{q_1(h+\epsilon)-q_1(h)}{\epsilon} \\
- & q_1(h+)+\lim_{\epsilon\to 0+}(1/\epsilon)\int_{h}^{h+\epsilon}q_1(y)dy \\
= & [V^{'}(\hat{q}(h))-h]\lim_{\epsilon\to 0+}\frac{q_1(h+\epsilon)-q_1(h)}{\epsilon}
\end{aligned}$$

for some $\hat{q}(h)\in[q_1(h+),q_1(h)]$. This is non-positive since $V^{'}(\hat{q}(h))\geq V^{'}(q_1(h))\geq h$ and $\lim_{\epsilon\to 0+}\frac{q_1(h+\epsilon)-q_1(h)}{\epsilon}\leq 0$. Because of left-continuity of $\Phi(h)$, it implies that $\Phi(h)$ is non-increasing in h.

Next we show that $\Phi(h)$ is not constant on $(\underline{h},\bar{h})$. First we consider the case that there exists $h\in(\underline{h},\bar{h})$ such that $q_1(h+)<q_1(h-)$. Then

$$\begin{aligned}
& \Phi(h+) \\
= & V(q_1(h+))-hq_1(h+)-\int_{h}^{\bar{h}}q_1(y)dy] \\
< & V(q_1(h-))-hq_1(h-)-\int_{h}^{\bar{h}}q_1(y)dy=\Phi(h-)
\end{aligned}$$

The inequality follows from $V^{'}(q_1(h+))>V^{'}(q_1(h-))\geq V^{'}(q^*(h))=h$. Therefore $\Phi(h)$ decreases discontinuously at h, implying that $\Phi(h)$ is not constant on $(\underline{h},\bar{h})$. Second we consider the case that $q(h)$ is continuous on $(\underline{h},\bar{h})$. Then from (ii) and (iii) above, there exists an interval (h^-,h^+) with the positive measure such that $q_1(h)$ is strictly decreasing and $V^{'}(q_1(h))>h$ on (h^-,h^+). $\Phi(h)$ is continuous and almost everywhere differentiable (because of monotonicity of $q_1(h)$). At any point of differentiability,

$$\Phi^{'}(h)=[V^{'}(q_1(h))-h]q_1^{'}(h).$$

This is negative almost everywhere on (h^-,h^+). Hence $\Phi(h)$ is strictly decreasing in h on (h^-,h^+).

(vi) Now consider the contracting problem with cost distribution $G_2(h)$. Since $q_1(h)$ is non-increasing in h, it is feasible for P to select this output schedule when the cost distribution is G_2. Hence $W(G_2) \geq \int_{\underline{h}}^{\bar{h}} \Phi(h)dG_2(h)$. Therefore if $\int_{\underline{h}}^{\bar{h}} \Phi(h)dG_2(h) > \int_{\underline{h}}^{\bar{h}} \Phi(h)dG_1(h) = W(G_1)$, it follows that $W(G_2) > W(G_1)$. Since $G_1(h)$ is right-continuous and $\Phi(h)$ is left-continuous and bounded, we can integrate by parts:

$$\int_{\underline{h}}^{\bar{h}} \Phi(h)dG_1(h) + \int_{\underline{h}}^{\bar{h}} G_1(h)d\Phi(h) = \Phi(\bar{h})G_1(\bar{h}) - \Phi(\underline{h})G_1(\underline{h}) = \Phi(\bar{h}).$$

Similarly for $G_2(h)$,

$$\int_{\underline{h}}^{\bar{h}} \Phi(h)dG_2(h) + \int_{\underline{h}}^{\bar{h}} G_2(h)d\Phi(h) = \Phi(\bar{h})G_2(\bar{h}) - \Phi(\underline{h})G_2(\underline{h}) = \Phi(\bar{h}).$$

Hence

$$\int_{\underline{h}}^{\bar{h}} \Phi(h)dG_2(h) - \int_{\underline{h}}^{\bar{h}} \Phi(h)dG_1(h) = \int_{\underline{h}}^{\bar{h}} [G_1(h) - G_2(h)]d\Phi(h).$$

By (iv) and $G_2(h) > G_1(h)$ for $h \in (\underline{h}, \bar{h})$, this is positive. ■

Proof of Proposition 4:

Step 1: For any $\eta \in \Pi$ and any closed interval $[\theta', \theta''] \subset \Theta(\eta)$ such that $\underline{\theta}(\eta) < \theta' < \theta'' < \bar{\theta}(\eta)$, there exists $\delta > 0$ such that $z(\cdot) \in Z(\eta)$ for any $z(\cdot)$ satisfying the following properties:

(i) $z(\theta)$ is increasing and differentiable with $|z(\theta) - \theta| < \delta$ and $|z'(\theta) - 1| < \delta$ for any $\theta \in \Theta(\eta)$

(ii) $z(\theta) = \theta$ for any $\theta \notin [\theta', \theta'']$.

Proof of Step 1

For arbitrary $\eta \in \Pi$ and arbitrary closed interval $[\theta', \theta''] \subset \Theta(\eta)$ such that $\underline{\theta}(\eta) < \theta' < \theta'' < \bar{\theta}(\eta)$, we choose ϵ_1 and ϵ_2 such that

$$\epsilon_1 \equiv \min_{\theta \in [\theta', \theta'']} f(\theta \mid \eta)$$

and

$$\epsilon_2 \equiv \max_{\theta \in [\theta', \theta'']} |f'(\theta \mid \eta)|.$$

From our assumptions that $f(\theta \mid \eta)$ is continuously differentiable and positive on $\Theta(\eta)$, $\epsilon_1 > 0$, and ϵ_2 is positive and bounded above. We choose $\delta > 0$ such that

$$\delta \in (0, \frac{\epsilon_1}{\epsilon_1 + \epsilon_2}).$$

For this δ, it is obvious that there exists $z(\theta)$ which satisfies conditions (i) and (ii) of the statement. Define

$$\Lambda(\theta \mid \eta) \equiv (\theta - z(\theta))f(\theta \mid \eta) + F(\theta \mid \eta).$$

Since $z(\theta)$ is differentiable on $\Theta(\eta)$, $\Lambda(\theta \mid \eta)$ is also so. It is equal to $\Lambda(\theta \mid \eta) = F(\theta \mid \eta)$ on $\theta \notin [\theta^{'}, \theta^{''}]$. For $\theta \in [\theta^{'}, \theta^{''}]$,

$$\begin{aligned} \frac{\partial \Lambda(\theta \mid \eta)}{\partial \theta} &= (2 - z^{'}(\theta))f(\theta \mid \eta) + (\theta - z(\theta))f^{'}(\theta \mid \eta) > (1-\delta)f(\theta \mid \eta) - \delta |f^{'}(\theta \mid \eta)| \\ &\geq (1-\delta)\epsilon_1 - \delta\epsilon_2. \end{aligned}$$

This is positive by the definition of $(\epsilon_1, \epsilon_2, \delta)$. Then $\Lambda(\theta \mid \eta)$ is increasing in θ on $\Theta(\eta)$ with $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ and $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$. Since $z(\theta)$ is increasing in θ by the definition, it is preserved even by ironing rule. Therefore $z(\cdot) \in Z(\eta)$. ■

Step 2: There exist $\eta \in \Pi$ and an interval of θ with positive measure such that $\frac{F(\theta|\eta)}{f(\theta|\eta)} / \frac{F(\theta)}{f(\theta)}$ is increasing in θ.

The proof of Step 2

Define

$$A(\theta \mid \eta) \equiv \frac{F(\theta \mid \eta)}{f(\theta \mid \eta)} / \frac{F(\theta)}{f(\theta)} \equiv \frac{\int_{\underline{\theta}(\eta)}^{\theta} f(y)a(\eta|y)dy}{a(\eta|\theta)F(\theta)}.$$

If the result is false, $A(\theta \mid \eta)$ is non-increasing in $\theta \in (\underline{\theta}(\eta), \bar{\theta}(\eta))$ for all η. Then

$$\begin{aligned} \partial A(\theta \mid \eta)/\partial\theta = &\frac{1}{F(\theta)^2 a(\eta \mid \theta)^2}[F(\theta)a(\eta \mid \theta)^2 f(\theta) \\ &- \int_{\underline{\theta}(\eta)}^{\theta} f(y)a(\eta \mid y)dy\{F(\theta)\partial a(\eta \mid \theta)/\partial\theta + f(\theta)a(\eta \mid \theta)\}] \leq 0 \end{aligned}$$

holds for $\theta \in (\underline{\theta}(\eta), \bar{\theta}(\eta))$. Equivalently

$$\partial a(\eta \mid \theta)/\partial\theta \geq \frac{f(\theta)}{F(\theta)}[1/A(\theta \mid \eta) - 1]a(\eta \mid \theta).$$

Define $\Pi(\theta) \equiv \{\eta \in \Pi \mid \theta \in (\underline{\theta}(\eta), \bar{\theta}(\eta))\}$. By $\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta) = 1$, $\Sigma_{\eta\in\Pi(\theta)} \partial a(\eta \mid \theta)/\partial\theta = 0$. This implies that

$$0 = \Sigma_{\eta\in\Pi(\theta)} \partial a(\eta \mid \theta)/\partial\theta \geq \frac{f(\theta)}{F(\theta)}[\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)/A(\theta \mid \eta) - 1],$$

or $\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)/A(\theta \mid \eta) \leq 1$ holds any for $\theta \in (\underline{\theta}, \bar{\theta})$. Since $1/A$ is convex in A and $\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)A(\theta \mid \eta) = 1$,

$$\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)/A(\theta \mid \eta) \geq 1/[\Sigma_{\eta\in\Pi(\theta)} a(\eta \mid \theta)A(\theta \mid \eta)] = 1$$

with strict inequality if there exists $\eta \in \Pi(\theta)$ such that $A(\theta \mid \eta) \neq 1$. This means that $A(\theta \mid \eta) = 1$ must hold for any $\eta \in \Pi(\theta)$ and any $\theta \in \Theta$. Then $h(\theta \mid \eta) = H(\theta)$ for any $(\theta, \eta) \in K$. This is a contradiction, since η is informative about θ. ■

Step 3:

From Step 2, we can choose $\eta^* \in \Pi$ and a closed interval $[\theta', \theta''] \subset \Theta(\eta^*)$ such that $\underline{\theta}(\eta^*) < \theta' < \theta'' < \bar{\theta}(\eta^*)$ and $A(\theta \mid \eta^*) \equiv \frac{F(\theta|\eta^*)}{f(\theta|\eta^*)}/\frac{F(\theta)}{f(\theta)}$ is increasing in θ on $[\theta', \theta'']$. According to the procedure in Step 1, we select $\delta > 0$ for η^* and $[\theta', \theta'']$. Then we also choose $\lambda > 0$, closed intervals $\Theta^L \subset [\theta', \theta'']$ and $\Theta^H \subset [\theta', \theta'']$,

$$\lambda < \frac{F(\theta)}{f(\theta)}/\frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} \text{ for } \theta \in \Theta^L \equiv [\underline{\theta}^L, \bar{\theta}^L] \subset [\theta', \theta'']$$

$$\lambda > \frac{F(\theta)}{f(\theta)}/\frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} \text{ for } \theta \in \Theta^H \equiv [\underline{\theta}^H, \bar{\theta}^H] \subset [\theta', \theta'']$$

with $\bar{\theta}^L < \underline{\theta}^H$. These conditions are equivalent to

$$H(\theta) - (1-\lambda)\theta - \lambda h(\theta \mid \eta^*) > 0 \text{ for } \theta \in \Theta^L$$

and

$$H(\theta) - (1-\lambda)\theta - \lambda h(\theta \mid \eta^*) < 0 \text{ for } \theta \in \Theta^H.$$

Step 4: Construction of $z(\theta \mid \eta)$

Now let us construct $z(\theta \mid \eta)$ which satisfies the following conditions.

(A) For $\eta \neq \eta^*$, $z(\theta \mid \eta) = \theta$ for any $\theta \in \Theta(\eta)$.

(B) For η^*, $z(\theta \mid \eta^*)$ satisfies

(i) $z(\theta \mid \eta^*)$ is increasing and differentiable with $|z(\theta \mid \eta^*) - \theta| < \delta$ and $|z^{'}(\theta \mid \eta^*) - 1| < \delta$ for any $\theta \in \Theta(\eta^*)$

(ii) $z(\theta \mid \eta^*) = \theta$ for any $\theta \notin \Theta^H \cup \Theta^L$

(iii) For $\theta \in \Theta^L$, $z(\theta \mid \eta^*)$ satisfies (a) $z(\theta \mid \eta^*) \leq \theta$ with strict inequality for some subinterval of Θ^L of positive measure, and (b) $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^*) > 0$ for any $z \in [z(\theta \mid \eta^*), \theta]$.

(iv) For $\theta \in \Theta^H$, $z(\theta \mid \eta^*)$ satisfies (a) $z(\theta \mid \eta^*) \geq \theta$ with strict inequality for some some subinterval of Θ^H of positive measure, (b) $z(\theta \mid \eta^*) < h(\theta \mid \eta^*)$ and (c) $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^*) < 0$ for any $z \in [\theta, z(\theta \mid \eta^*)]$.

(v) $E[(z(\theta \mid \eta^*) - h(\theta \mid \eta^*))q^{NS}(z(\theta \mid \eta^*)) + \int_{z(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] = 0$.

We now argue there exists $z^*(\theta \mid \eta^*)$ which satisfies (B(i)-(v)). Step 3 guarantees that we can select $z(\theta \mid \eta^*)$ which satisfies (B(i)-(iv)). Since

$$(z - h(\theta \mid \eta^*))q^{NS}(z) + \int_z^{\bar{\theta}(\eta^*)} q^{NS}(y)dy$$

is increasing in z for $z < h(\theta \mid \eta^*)$, and

$$E[(\theta - h(\theta \mid \eta^*))q^{NS}(\theta) + \int_\theta^{\bar{\theta}(\eta^*)} q^{NS}(y)dy \mid \eta^*] = 0,$$

the choice of $z(\theta \mid \eta^*) \leq \theta$ on Θ_L (or $z(\theta \mid \eta^*) \geq \theta$ on Θ_H) reduces (or raises)

$$E[(z(\theta \mid \eta^*) - h(\theta \mid \eta^*))q^{NS}(z(\theta \mid \eta^*)) + \int_{z(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*]$$

away from zero. For any pair of parameters α_H, α_L lying in $[0, 1]$, define a function $z_{\alpha_L,\alpha_H}(\theta|\eta^*)$ which equals $(1-\alpha_L)z(\theta|\eta^*) + \alpha_L\theta$ on Θ_L, equals $(1-\alpha_H)z(\theta|\eta^*) + \alpha_H\theta$ on Θ_H and equals θ elsewhere. It is easily checked that any such function also satisfies conditions (B(i)–(iv)). Define

$$Q(\alpha_L, \alpha_U) \equiv E[(z_{\alpha_L,\alpha_H}(\theta \mid \eta^*) - h(\theta \mid \eta^*))q^{NS}(z_{\alpha_L,\alpha_H}(\theta \mid \eta^*)) + \int_{z_{\alpha_L,\alpha_H}(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*].$$

Then Q is continuously differentiable, strictly increasing in α_L and strictly decreasing in α_H. By (B(v)), $Q(1, 1) = 0$. The Implicit Function Theorem ensures existence of α_L^*, α_H^* both smaller than 1 such that $Q(\alpha_L^*, \alpha_H^*) = 0$. Hence the function $z_{\alpha_L^*,\alpha_H^*}(\theta|\eta^*)$ satisfies (B(i)-(v)).

Step 5

By Step 1, $z(\cdot \mid \eta)$ constructed in Step 4 is in $Z(\eta)$ for any $\eta \in \Pi$. Consider the following allocation (u_A, u_S, q):

$$q(\theta,\eta) = q^{NS}(z(\theta \mid \eta))$$

$$u_A(\theta,\eta) = \int_\theta^{\bar{\theta}} q^{NS}(z(y \mid \eta))dy$$

$$u_S(\theta,\eta) = X^{NS}(z(\theta \mid \eta)) - \theta q^{NS}(z(\theta \mid \eta)) - \int_\theta^{\bar{\theta}(\eta)} q^{NS}(z(y \mid \eta))dy - \int_{\bar{\theta}(\eta)}^{\bar{\theta}} q^{NS}(y)dy.$$

where

$$X^{NS}(z(\theta \mid \eta)) \equiv z(\theta \mid \eta)q^{NS}(z(\theta \mid \eta)) + \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(z)dz.$$

The construction of $z(\theta \mid \eta)$ implies that $z(\bar{\theta}(\eta) \mid \eta) \leq \bar{\theta}$ for any $\eta \in \Pi$. Hence

$$X^{NS}(z(\theta \mid \eta)) - z(\theta \mid \eta)q^{NS}(z(\theta \mid \eta)) \geq 0$$

for any $(\theta, \eta) \in K$ and

$$E[u_S(\theta,\eta) \mid \eta] = 0$$

from (A) and (B(v)). Then (u_A, u_S, q) is a WCP allocation satisfying interim PCs. Now we show that this allocation generates a higher payoff to P than the optimal allocation in NS. P's resulting expected payoff conditional on η^* (maintaining the expected payoff conditional on $\eta \neq \eta^*$ unchanged) is:

$$E[V(q^{NS}(z(\theta \mid \eta^*))) - z(\theta \mid \eta^*)q^{NS}(z(\theta \mid \eta^*)) - \int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*].$$

With $E[u_S(\theta, \eta^*) \mid \eta^*] = 0$, this is equal to

$$\begin{aligned}
& E[V(q^{NS}(z(\theta \mid \eta^*))) - z(\theta \mid \eta^*)q^{NS}(z(\theta \mid \eta^*)) - \int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
+ \quad & \lambda E[(z(\theta \mid \eta^*) - h(\theta \mid \eta^*))q^{NS}(z(\theta \mid \eta^*)) + \int_{z(\theta|\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] \\
= \quad & E[V(q^{NS}(z(\theta \mid \eta^*))) - [(1-\lambda)z(\theta \mid \eta^*) + \lambda h(\theta \mid \eta^*)]q^{NS}(z(\theta \mid \eta^*)) \\
- \quad & (1-\lambda)\int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \mid \eta^*] \\
- \quad & \lambda \int_{\bar{\theta}(\eta^*)}^{\bar{\theta}} q^{NS}(z)dz
\end{aligned}$$

On the other hand,

$$E[(\theta - h(\theta \mid \eta^*))q^{NS}(\theta) + \int_{\theta}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] = 0.$$

P's expected payoff conditional on η^* in the optimal allocation in NS is:

$$\begin{aligned}
& E[V(q^{NS}(\theta)) - \theta q^{NS}(\theta) - \int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
= \ & E[V(q^{NS}(\theta)) - \theta q^{NS}(\theta) - \int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
+ \ & \lambda E[(\theta - h(\theta \mid \eta^*))q^{NS}(\theta) + \int_{\theta}^{\bar{\theta}(\eta^*)} q^{NS}(z)dz \mid \eta^*] \\
= \ & E[V(q^{NS}(\theta)) - [(1-\lambda)\theta + \lambda h(\theta \mid \eta)]q^{NS}(\theta) - (1-\lambda)\int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
- \ & \lambda \int_{\bar{\theta}(\eta^*)}^{\bar{\theta}} q^{NS}(z)dz
\end{aligned}$$

The difference between two payoffs is

$$\begin{aligned}
& E[V(q^{NS}(z(\theta \mid \eta^*))) - [(1-\lambda)z(\theta \mid \eta^*) + \lambda h(\theta \mid \eta^*)]q^{NS}(z(\theta \mid \eta^*)) \\
- \ & (1-\lambda)\int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
- \ & E[V(q^{NS}(\theta)) - [(1-\lambda)\theta + \lambda h(\theta \mid \eta^*)]q^{NS}(\theta) - (1-\lambda)\int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
= \ & E[\int_{\theta}^{z(\theta|\eta^*)} [V'(q^{NS}(z)) - \{(1-\lambda)z + \lambda h(\theta \mid \eta^*)\}]q^{NS'}(z)dz \mid \eta^*] \\
= \ & E[\int_{\theta}^{z(\theta|\eta^*)} [H(z) - \{(1-\lambda)z + \lambda h(\theta \mid \eta^*)\}]q^{NS'}(z)dz \mid \eta^*].
\end{aligned}$$

The second equality uses $V'(q^{NS}(z)) = H(z)$. From the construction of $z(\theta \mid \eta^*)$ in Step 4 and $q^{NS'}(z) < 0$, this is positive. We have thus found an implementable allocation generating a higher payoff to P in CS compared to the optimal allocation in NS. ■

Proof of Proposition 5:

Since $f(\theta \mid \eta^*)$ is decreasing in θ, $h(\theta \mid \eta^*)$ is increasing in θ, implying $h(\theta \mid \eta^*) = \hat{h}(\theta \mid \eta^*)$. Since $\frac{f(\theta|\eta^*)}{f(\theta|\eta)}$ is strictly decreasing in θ for any $\eta \neq \eta^*$, $\frac{f(\theta'|\eta^*)}{f(\theta|\eta^*)} > \frac{f(\theta'|\eta)}{f(\theta|\eta)}$ for $\theta > \theta'$. $\Theta(\eta) = \Theta(\eta^*) = \Theta$ then implies

$$\frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} = \int_{\underline{\theta}}^{\theta} \frac{f(\theta' \mid \eta^*)}{f(\theta \mid \eta^*)} d\theta' > \int_{\underline{\theta}}^{\theta} \frac{f(\theta' \mid \eta)}{f(\theta \mid \eta)} d\theta' = \frac{F(\theta \mid \eta)}{f(\theta \mid \eta)}.$$

Hence $h(\theta \mid \eta^*) > h(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta}]$ and $h(\underline{\theta} \mid \eta^*) = h(\underline{\theta} \mid \eta) = \underline{\theta}$. The ironing procedure then ensures that $\hat{h}(\theta \mid \eta^*) > \hat{h}(\theta \mid \eta)$ for any $\theta > \underline{\theta}$ and any $\eta \neq \eta^*$. Thus $\hat{h}(\bar{\theta}|\eta^*) > \hat{h}(\bar{\theta}|\eta)$ while $\hat{h}(\underline{\theta}|\eta^*) = \hat{h}(\underline{\theta}|\eta) = \underline{\theta}$ for $\eta \neq \eta^*$, i.e., the range of $\hat{h}$ conditional on η^* includes the range of $\hat{h}$ conditional on η. Since $h(\theta \mid \eta^*) = \hat{h}(\theta \mid \eta^*)$ is strictly increasing and continuously differentiable, $q^*(\hat{h}(\theta \mid \eta^*))$ is also continuously differentiable and strictly decreasing in θ.

Suppose the result is false, and the second best allocation

$$(u_A^{SB}(\theta, \eta), u_S^{SB}(\theta, \eta), q^{SB}(\theta, \eta))$$

is implementable with weak collusion. Then Proposition 2 implies existence of $\pi(\cdot \mid \eta) \in Y(\eta)$ such that for any $(\theta, \eta), (\theta^{'}, \eta^{'})$,

$$q^{SB}(\theta, \eta) = q^*(\hat{h}(\theta \mid \eta))$$

$$X^{SB}(\theta, \eta) - z(\theta \mid \eta) q^{SB}(\theta, \eta) \geq 0$$

$$X^{SB}(\theta, \eta) - z(\theta \mid \eta) q^{SB}(\theta, \eta) \geq X^{SB}(\theta^{'}, \eta^{'}) - z(\theta \mid \eta) q^{SB}(\theta^{'}, \eta^{'})$$

where $z(\theta \mid \eta) \equiv z(\theta, \pi(\theta \mid \eta), \eta)$ and

$$X^{SB}(\theta, \eta) \equiv u_A^{SB}(\theta, \eta) + u_S^{SB}(\theta, \eta) + \theta q^{SB}(\theta, \eta).$$

Step 1: $z(\theta \mid \eta) \in [z(\underline{\theta} \mid \eta^*), z(\bar{\theta} \mid \eta^*)]$ holds for any (θ, η).

The proof is as follows. Since $\hat{h}(\theta \mid \eta) < \hat{h}(\theta \mid \eta^*)$ for any $\theta > \underline{\theta}$ and $\eta \neq \eta^*$,

$$q^{SB}(\theta, \eta^*) = q^*(\hat{h}(\theta \mid \eta^*)) < q^*(\hat{h}(\theta \mid \eta)) = q^{SB}(\theta, \eta).$$

Then $z(\theta \mid \eta^*) \geq z(\theta \mid \eta)$ follows from the coalitional incentive constraints.

If on the other hand $z(\underline{\theta}|\eta) < z(\underline{\theta}|\eta^*)$, there exists a non-degenerate interval T of θ for which $z(\theta|\eta) \in (z(\underline{\theta}|\eta), z(\underline{\theta}|\eta^*))$. The second-best output in either state $(\underline{\theta}, \eta)$ or $(\underline{\theta}, \eta^*)$ is the first-best level $q^*(\underline{\theta})$ corresponding to cost $\underline{\theta}$. The coalitional incentive constraints imply output must be constant over T given η, so must equal the first-best $q^*(\underline{\theta})$ corresponding to cost $\underline{\theta}$. But $\hat{h}(\theta, \eta) \geq \theta$ for every $\theta \in T$, implying $q^{SB}(\theta, \eta) = q^*(\hat{h}(\theta, \eta)) \leq q^*(\theta) < q^*(\underline{\theta})$, and we obtain a contradiction.

In what follows, we denote $[z(\underline{\theta} \mid \eta^*), z(\bar{\theta} \mid \eta^*)]$ by $[\underline{z}, \bar{z}]$.

Step 2:

Now we claim that there exists $\phi(\cdot) : [\underline{h}, \bar{h}] \to [\underline{z}, \bar{z}]$ which satisfies

(i) $z(\theta \mid \eta) = \phi(\hat{h}(\theta \mid \eta))$.

(ii) $\phi(h)$ is continuous, and non-decreasing in h.

(iii) $h - \phi(h)$ is non-negative and increasing in h.

First we show that for any (θ, η) and $(\theta^{'}, \eta^{'})$ such that $\hat{h}(\theta \mid \eta) = \hat{h}(\theta^{'} \mid \eta^{'})$, $z(\theta \mid \eta) = z(\theta^{'} \mid \eta^{'})$. Otherwise, there exists $(\theta^{'}, \eta^{'})$ and $(\theta^{''}, \eta^{''})$ such that $\hat{h}(\theta^{'} \mid \eta^{'}) = \hat{h}(\theta^{''} \mid \eta^{''})$ and $z(\theta^{'} \mid \eta^{'}) \neq z(\theta^{''} \mid \eta^{''})$. Suppose $z(\theta^{'} \mid \eta^{'}) < z(\theta^{''} \mid \eta^{''})$ without loss of generality. By Step 1 and the continuity of $z(\theta \mid \eta^*)$, there exists θ_1 and θ_2 ($\theta_1 < \theta_2$) such that

$$z(\theta_1 \mid \eta^*) = z(\theta^{'} \mid \eta^{'}) < z(\theta^{''} \mid \eta^{''}) = z(\theta_2 \mid \eta^*).$$

Since $z(\theta \mid \eta^*)$ is continuous in θ and non-decreasing in θ,

$$z(\theta^{'} \mid \eta^{'}) \leq z(\theta \mid \eta^*) \leq z(\theta^{''} \mid \eta^{''})$$

for any $\theta \in [\theta_1, \theta_2]$. The coalitional incentive constraints imply

$$q^{SB}(\theta^{'}, \eta^{'}) \geq q^{SB}(\theta, \eta^*) \geq q^{SB}(\theta^{''}, \eta^{''})$$

for any $\theta \in [\theta_1, \theta_2]$. On the other hand $\hat{h}(\theta^{'} \mid \eta^{'}) = \hat{h}(\theta^{''} \mid \eta^{''})$ implies $q^{SB}(\theta^{'}, \eta^{'}) = q^{SB}(\theta^{''}, \eta^{''})$. Therefore $q^{SB}(\theta, \eta^*) = q^{SB}(\theta^{'}, \eta^{'}) = q^{SB}(\theta^{''}, \eta^{''})$ for any $\theta \in [\theta_1, \theta_2]$. This contradicts the property that $q^{SB}(\theta, \eta^*)$ must be strictly decreasing in θ.

Hence there exists a function $\phi(\cdot) : [\underline{h}, \bar{h}] \to [\underline{z}, \bar{z}]$ such that $z(\theta \mid \eta) = \phi(\hat{h}(\theta \mid \eta))$. Since $z(\theta \mid \eta^*)$ and $\hat{h}(\theta \mid \eta^*)$ are continuous in θ, $\phi(h)$ must be continuous.

Second we show that $\phi(h)$ is non-decreasing in h. For any (θ, η) and $(\theta^{'}, \eta^{'})$ such that $\hat{h}(\theta \mid \eta) < \hat{h}(\theta^{'} \mid \eta^{'})$,

$$q^{SB}(\theta, \eta) = q^*(\hat{h}(\theta \mid \eta)) > q^*(\hat{h}(\theta^{'} \mid \eta^{'})) = q^{SB}(\theta^{'}, \eta^{'}).$$

The coalitional incentive constraints then imply $z(\theta \mid \eta) \leq z(\theta^{'} \mid \eta^{'})$.

Third we show $h - \phi(h)$ is non-negative and increasing in h. Since $q^{SB}(\theta, \eta^*) = q^*(\hat{h}(\theta \mid \eta^*))$ is strictly decreasing in θ, the pooling region $\Theta(\pi(\cdot \mid \eta^*), \eta^*)$ must be empty. Hence it must be the case that

$$z(\theta \mid \eta^*) = \theta + \frac{F(\theta \mid \eta^*) - \Lambda(\theta \mid \eta^*)}{f(\theta \mid \eta^*)},$$

implying

$$\frac{\Lambda(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} = \hat{h}(\theta \mid \eta^*) - \phi(\hat{h}(\theta \mid \eta^*)).$$

The LHS is non-negative and increasing in θ, since $f(\theta \mid \eta^*)$ is decreasing in θ and $\Lambda(\theta \mid \eta^*)$ is non-negative and non-decreasing in θ. So $h - \phi(h)$ must be non-negative and increasing in $h \in [\underline{h}, \bar{h}]$.

Step 3:

Define $R(z) \equiv \max_{(\tilde{\theta},\tilde{\eta})\in K}[X^{SB}(\tilde{\theta},\tilde{\eta}) - zq^{SB}(\tilde{\theta},\tilde{\eta})]$ for any $z \in [\underline{z}, \bar{z}]$. Then

$$R(z(\theta \mid \eta)) = X^{SB}(\theta,\eta) - z(\theta \mid \eta)q^{SB}(\theta,\eta)$$

and by the Envelope Theorem, $R^{'}(z(\theta \mid \eta)) = -q^{SB}(\theta,\eta) = -q^*(\hat{h}(\theta \mid \eta))$. It also implies $R^{'}(\phi(h)) = -q^*(h)$. Then S's interim payoff is

$$\begin{aligned}
& E[X^{SB}(\theta,\eta) - h(\theta \mid \eta)q^{SB}(\theta,\eta) \mid \eta] \\
= \ & E[X^{SB}(\theta,\eta) - z(\theta \mid \eta)q^{SB}(\theta,\eta) + (z(\theta \mid \eta) - h(\theta \mid \eta))q^{SB}(\theta,\eta) \mid \eta] \\
= \ & E[R(\phi(\hat{h}(\theta \mid \eta))) + (\phi(\hat{h}(\theta \mid \eta)) - \hat{h}(\theta \mid \eta))q^*(\hat{h}(\theta \mid \eta)) \mid \eta]
\end{aligned}$$

with the last equality using the property of the ironing rule.

Next define

$$L(h) \equiv R(\phi(h)) + (\phi(h) - h)q^*(h).$$

$L(h)$ is continuous and differentiable almost everywhere, since the monotonicity implies the differentiability of $\phi(h)$ almost everywhere. If the second best allocation is implementable with weak collusion, $E[L(\hat{h}(\theta \mid \eta)) \mid \eta] = 0$ holds for any η. The first derivative of $L(h)$ is

$$L^{'}(h) = (\phi(h) - h)q^{*'}(h) - q^*(h).$$

Since $q^*(h)$ is continuously differentiable, $L^{'}(h)$ is continuous and also differentiable almost everywhere and

$$L^{''}(h) = (\phi^{'}(h) - 1)q^{*'}(h) + (\phi(h) - h)q^{*''}(h) - q^{*'}(h).$$

By using $V^{'}(q^*(h)) = h$, we can show that $V^{'''}(q) \le 0$ implies $q^{*''}(h) \le 0$, and $0 < V^{'''}(q) \le \frac{(V^{''}(q))^2}{V^{'}(q)}$ implies $q^{*''}(h) > 0$ and $hq^{*''}(h) + q^{*'}(h) < 0$. By $\phi^{'}(h) - 1 < 0$ and $\phi(h) - h \le 0$, it follows that $L^{''}(h) > 0$.

The strict convexity of L then implies $L(h) > L(h^{'}) - (h^{'} - h)L^{'}(h^{'})$ for any $h \neq h^{'}$. Hence

$$\begin{aligned}
& E[L(\hat{h}(\theta \mid \eta^*)) \mid \eta^*] = E[L(h(\theta \mid \eta^*)) \mid \eta^*] \\
> \;& E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - h(\theta \mid \eta^*)]L^{'}(\hat{h}(\theta \mid \eta)) \mid \eta^*] \\
= \;& E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L^{'}(\hat{h}(\theta \mid \eta)) + \int_{\theta}^{\bar{\theta}} L^{'}(\hat{h}(y \mid \eta))dy \mid \eta^*]
\end{aligned}$$

for any $\eta \neq \eta^*$. $L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L^{'}(\hat{h}(\theta \mid \eta)) + \int_{\theta}^{\bar{\theta}} L^{'}(\hat{h}(y \mid \eta))dy$ is non-increasing in θ, since

$$-[\hat{h}(\theta \mid \eta) - \theta]L^{''}(\hat{h}(\theta \mid \eta)) < 0$$

and is strictly decreasing in θ over some interval (since the ironing rule ensures $\hat{h}(\theta \mid \eta)$ is continuous with $\hat{h}(\underline{\theta} \mid \eta) = \underline{\theta}$ and $\hat{h}(\bar{\theta} \mid \eta) > \bar{\theta}$). Then property (ii) implies $F(\theta \mid \eta^*) > F(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta})$ and for any $\eta \neq \eta^*$. A first order stochastic dominance argument then ensures

$$\begin{aligned}
& E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L^{'}(\hat{h}(\theta \mid \eta)) + \int_{\theta}^{\bar{\theta}} L^{'}(\hat{h}(y \mid \eta))dy \mid \eta^*] \\
> \;& E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - \theta]L^{'}(\hat{h}(\theta \mid \eta)) + \int_{\theta}^{\bar{\theta}} L^{'}(\hat{h}(y \mid \eta))dy \mid \eta] \\
= \;& E[L(\hat{h}(\theta \mid \eta)) - [\hat{h}(\theta \mid \eta) - h(\theta \mid \eta)]L^{'}(\hat{h}(\theta \mid \eta)) \mid \eta] \\
= \;& E[L(\hat{h}(\theta \mid \eta)) \mid \eta].
\end{aligned}$$

where the last equality utilizes a property of the ironing transformation. Therefore S must earn a positive rent in state η^*, as $E[L(h(\theta \mid \eta^*)) \mid \eta^*] > E[L(\hat{h}(\theta \mid \eta)) \mid \eta] \geq 0$. This is a contradiction. ■

Proof of Propositions 6, 7, 8: sketched in the text.

Proof of Proposition 9: We start with some characterization of the optimal WCP allocation in the ex-ante collusion. For the convenience of the proof of later Proposition 11, our argument is based on general information structure satisfying Assumption 1 and 2.

The problem set up in the text can be represented more compactly using the following notation. Define $Z(\theta_1, \theta_2)$ as the set of z so that (θ_1, θ_2, b) satisfies the coalitional incentive constraint for a given (θ_1, θ_2). $Z(\theta_1, \theta_2)$ is non-empty if and only if $(\theta_1, \theta_2) \in L$ where L is defined as the set of (θ_1, θ_2) such that (θ_1, θ_2, b) satisfies the coalitional incentive constraints for some b. Also define $\bar{l}_i(\theta_i)$ such that $\bar{l}_i(\theta_i) = l_i(\theta_i)$ for $\theta_i \in (\underline{\theta}_i, \bar{\theta}_i]$ and $-\infty$ for $\theta_i = \underline{\theta}_i$ and $\bar{h}_i(\theta_i)$ such that $\bar{h}_i(\theta_i) = h_i(\theta_i)$ for $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i)$ and $+\infty$ for $\theta_i = \bar{\theta}_i$. Then

$$Z(\theta_1, \theta_2) = [\max\{\bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\}, \min\{\bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\}]$$

and

$$L \equiv \{(\theta_1, \theta_2) \mid Z(\theta_1, \theta_2) \neq \phi\}.$$

It is evident that in the solution of the ex ante collusion problem, X_0 is set equal to:

$$X_0 = \max\{0, F(\theta_1 \mid \eta_1)(\theta_1 - b), F(\theta_2 \mid \eta_2)(\theta_2 - b)\}.$$

Hence the ex ante collusion problem reduces to

$$\max p_1 F(\theta_1 \mid \eta_1)[V - b] + p_2 F(\theta_2 \mid \eta_2)[V - b] - \max\{0, F(\theta_1 \mid \eta_1)(\theta_1 - b), F(\theta_2 \mid \eta_2)(\theta_2 - b)\}$$

subject to

$$b \in Z(\theta_1, \theta_2)$$

and is hereafter denoted problem P_E. Let $(\theta_1^E, \theta_2^E, b^E, X_0^E)$ be the solution for the ex-ante collusion (P_E).

The following lemma provides one characterization of the optimal WCP allocation in the ex-ante collusion.

Lemma 2 *(i) Either $F(\theta_1^E \mid \eta_1) > F(\theta_2^E \mid \eta_2)$ and $\theta_1^E \leq \theta_2^E$ or $(\theta_1^E, \theta_2^E) = (\bar{\theta}_1, \bar{\theta}_2)$ holds.*

(ii) Let us set up the problem (denoted by $\bar{P}_E$) as follows:

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \max\{B(\theta_1, \theta_2), l_1(\theta_1), \bar{l}_2(\theta_2)\},$$

$$\theta_1 \leq \theta_2$$

and

$$F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$$

where

$$B(\theta_1, \theta_2) \equiv \frac{\theta_1 F(\theta_1 \mid \eta_1) - \theta_2 F(\theta_2 \mid \eta_2)}{F(\theta_1 \mid \eta_1) - F(\theta_2 \mid \eta_2)}.$$

Then if the problem $\bar{P}_E$ has a solution, it is a pair of thresholds in the optimal WCP allocation which is associated with

$$b^E = \max\{B(\theta_1^E, \theta_2^E), l_1(\theta_1^E), \bar{l}_2(\theta_2^E)\}$$

and

$$X_0^E = F(\theta_2^E \mid \eta_2)(\theta_2^E - b^E).$$

If the problem $\bar{P}_E$ does not have a solution, $(\theta_1^E, \theta_2^E, b^E, X_0^E) = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_2, 0)$.

Proof of Lemma 2

Proof of (i)

Suppose otherwise that the solution of P_E satisfies $F(\theta_1 \mid \eta_1) \le F(\theta_2 \mid \eta_2)$ and $(\theta_1, \theta_2) \neq (\bar{\theta}_1, \bar{\theta}_2)$. Assumption 1 implies $\theta_1 < \theta_2$. It also implies that $\theta_1 < \bar{\theta}_1$. Then the objective function of P

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - \max\{0, F(\theta_1 \mid \eta_1)(\theta_1 - b), F(\theta_2 \mid \eta_2)(\theta_2 - b)\}$$

is non-decreasing in b for $b \le \theta_2$, and is non-increasing in b for $b > \theta_2$, implying that it is maximized at $b = \theta_2$. $b \in Z(\theta_1, \theta_2)$ must be satisfied in the optimal allocation. Feasibility requires that $\max\{\bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\} \le \theta_2$, since $\bar{l}_1(\theta_1) \le \theta_1 < \theta_2$. Hence $b = \min\{\theta_2, \bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\}$ in the optimal solution, implying P's payoff is:

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b).$$

But this is less than the P's payoff in the optimal solution to NS, since

$$\begin{aligned} & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b) \\ \le \; & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2). \\ < \; & F(\theta^{NS})(V - \theta^{NS}) \end{aligned}$$

The first inequality comes from $b \le \theta_2$ and $F(\theta_1 \mid \eta_1) \le F(\theta_2 \mid \eta_2)$. The second inequality comes from the fact that (i) if $V > \theta_2$,

$$\begin{aligned} & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2) < [p_1 F(\theta_2 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2) \\ = \; & F(\theta_2)(V - \theta_2) \le F(\theta^{NS})(V - \theta^{NS}) \end{aligned}$$

and (ii) if $V \leq \theta_2$,

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_2) \leq 0 < F(\theta^{NS})(V - \theta^{NS}).$$

Hence we obtain a contradiction, since the P's payoff in the optimal NS is always attainable in the weak collusion.

Next suppose $\theta_1 > \theta_2$. Then it is evident that $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ from Assumption 1. Then the objective function in the problem P_E is maximized at θ_1. Since $\theta_1 > \theta_2$ implies $\theta_1 > \bar{l}_2(\theta_2)$, $\theta_1 \geq \max\{\bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\}$. It implies that $b = \min\{\theta_1, \bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\}$ in the solution, bringing the P's payoff:

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_1 \mid \eta_1)(\theta_1 - b).$$

But it is shown that this is less than the P's payoff in the optimal NS, since

$$\begin{aligned} & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_1 \mid \eta_1)(\theta_1 - b) \\ \leq\ & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_1). \\ <\ & F(\theta^{NS})(V - \theta^{NS}) \end{aligned}$$

The first inequality comes from $b \leq \theta_1$ and $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$. The second inequality comes from the fact that (i) if $V > \theta_1$,

$$\begin{aligned} & [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_1) < [p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_1 \mid \eta_2)](V - \theta_1) \\ =\ & F(\theta_1)(V - \theta_1) \leq F(\theta^{NS})(V - \theta^{NS}) \end{aligned}$$

and (ii) if $V \leq \theta_1$,

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - \theta_1) \leq 0 < F(\theta^{NS})(V - \theta^{NS}).$$

This is the contradiction, since the P's payoff in the optimal NS is always attainable in the weak collusion.

Proof of (ii)

For (θ_1, θ_2) which satisfies $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ and $\theta_1 \leq \theta_2$, $B(\theta_1, \theta_2)$ in the problem $\bar{P}_E$ is well-defined. This is the value of b which satisfies $F(\theta_1 \mid \eta_1)(\theta_1 - b) = F(\theta_2 \mid \eta_2)(\theta_2 - b)$. It is evident that $B(\theta_1, \theta_2) \leq \theta_1 \leq \theta_2$.

Suppose that the optimal threshold (θ_1^E, θ_2^E), which is a solution of the problem P_E, satisfies $F(\theta_1^E \mid \eta_1) > F(\theta_2^E \mid \eta_2)$ and $\theta_1^E \leq \theta_2^E$. Since $b^E \in Z(\theta_1^E, \theta_2^E)$, it is evident that $Z(\theta_1^E, \theta_2^E) \neq \phi$ or $(\theta_1^E, \theta_2^E) \in L$. Then (θ_1^E, θ_2^E) must be also a solution of the revised problem of P_E where the constraints $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$, $\theta_1 \leq \theta_2$ and $(\theta_1, \theta_2) \in L$ are added, we hereafter refer to this as problem $P_E^{'}$. Consider now the solution of $P_E^{'}$. For (θ_1, θ_2) which satisfies the additional constraint $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ and $\theta_1 \leq \theta_2$, the objective function in $P_E^{'}$:

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - \max\{0, F(\theta_1 \mid \eta_1)(\theta_1 - b), F(\theta_2 \mid \eta_2)(\theta_2 - b)\}$$

is non-decreasing in b for $b < B(\theta_1, \theta_2)$ and is non-increasing in b for $b \in [B(\theta_1, \theta_2), \theta_2]$ and is non-increasing in b for $b > \theta_2$. Therefore it is maximized at $b = B(\theta_1, \theta_2)$. Since $\min\{\bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\} > \theta_1 \geq B(\theta_1, \theta_2)$ for any (θ_1, θ_2) such that $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ and $\theta_1 \leq \theta_2$, the constraint $b \in Z(\theta_1, \theta_2)$ implies that for a given (θ_1, θ_2), the optimal b must satisfy

$$b = \max\{B(\theta_1, \theta_2), \bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\}.$$

Since $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ also implies $\theta_1 > \underline{\theta}_1$, we can replace $\bar{l}_1(\theta_1)$ by $l_1(\theta_1)$ without loss of generality. $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ also implies $\theta_2 < \bar{\theta}_2$ and $\bar{l}(\theta_2) < \theta_2$. Then since $\max\{\bar{l}(\theta_1), \bar{l}_2(\theta_2)\} \leq \theta_2$ and $B(\theta_1, \theta_2) \leq \theta_2$, this optimal choice of b must be in $[B(\theta_1, \theta_2), \theta_2]$, implying that the optimal choice of X_0 satisfies

$$X_0 = F(\theta_2 \mid \eta_2)(\theta_2 - b).$$

With these choices of b and X_0, the problem $P_E^{'}$ reduces to

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \max\{B(\theta_1, \theta_2), l_1(\theta_1), \bar{l}_2(\theta_2)\}$$

$$\theta_1 \leq \theta_2,$$

$$F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$$

and

$$(\theta_1, \theta_2) \in L,$$

where, by hypothesis, the optimal pair of thresholds (θ_1^E, θ_2^E), is also the solution of this problem.

Next we show that (θ_1^E, θ_2^E) is also the solution of the problem $\bar{P}_E$ where the last constraint $(\theta_1, \theta_2) \in L$ is dropped from the above problem. To show it by the contradiction, suppose that (θ_1^E, θ_2^E) is the solution of $\bar{P}_E$ with additional constraint $(\theta_1, \theta_2) \in L$, but is not the solution of $\bar{P}_E$. Then we can find $(\theta_1^{'}, \theta_2^{'}) \notin L$ such that the objective function of $\bar{P}_E$ can take a higher value, satisfying $\theta_2^{'} \geq \theta_1^{'}$ and $F(\theta_1^{'} \mid \eta_1) > F(\theta_2^{'} \mid \eta_2)$. $(\theta_1^{'}, \theta_2^{'}) \notin L$ implies that at least one of either $\bar{l}_2(\theta_2^{'}) > \bar{h}_1(\theta_1^{'})$ or $\bar{l}_1(\theta_1^{'}) > \bar{h}_2(\theta_2^{'})$ holds. But it is evident that both of them never hold. If $\bar{l}_2(\theta_2^{'}) > \bar{h}_1(\theta_1^{'})$ (or $\bar{l}_1(\theta_1^{'}) > \bar{h}_2(\theta_2^{'})$), then $\bar{l}_1(\theta_1^{'}) < \bar{h}_2(\theta_2^{'})$ (or $\bar{l}_2(\theta_2^{'}) < \bar{h}_1(\theta_1^{'})$). In addition, $\theta_1^{'} \leq \theta_2^{'}$ implies that

$$\bar{l}_1(\theta_1^{'}) \leq \theta_1^{'} \leq \theta_2^{'} \leq \bar{h}_2(\theta_2^{'}).$$

Therefore without loss of generality, we can suppose $\bar{l}_2(\theta_2^{'}) > \bar{h}_1(\theta_1^{'})$. It is equivalent to $h_1(\theta_1^{'}) < l_2(\theta_2^{'})$, $\theta_2^{'} > \underline{\theta}_2$ and $\theta_1^{'} < \bar{\theta}_1$. Then since

$$\max\{B(\theta_1^{'}, \theta_2^{'}), l_1(\theta_1^{'})\} < \theta_1^{'} \leq h_1(\theta_1^{'}) < l_2(\theta_2^{'}),$$

the choice of b must be $b^{'} = l_2(\theta_2^{'}) \leq \theta_2^{'}$. Since the value of the objective function

$$[p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)](V - l_2(\theta_2^{'})) - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - l_2(\theta_2^{'}))$$

is positive (as P earns the positive payoff under (θ_1^E, θ_2^E)), $V > l_2(\theta_2^{'})$ must be satisfied. Now define

$$\theta_1^{''} \equiv \max\{\theta_1 \mid h_1(\theta_1) \leq l_2(\theta_2^{'}), \theta_1 \leq \bar{\theta}_1, \bar{l}_1(\theta_1) \leq \bar{h}_2(\theta_2^{'})\},$$

which is strictly larger than $\theta_1^{'}$. Since $\theta_1^{''} < h_1(\theta_1^{''}) \leq l_2(\theta_2^{'}) \leq \theta_2^{'}$, $\theta_1^{''} < \theta_2^{'}$. Since $\bar{l}_1(\theta_1^{''}) \leq \bar{h}_2(\theta_2^{'})$, $(\theta_1^{''}, \theta_2^{'}) \in L$ and $F(\theta_1^{''} \mid \eta_1) > F(\theta_2^{'} \mid \eta_2)$. Since

$$\max\{B(\theta_1^{''}, \theta_2^{'}), l_1(\theta_1^{''})\} \leq \theta_1^{''} < h_1(\theta_1^{''}) \leq l_2(\theta_2^{'}),$$

the choice of b is still equal to $l_2(\theta_2^{'})$. It is evident that this choice $(\theta_1^{''}, \theta_2^{'})$ generates a higher value of the objective function:

$$\begin{aligned} & [p_1 F(\theta_1^{''} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)](V - l_2(\theta_2^{'})) - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - l_2(\theta_2^{'})) \\ > \; & [p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)](V - l_2(\theta_2^{'})) - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - l_2(\theta_2^{'})). \end{aligned}$$

Since the left side hand cannot be larger than the maximum value in the problem $\bar{P}_E$ with additional constraint $(\theta_1, \theta_2) \in L$, we obtain a contradiction. We conclude that if (θ_1^E, θ_2^E)

which is the solution of the problem P_E satisfy $F(\theta_1^E \mid \eta_1) > F(\theta_2^E \mid \eta_2)$ and $\theta_1^E \leq \theta_2^E$, the problem $\bar{P}_E$ has a solution (θ_1^E, θ_2^E). Hence if $\bar{P}_E$ does not have a solution, the solution of P_E does not satisfy either $F(\theta_1^E \mid \eta_1) > F(\theta_2^E \mid \eta_2)$ or $\theta_1^E \leq \theta_2^E$. Then from (i) of this lemma, $(\theta_1^E, \theta_2^E) = (\bar{\theta}_1, \bar{\theta}_2)$ is the optimal thresholds in the solution of P_E.

Finally we show that if $\bar{P}_E$ has a solution, it must be always a solution of P_E. Suppose that $\bar{P}_E$ has a solution $(\theta_1^{'}, \theta_2^{'})$, but it is not a solution of P_E. Then from (i) of this lemma, $(\theta_1, \theta_2) = (\bar{\theta}_1, \bar{\theta}_2)$ must be solution of P_E and the P's payoff is $V-1$. Then with $(\theta_1^{'}, \theta_2^{'})$, the objective function in the problem $\bar{P}_E$ must take strictly lower value than $V-1$. However the value of the objective function in the problem $\bar{P}_E$ can approximate $V-1$ by selecting (θ_1, θ_2) which is sufficiently close to $(\bar{\theta}_1, \bar{\theta}_2)$ without violating all the constraints. This is the contradiction. ■

Now we provide the proof of Proposition 9 in the case of a partition information structure. We define $\bar{H}(\theta)$ on $\theta \leq 1$ such that $\bar{H}(\theta) = 0$ for $\theta \leq 0$ and $\bar{H}(\theta) = H(\theta)$ for $\theta \in (0,1]$. As a first step, let us show that it is beneficial to hire S for any $V \in (\max\{c, \bar{H}(l_2(c))\}, H(1))$. For any $V \in (\max\{c, \bar{H}(l_2(c))\}, H(1))$, since $h_1(0) = 0 < V$ and $h_2(c) = c < V$, $\theta_1^{SB} > 0$, $\theta_2^{SB} > c$ and $\theta^{NS} \in (0,1)$ where $V = H(\theta^{NS})$. P's payoff without S is $\Pi_{NS} \equiv F(\theta^{NS})[V - \theta^{NS}] > 0$. In the third-best problem P_E defined above, Π_{NS} can be achieved if we select $X_0 = 0$, $b = \theta_1 = \theta^{NS}$ and $\theta_2 = c$ in the case of $c < V < H(c)$, and $X_0 = 0$, $\theta_2 = b = \theta^{NS}$ and $\theta_1 = c$ in the case of $H(c) \leq V \leq H(1)$.

First consider the case $\max\{c, \bar{H}(l_2(c))\} < V < H(c)$. Then we observe the following relationship among the thresholds in NS and SB:

$$\theta_1^{SB} = \theta^{NS} < c < \theta_2^{SB}.$$

Let us create a small variation from the optimal NS ($X_0 = 0$, $b = \theta_1 = \theta^{NS}$ and $\theta_2 = c$) to $(\theta_1^{'}, \theta_2^{'}, b^{'}, X_0^{'})$ which satisfies

(i) $\theta_1^{'} = \theta_1 = \theta^{NS}$

(ii) $\theta_2^{'}$ is selected such that $\theta_2^{'} \in (\theta_2, \theta_2^{SB})$ and $F(\theta_1 \mid \eta_1) > F(\theta_2^{'} \mid \eta_2)$

(iii) $b^{'} = \frac{\theta_1 F(\theta_1|\eta_1) - \theta_2^{'} F(\theta_2^{'}|\eta_2)}{F(\theta_1|\eta_1) - F(\theta_2^{'}|\eta_2)} < \theta_1 < \theta_2^{'}$

(iv) $X_0^{'} = F(\theta_1 \mid \eta_1)(\theta_1 - b^{'}) = F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - b^{'}) > 0$.

With this variation, the threshold pair moves closer to the second best one, while maintaining S's zero information rent owing to (iv). Hence P's payoff is strictly improved. In order for this allocation to be implementable under weak collusion, we need to check that

$$\max\{l_1(\theta_1), l_2(\theta_2^{'})\} \leq b^{'} \leq \min\{h_1(\theta_1), h_2(\theta_2^{'})\}.$$

It is evident that

$$b^{'} < \theta_1 \leq \min\{h_1(\theta_1), h_2(\theta_2^{'})\}.$$

$\max\{c, \bar{H}(l_2(c))\} < V = H(\theta^{NS})$ implies $l_2(c) < \theta_1 = \theta^{NS}$. Since $\lim_{\theta_2^{'} \to c} b^{'} = \theta_1 = \theta^{NS}$, we can find $\theta_2^{'}$ sufficiently close to c such that

$$\max\{l_1(\theta_1), l_2(\theta_2^{'})\} \leq b^{'}.$$

Next consider the case $H(c) \leq V < H(1)$. Then

$$\theta_1^{SB} = c < \theta^{NS} < \theta_2^{SB}$$

Construct the following small variation from the optimal NS allocation $((\theta_1, \theta_2, b, X_0) = (c, \theta^{NS}, \theta^{NS}, 0))$ to $(\theta_1^{'}, \theta_2^{'}, b^{'}, X_0^{'})$ which satisfies

(i) $X_0^{'} = F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - b^{'})$

(ii) $\theta_1^{'} = \theta_1 = c$

(iii) $\theta_2^{'}$ satisfies $\theta^{NS} < \theta_2^{'} < \theta_2^{SB}$

(iv) $b^{'} = \frac{\theta^{NS} - F(\theta_2^{'}|\eta_2)\theta_2^{'}}{(1-F(\theta_2^{'}|\eta_2))} < \theta^{NS}$.

Since $b^{'} < \theta^{NS} < \theta_2^{'}$, $X_0^{'} > 0$, and the coalitional participation constraint is satisfied. $b^{'} \leq \min\{\bar{h}_1(c), \bar{h}_2(\theta_2^{'})\}$ is obviously satisfied. $b^{'} \geq \max\{\bar{l}_1(c), \bar{l}_2(\theta_2^{'})\} = \max\{c, l_2(\theta_2^{'})\}$ is also satisfied for $\theta_2^{'}$ sufficiently close to θ^{NS}, since $\lim_{\theta_2^{'} \to \theta^{NS}} b^{'} = \theta^{NS}$ and $\theta^{NS} > \max\{c, l_2(\theta^{NS})\}$. P's payoff is strictly improved with this allocation, since it moves closer to the second best, while S's interim payoff is unchanged as $(b^{'} - c) + X_0^{'} = (\theta^{NS} - c)$ in state η_1 and S earns zero rent in state η_2 owing to (i) above.

To proceed with the necessity part of the result, we use Lemma 2 which help characterize the optimal allocation. We show that if $c < \bar{H}(l_2(c))$, there exists V_1 such that $c < V_1 \leq \bar{H}(l_2(c))$ and S is not valuable for $V \in (c, V_1]$ and valuable for $V \in (V_1, \bar{H}(l_2(c))$. In order to show it by contradiction, suppose that the supervisor is valuable for any $V \in (c, \bar{H}(l_2(c))]$.

Then we can show that $c < \theta_2 < 1$ must hold in the optimal allocation for any $V \in (c, H(l_2(c))]$. The argument is as follows. Consider the problem with the restriction to $\theta_2 = c$. Then $B(\theta_1, c) = \theta_1$ and $b = \theta_1$, and the maximum value in the problem $\bar{P}_E$ (in Lemma 2) is Π_{NS} under $\theta_1 = \theta^{NS}$. It implies that if S is valuable, we must have $\theta_2 > c$. With the choice of $\theta_2 = 1$, P's possible maximum payoff is $V - 1$ with the choice of $\theta_1 = c$, which is lower than Π^{NS} since $\bar{H}(l_2(c)) < H(1)$, implying $\theta_2 < 1$.

Therefore the following problem (with the additional constraint $\theta_2 > c$) has a solution and its maximum value is larger than Π_{NS} for any $V \in (c, \bar{H}(l_2(c))]$:

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - b) - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \max\{B(\theta_1, \theta_2), l_1(\theta_1), l_2(\theta_2)\}$$

and

$$F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2).$$

$$\theta_2 > c.$$

Let (θ_1^*, θ_2^*) be the solution of the above problem. First we show that $\theta_1^* > l_2(c)$ by contradiction. Suppose $\theta_1^* \leq l_2(c)$. Since it implies $\theta_1^* < c < \theta_2^*$,

$$l_1(\theta_1^*) < \theta_1^* \leq l_2(c) < l_2(\theta_2^*)$$

and

$$B(\theta_1^*, \theta_2^*) < \theta_1^* \leq l_2(c) < l_2(\theta_2^*).$$

It implies $b^* = l_2(\theta_2^*)$. The objective function in the above problem takes a value of

$$[p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - l_2(\theta_2^*)) - F(\theta_2^* \mid \eta_2)(\theta_2^* - l_2(\theta_2^*)).$$

Since this must be larger than $\Pi_{NS} > 0$ and $\theta_2^* > l_2(\theta_2^*)$, $V > l_2(\theta_2^*)$ must hold. But P's payoff can be improved with the small increase in θ_1 from θ_1^* without violating all constraints of the above problem, which is a contradiction.

With $b^* = \max\{B(\theta_1^*, \theta_2^*), l_1(\theta_1^*), l_2(\theta_2^*)\}$ and $F(\theta_1^* \mid \eta_1) > F(\theta_2^* \mid \eta_2)$,

$$F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \geq F(\theta_1^* \mid \eta_1)(\theta_1^* - b^*).$$

It implies that

$$\begin{aligned} & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \\ \leq \; & p_1 F(\theta_1^* \mid \eta_1)(V - \theta_1^*) + p_2 F(\theta_2^* \mid \eta_2)(V - \theta_2^*). \end{aligned}$$

Then P's payoff in the optimal allocation cannot be larger than the maximum value of the problem:

$$\max p_1 F(\theta_1 \mid \eta_1)(V - \theta) + p_2 F(\theta_2 \mid \eta_2)(V - \theta_2)$$

subject to

$$\theta_1 \geq l_2(c)$$

$$\theta_2 \geq c.$$

Let $\bar{\Pi}(V)$ be the maximum value of the above problem, and $\Pi_{NS}(V)$ be the optimal payoff in NS for V. It is evident that both $\bar{\Pi}(V)$ and $\Pi_{NS}(V)$ are continuous in V. By hypothesis, $\bar{\Pi}(V) > \Pi_{NS}(V)$ for any $V \in (c, \bar{H}(l_2(c))]$. But $\lim_{+V \to c} \bar{\Pi}(V) = F(l_2(c))[V - l_2(c)] < \lim_{+V \to c} \Pi_{NS}(V)$, since $\theta^{NS} < l_2(c)$ at $V = c$. This is the contradiction, implying that there exists some interval of V on $(c, \bar{H}(l_2(c))]$ such that S is not valuable.

Next we show that if there exists $V \in (c, \bar{H}(l_2(c)))$ such that S is valuable, S is also valuable for any $V^{'} \in (V, \bar{H}(l_2(c))]$. Otherwise, suppose there exists $V^{'} \in (V, \bar{H}(l_2(c))]$ such that $(\theta_1, \theta_2) = (\theta^{NS}(V^{'}), c)$ is the solution of $\bar{P}_E$, even though (θ_1^*, θ_2^*), which is the solution of $\bar{P}_E$ for V satisfies $\theta_1^* > l_2(c)$ and $\theta_2^* > c$ (by the reason explained above). It implies that

$$\begin{aligned} & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \\ > \; & p_1 F(\theta^{NS}(V^{'}) \mid \eta_1)(V - \theta^{NS}(V^{'})). \end{aligned}$$

and

$$\begin{aligned} & p_1 F(\theta^{NS}(V^{'}) \mid \eta_1)(V^{'} - \theta^{NS}(V^{'})) \\ > \; & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V^{'} - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*). \end{aligned}$$

This implies

$$p_1 F(\theta^{NS}(V^{'}) \mid \eta_1) > p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2).$$

But this is inconsistent with $\theta^{NS}(V') < l_2(c) < \theta_1^*$ and $c < \theta_2^*$, a contradiction. This argument guarantees the existence of a critical value V_1 of V in $(c, \bar{H}(l_2(c))]$, such that S is not valuable (or valuable) for lower (or higher) V than V_1.

Finally let us show that there exists a critical value of V, V_2, with $V_2 \geq H(1)$ such that S is not valuable for V higher than V_2. Otherwise suppose that S is valuable for any $V \geq H(1)$. This implies that $\bar{P}_E$ has a solution with $\theta_2^* < 1$ for any $V \geq H(1)$ and its maximum value is higher than $V - 1$. Let $(\theta_1^*, \theta_2^*, b^*)$ be the solution of $\bar{P}_E$ for $V \geq H(1)$. Then

$$\begin{aligned} & [p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)](V - b^*) - F(\theta_2^* \mid \eta_2)(\theta_2^* - b^*) \\ \leq \quad & [p_1 + p_2 F(\theta_2^* \mid \eta_2)](V - l_2(\theta_2^*)) - F(\theta_2^* \mid \eta_2)(\theta_2^* - l_2(\theta_2^*)), \end{aligned}$$

since $V \geq H(1) > l_2(\theta_2^*)$, $b^* \geq l_2(\theta_2^*)$ and $F(\theta_1^* \mid \eta_1) > F(\theta_2^* \mid \eta_2)$. By hypothesis, there must exist $\theta_2^* < 1$ such that

$$[p_1 + p_2 F(\theta_2^* \mid \eta_2)](V - l_2(\theta_2^*)) - F(\theta_2^* \mid \eta_2)(\theta_2^* - l_2(\theta_2^*)) > V - 1$$

for any $V \geq H(1)$. It also implies that there exists $\theta_2 < 1$ such that

$$\frac{1 - [p_1 + p_2 F(\theta_2 \mid \eta_2)]l_2(\theta_2) - F(\theta_2 \mid \eta_2)(\theta_2 - l_2(\theta_2))}{p_2[1 - F(\theta_2 \mid \eta_2)]} > V$$

for any $V \geq H(1)$. But this is impossible, since the left hand side is bounded above on $[c, 1]$, because $f(\theta \mid \eta_2)$ is continuous on $[c, 1]$ and

$$\lim_{\theta_2 \to 1} \frac{1 - [p_1 + p_2 F(\theta_2 \mid \eta_2)]l_2(\theta_2) - F(\theta_2 \mid \eta_2)(\theta_2 - l_2(\theta_2))}{p_2[1 - F(\theta_2 \mid \eta_2)]} = 1 + \frac{1}{f(1)}$$

by using l'Hopital's rule. This is a contradiction. Therefore if V is sufficiently large, S cannot generate any value. Finally it is easy to show that if S is not valuable for some $V \geq H(1)$, the same must be true for any larger V, since we can make sure that $p_1 F(\theta_1^* \mid \eta_1) + p_2 F(\theta_2^* \mid \eta_2)$ is monotone for V by the same method as in the previous paragraph. This guarantees the existence of the critical value of $V_2 \geq H(1)$. ■

Proof of Proposition 10: sketched in the text.

Proof of Proposition 11:

We have already shown some characterizations of optimal WCP allocation in ex-ante collusion in Lemma 2. Here we provide some characterizations in interim collusion. In interim collusion, we can drop the coalitional participation constraint. On the other hand, for a pair of thresholds (θ_1, θ_2) such that $(\theta_1, \theta_2) \neq (\underline{\theta}_1, \underline{\theta}_2)$ and $(\theta_1, \theta_2) \neq (\bar{\theta}_1, \bar{\theta}_2)$, z needs to satisfy the same coalitional incentive constraint. Hence the interim collusion problem (denoted P_I hereafter) is obtained by dropping the non-negativity constraint for X_0 from the ex ante collusion problem:

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)[V - b] - \max\{F(\theta_1 \mid \eta_1)(\theta_1 - b), F(\theta_2 \mid \eta_2)(\theta_2 - b)\}$$

subject to

$$b \in Z(\theta_1, \theta_2)$$

Let $(\theta_1^I, \theta_2^I, b^I, X_0^I)$ be the solution for the interim collusion (P_I). The following lemma shows some properties in the optimal WCP allocation in interim collusion.

Lemma 3 *(i) If and only if $F(\theta_1^I \mid \eta_1) < F(\theta_2^I \mid \eta_2)$, $(\theta_1^I, \theta_2^I, b^I, X_0^I) \neq (\theta_1^E, \theta_2^E, b^E, X_0^E)$.*

(ii) If $(\theta_1^I, \theta_2^I, b^I, X_0^I) \neq (\theta_1^E, \theta_2^E, b^E, X_0^E)$, $(\theta_1^I, \theta_2^I, b^I, X_0^I)$ can be attained with the pure delegation to S

Proof of Lemma 3

Proof of (i)

Step 1: $\theta_1^I \leq \theta_2^I$

Since it is evident if we have $\bar{\theta}_1 = \underline{\theta}_2$ (or the partition case), let us consider the case of $\bar{\theta}_1 > \underline{\theta}_2$. Suppose $\theta_1 > \theta_2$ in the solution of the problem P_I. For simplicity of the exposition, we omit superscript I for later part of this proof. Then it is evident that $F(\theta_1 \mid \eta_1) > F(\theta_2 \mid \eta_2)$ from Assumption 1, implying $B(\theta_1, \theta_2) > \theta_1 > \theta_2$. Then the objective function in the problem P_I is maximized at $B(\theta_1, \theta_2)$ if the constraint is ignored. Since $\theta_1 > \theta_2$ implies $\theta_1 > \bar{l}_2(\theta_2)$, $B(\theta_1, \theta_2) > \theta_1 \geq \max\{\bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\}$. It implies that $b = \min\{B(\theta_1, \theta_2), \bar{h}_1(\theta_1), \bar{h}_2(\theta_2)\}$ in the solution. Since $\theta_1 > \theta_2$ implies $\theta_2 < \bar{\theta}_2$ and $h_1(\theta_1) > h_2(\theta_2)$ (from Assumption 1 and 2), $b = \min\{B(\theta_1, \theta_2), h_2(\theta_2)\}$.

Now well show that there is a scope of small change of (θ_1, θ_2) which improves the value of the objective function in P_I in order to show that this is not optimal solution. First let us show that $B(\theta_1, \theta_2) > h_2(\theta_2)$ must hold if (θ_1, θ_2) is the solution of P_I. Suppose $B(\theta_1, \theta_2) \leq h_2(\theta_2)$ or equivalently

$$F(\theta_1 \mid \eta_1)(\theta_1 - h_2(\theta_2)) \leq F(\theta_2 \mid \eta_2)(\theta_2 - h_2(\theta_2)). \tag{14}$$

Then $b = B(\theta_1, \theta_2)$, and the objective function reduces to

$$p_1 F(\theta_1 \mid \eta_1)(V - \theta_1) + p_2 F(\theta_2 \mid \eta_2)(V - \theta_2),$$

which is equal to the objective function of the second best problem. This is maximized at $(\theta_1^{SB}, \theta_2^{SB})$ which satisfies $\theta_1^{SB} < \theta_2^{SB}$ from Assumption 1. It means that at least one of either $\theta_1^{SB} < \theta_1$ or $\theta_2^{SB} > \theta_2$ holds. If $\theta_1^{SB} < \theta_1$, consider small decrease of θ_1, taking θ_2 as given. $B(\theta_1, \theta_2) \leq h_2(\theta_2)$ is maintained with this change, since the left hand side of (14) is increasing in θ_1 because of $h_1(\theta_1) > h_2(\theta_2)$. If $\theta_2^{SB} > \theta_2$, consider small increase in θ_2, taking θ_1 as given. $B(\theta_1, \theta_2) \leq h_2(\theta_2)$ is maintained with this change, since the left hand side of (14) decreases more than the right hand side. Since both changes improve the value of the objective function, this is the contradiction.

The above argument means that $b = h_2(\theta_2) < B(\theta_1, \theta_2)$ in the solution of P_E. Then the maximum value is represented by

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)](V - h_2(\theta_2)) - F(\theta_1 \mid \eta_1)(\theta_1 - h_2(\theta_2)),$$

since $F(\theta_1 \mid \eta_1)(\theta_1 - h_2(\theta_2)) > F(\theta_2 \mid \eta_2)(\theta_2 - h_2(\theta_2))$. But if $V > h_2(\theta_2)$, we can take small increase of θ_2 such that $h_2(\theta_2) < B(\theta_1, \theta_2)$ and $\theta_1 > \theta_2$ are not violated. Similarly if $V \leq h_2(\theta_2)$, we can take small decrease of θ_1 such that the same inequalities are not violated. These changes increase the value of the objective function. This is the contradiction, which completes the proof of the statement in Step 1.

Step 2:

Since (If) part of (i) is obvious from Lemma 2 (i), we show (Only if) part. Suppose that $F(\theta_1^I \mid \eta_1) \geq F(\theta_2^I \mid \eta_2)$ in the solution of the problem P_I. We will show that (θ_1^I, θ_2^I) must be the solution of P_E, which is the problem in the ex-ante collusion. As we know from Step 1 that $\theta_1^I \leq \theta_2^I$, our attention is provided to (θ_1, θ_2) which satisfies $\theta_1 \leq \theta_2$ without loss of generality. For (θ_1, θ_2) such that $F(\theta_1 \mid \eta_1) \geq F(\theta_2 \mid \eta_2)$ and $\theta_1 \leq \theta_2$, the objective function in P_I is non-increasing in b for $b \geq \theta_2$. Since $\max\{\bar{l}_1(\theta_1), \bar{l}_2(\theta_2)\} \leq \theta_2$ from $\bar{l}_1(\theta_1) \leq \theta_1 \leq \theta_2$, the optimal choice of b in the problem P_I satisfies $b^I \leq \theta_2^I$.[37] Therefore in the solution of P_I,

$$X_0^I = \max\{F(\theta_1^I \mid \eta_1)(\theta_1^I - b^I), F(\theta_2^I \mid \eta_2)(\theta_2^I - b^I)\} \geq 0.$$

It implies that (θ_1^I, θ_2^I) must be the solution of P_E.

Proof of (ii)

Next let us consider small modified version of the problem P_I with the additional constraints $F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$ and $(\theta_1, \theta_2) \in L$, which is called P_I'. It is evident that for any (θ_1, θ_2) such that $F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$,

$$\theta_1 < \theta_2 < B(\theta_1, \theta_2).$$

Given (θ_1, θ_2) which satisfies $F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$, the objective function in the problem P_I' is increasing in b if $b < B(\theta_1, \theta_2)$ and decreasing in $b > B(\theta_1, \theta_2)$, taking a maximum value at $b = B(\theta_1, \theta_2)$. Then the optimal selection of b in the problem P_I' is

$$b = \min\{h_1(\theta_1), \bar{h}_2(\theta_2), B(\theta_1, \theta_2)\}.$$[38]

With this choice of b, $F(\theta_1 \mid \eta_1)(\theta_1 - b) \leq F(\theta_2 \mid \eta_2)(\theta_2 - b)$. Therefore the problem P_I' reduces to

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)][V - b] - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \min\{h_1(\theta_1), \bar{h}_2(\theta_2), B(\theta_1, \theta_2)\}$$

$$F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$$

and

$$(\theta_1, \theta_2) \in L$$

Now let us consider more relaxed problem (called $\bar{P}_I$) where $(\theta_1, \theta_2) \in L$ is dropped from this problem.

Step 1:

[37] In the case of $F(\theta_1^I \mid \eta_1) = F(\theta_2^I \mid \eta_2)$, the objective function does not depend on b. There is no loss of generality that our focus is provided to the selection of $b^I \leq \theta_2^I$.

[38] Since $F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$ implies $\theta_1 < \bar{\theta}_1$, $\bar{h}_1(\theta_1)$ can be replaced by $h_1(\theta_1)$.

As first step of the proof, we obtain the following statement: If the optimal allocation differs between the interim collusion and the ex-ante collusion, the optimal threshold (θ_1^I, θ_2^I) in the interim collusion is the solution of $\bar{P}_I$, and (b^I, X_0^I) in the optimal allocation satisfies

$$b^I = \min\{h_1(\theta_1^I), \bar{h}_2(\theta_2^I), B(\theta_1^I, \theta_2^I)\}$$

and

$$X_0^I = F(\theta_2^I \mid \eta_2)(\theta_2^I - b^I).$$

This can be proven as follows. From the above argument, if the optimal allocation differs between the interim collusion and the ex-ante collusion, the optimal threshold (θ_1^I, θ_2^I) in the interim collusion is the solution of

$$\max[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)][V - b] - F(\theta_2 \mid \eta_2)(\theta_2 - b)$$

subject to

$$b = \min\{h_1(\theta_1), \bar{h}_2(\theta_2), B(\theta_1, \theta_2)\}$$

$$F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$$

and

$$(\theta_1, \theta_2) \in L.$$

Here we just show that the last constraint $(\theta_1, \theta_2) \in L$ can be dropped without loss of generality. Suppose that we can find $(\theta_1^{'}, \theta_2^{'}) \notin L$ such that $F(\theta_1^{'} \mid \eta_1) < F(\theta_2^{'} \mid \eta_2)$ and the objective function takes a higher value than in the optimal threshold (θ_1^I, θ_2^I). $(\theta_1^{'}, \theta_2^{'}) \notin L$ implies $\bar{l}_2(\theta_2^{'}) > \bar{h}_1(\theta_1^{'})$ or equivalently $h_1(\theta_1^{'}) < l_2(\theta_2^{'})$, $\theta_2^{'} > \underline{\theta}_2$ and $\theta_1^{'} < \bar{\theta}_1$. Since $h_1(\theta_1^{'}) < l_2(\theta_2^{'}) < \min\{\bar{h}_2(\theta_2^{'}), B(\theta_1^{'}, \theta_2^{'})\}$,

$$b^{'} = h_1(\theta_1^{'}) < \theta_2^{'}.$$

The objective function is equal to

$$[p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)][V - h_1(\theta_1^{'})] - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - h_1(\theta_1^{'})).$$

This must be positive, implying $V > \theta_2^{'}$, since if $V \leq \theta_2^{'}$,

$$\begin{aligned} & [p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)][V - h_1(\theta_1^{'})] - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - h_1(\theta_1^{'})) \\ \leq \; & [p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)][\theta_2^{'} - h_1(\theta_1^{'})] - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - h_1(\theta_1^{'})) < 0. \end{aligned}$$

Then by $h_1(\theta_1^{'}) < \theta_2^{'} < V$ and $F(\theta_1^{'} \mid \eta_1) < F(\theta_2^{'} \mid \eta_2)$,

$$\begin{aligned} & [p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)][V - h_1(\theta_1^{'})] - F(\theta_2^{'} \mid \eta_2)(\theta_2^{'} - h_1(\theta_1^{'})) \\ < \; & [p_1 F(\theta_1^{'} \mid \eta_1) + p_2 F(\theta_2^{'} \mid \eta_2)][V - \theta_2^{'}] \\ \leq \; & F(\theta_2^{'})[V - \theta_2^{'}] \leq \Pi_{NS}. \end{aligned}$$

Since Π_{NS} can be achieved with the ex-ante collusion, (θ_1^I, θ_2^I) cannot be the optimal allocation in the interim collusion which is not achieved with the ex-ante collusion. we obtain a contradiction. Therefore the optimal threshold must be the solution of the problem $\bar{P}_I$.

Step 2:

Now let our focus be provided to the problem $\bar{P}_I$. We will start with showing that S receives the positive rent in η_1 in $(\theta_1^I, \theta_2^I, b^I, X_0^I)$. First we will show that for any (θ_1, θ_2) such that $F(\theta_2 \mid \eta_2) > F(\theta_1 \mid \eta_1)$, $B(\theta_1, \theta_2) > h_1(\theta_1)$. Let us begin with the proof of

$$F(\theta_1 \mid \eta_1)(h_1(\theta_1) - \theta_1) > F(\theta_2 \mid \eta_2)(h_1(\theta_1) - \theta_2)$$

for any (θ_1, θ_2) such that $F(\theta_2 \mid \eta_2) > F(\theta_1 \mid \eta_1)$. Define $J(\theta_1, \theta_2)$ as

$$J(\theta_1, \theta_2) \equiv F(\theta_1 \mid \eta_1)(h_1(\theta_1) - \theta_1) - F(\theta_2 \mid \eta_2)(h_1(\theta_1) - \theta_2).$$

Then

$$\partial J(\theta_1, \theta_2)/\partial \theta_1 = [F(\theta_1 \mid \eta_1) - F(\theta_2 \mid \eta_2)]h_1^{'}(\theta_1) < 0$$

for (θ_1, θ_2) such that $F(\theta_2 \mid \eta_2) > F(\theta_1 \mid \eta_1)$. For a given $\theta_2 > \bar{\theta}_2$,

$$J(\hat{\theta}_1, \theta_2) \equiv F(\theta_2 \mid \eta_2)(\theta_2 - \hat{\theta}_1) > 0$$

in $\hat{\theta}_1$ which satisfies $F(\hat{\theta}_1 \mid \eta_1) = F(\theta_2 \mid \eta_2)$. It implies that $J(\theta_1, \theta_2) > 0$.

Since $B(\theta_1, \theta_2)$ is b which satisfies

$$F(\theta_1 \mid \eta_1)(b - \theta_1) = F(\theta_2 \mid \eta_2)(b - \theta_2),$$

$F(\theta_2 \mid \eta_2) > F(\theta_1 \mid \eta_1)$ implies $B(\theta_1, \theta_2) > h_1(\theta_1)$.

By using this result, since the coalitional incentive constraint implies $b^I \leq h_1(\theta_1^I)$,

$$F(\theta_1^I \mid \eta_1)(b^I - \theta_1^I) > F(\theta_2^I \mid \eta_2)(b^I - \theta_2^I).$$

It is concluded that $b^I = \min\{h_1(\theta_1^I), \bar{h}_2(\theta_2^I)\} < B(\theta_1^I, \theta_2^I)$ and $X_0^I = F(\theta_2^I \mid \eta_2)(\theta_2^I - b^I) < 0$. S receives the positive rent in η_1:

$$F(\theta_1^I \mid \eta_1)(b^I - \theta_1^I) - F(\theta_2^I \mid \eta_2)(b^I - \theta_2^I) > 0.$$

Next we argue the properties of S's rent. Define the S's rent in η_1 as $u_S(\theta_1, \theta_2)$ such as

$$u_S(\theta_1, \theta_2) \equiv F(\theta_1 \mid \eta_1)(\min\{h_1(\theta_1), \bar{h}_2(\theta_2)\} - \theta_1) - F(\theta_2 \mid \eta_2)(\min\{h_1(\theta_1), \bar{h}_2(\theta_2)\} - \theta_2).$$

For (θ_1, θ_2) which satisfies $F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$, we obtain the following properties of $u_S(\theta_1, \theta_2)$:

(i) $u_S(\theta_1, \theta_2)$ is decreasing in θ_1.

(ii)$u_S(\theta_1, \theta_2)$ is decreasing in θ_2 on the region of (θ_1, θ_2) such that $h_1(\theta_1) > h_2(\theta_2)$ and increasing in θ_2 on the region of (θ_1, θ_2) such that $h_1(\theta_1) < h_2(\theta_2)$.

These properties are obtained with the derivative of $u_S(\theta_1, \theta_2)$ with respect to each of θ_1 and θ_2:

- Consider (θ_1, θ_2) such that $h_1(\theta_1) < h_2(\theta_2)$ and $\theta_2 < \bar{\theta}_2$. It implies $b = h_1(\theta_1)$ and

$$\partial u_S(\theta_1, \theta_2)/\partial \theta_2 = -f(\theta_2 \mid \eta_2)[h_1(\theta_1) - h_2(\theta_2)] > 0$$

$$\partial u_S(\theta_1, \theta_2)/\partial \theta_1 = -[F(\theta_2 \mid \eta_2) - F(\theta_1 \mid \eta_1)]h_1^{'}(\theta_1) < 0$$

- Consider (θ_1, θ_2) such that $h_1(\theta_1) > h_2(\theta_2)$ and $\theta_2 < \bar{\theta}_2$. It implies $b = h_2(\theta_2)$ and

$$\partial u_S(\theta_1, \theta_2)/\partial \theta_1 = f(\theta_1 \mid \eta_1)[h_2(\theta_2) - h_1(\theta_1)] < 0$$

$$\partial u_S(\theta_1, \theta_2)/\partial \theta_2 = -[F(\theta_2 \mid \eta_2) - F(\theta_1 \mid \eta_1)]h_2^{'}(\theta_2) < 0$$

- Consider the case of $\theta_2 = \bar{\theta}_2$. Then $b = h_1(\theta_1)$, and

$$\partial u_S(\theta_1, \bar{\theta}_2)/\partial \theta_1 = [F(\theta_1 \mid \eta_1) - 1]h_1^{'}(\theta_1) < 0.$$

As a next step, we will show that (a) $h_1(\theta_1^I) = h_2(\theta_2^I)$ and $\theta_2^I < \bar{\theta}_2$ or (b) $h_1(\theta_1^I) \geq h_2(\bar{\theta}_2)$ and $\theta_2^I = \bar{\theta}_2$ in the optimal WCP allocation of interim collusion. The P's payoff is represented by

$$\Pi(\theta_1, \theta_2) \equiv p_1 F(\theta_1 \mid \eta_1)(V - \theta_1) + p_2 F(\theta_2 \mid \eta_2)(V - \theta_2) - p_1 u_S(\theta_1, \theta_2).$$

Since $u_S(\theta_1, \theta_2)$ is decreasing in θ_1,

$$\partial\Pi(\theta_1, \theta_2)/\partial\theta_1 > p_1 f(\theta_1 \mid \eta_1)[V - h_1(\theta_1)]$$

implying $\theta_1^I > \theta_1^{SB}$ if $\theta_1^{SB} < \bar{\theta}_1$. From the property (ii), θ_2^I must satisfy $V < h_2(\theta_2^I) \leq h_1(\theta_1^I)$, since $\partial\Pi(\theta_1^I, \theta_2)/\partial\theta_2 > 0$ (or < 0) for $h_2(\theta_2) \leq V$ (or $h_2(\theta_2) > h_1(\theta_1^I)$).

Suppose $\theta_2^I < \bar{\theta}_2$. Then $h_2(\theta_2^I) \leq h_1(\theta_1^I)$ implies $b^I = h_2(\theta_2^I) > V$. If $h_2(\theta_2^I) < h_1(\theta_1^I)$, the P's payoff is improved with the decrease of θ_1 from θ_1^I, since

$$[p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2^I \mid \eta_2)][V - b^I] - F(\theta_2^I \mid \eta_2)[\theta_2^I - b^I]$$

is decreasing in θ_1. These arguments imply that (θ_1^I, θ_2^I) satisfies either (a) or (b).

Finally we examine that the optimal WCP allocation is achieved by pure delegation to S. From the above arguments, $(\theta_1^I, \theta_2^I, b^I, X_0^I)$ must satisfy either (a) $b^I = h_1(\theta_1^I) = h_2(\theta_2^I)$, $X_0^I = F(\theta_2^I \mid \eta_2)(\theta_2^I - h_2(\theta_2^I)) < 0$ and $\theta_2^I < \bar{\theta}_2$ or (b)$b^I = h_1(\theta_1^I) > h_2(\bar{\theta}_2)$, $X_0^I = \bar{\theta}_2 - h_1(\theta_1^I) < 0$ and $\theta_2^I = \bar{\theta}_2$. Suppose a prime contract where P pays $\bar{X}_0$ and $\bar{X}_1$ to S for each of the output 0 and 1. With the pure delegation to S, S will select (θ_1, θ_2) which maximizes

$$F(\theta_1 \mid \eta_1)(\bar{X}_1 - \theta_1) + [1 - F(\theta_1 \mid \eta_1)]\bar{X}_0$$

for η_1 and

$$F(\theta_2 \mid \eta_2)(\bar{X}_1 - \theta_2) + [1 - F(\theta_2 \mid \eta_2)]\bar{X}_0$$

for η_2 unless the maximum value is negative. It is easy to show that the above allocation can be implemented as a solution of these problems with the selection of $(\bar{X}_1, \bar{X}_0) = (X_0 + b, X_0)$. It implies (ii) in the lemma. ■

Now using Lemma 2 and 3, we show the proof of Proposition 11.

Proof of Proposition 11(i)

With Lemma 3, if interim and ex-ante solutions differ, interim solution can be attained via pure delegation. In order to show the converse, suppose that the interim solution is attained via pure delegation, but the interim and ex-ante solutions are the same. It implies that the coalitional participation constraint ($X_0^I \geq 0$) is satisfied in the interim solution, and the optimal WCP allocation in the ex-ante collusion is attained with the pure delegation to S. But since the pure delegation always bring a lower payoff than the organization in the absence of S whenever $V > 0$, we obtain a contradiction. ■

Proof of Proposition 11(ii) and (iii)

Step 1

To prove (ii) and (iii), we start with the proof of relatively more general statement:

(i) Suppose that $a(\eta_1 \mid \bar{\theta}_1) > 0$. Then if $h_1(\bar{\theta}_1) > h_2(\bar{\theta}_2)$, there exists the region of a non-degenerate V (which includes the interval $(p_1 h_1(\bar{\theta}_1) + p_2 h_2(\bar{\theta}_2), h_1(\bar{\theta}_1))$) such that the pure delegation to S is optimal in the interim collusion.

(ii) If $F(\theta_1^{SB} \mid \eta_1) \geq F(\theta_2^{SB} \mid \eta_2)$ for any V, the interim collusion and the ex-ante collusion induce the same optimal WCP allocation.

We will show the proof of this statement. With $a(\eta_1 \mid \bar{\theta}_1) > 0$, $h_1(\bar{\theta}_1)$ is bounded above. Since $h_1(\bar{\theta}_1) > h_2(\bar{\theta}_2)$, $p_1 h_1(\bar{\theta}_1) + p_2 h_2(\bar{\theta}_2) < h_1(\bar{\theta}_1)$. Suppose $V \in (p_1 h_1(\bar{\theta}_1) + p_2 h_2(\bar{\theta}_2), h_1(\bar{\theta}_1))$. Then $h_1(\bar{\theta}_1) > V > h_2(\bar{\theta}_2)$ implies $\theta_1^{SB} < \bar{\theta}_1$ and $\theta_2^{SB} = \bar{\theta}_2$, and also $F(\theta_1^{SB} \mid \eta_1) < F(\theta_2^{SB} \mid \eta_2)$. First let us consider the maximization of $\bar{P}^I$. If $(\theta_1, \theta_2) = (\bar{\theta}_1, \bar{\theta}_2)$ is selected in this problem, the value of the objective function is $V - 1$. But it is shown that the maximum value is larger than $V - 1$. Now let us consider a small decrease in θ_1 from $\bar{\theta}_1$, taking $\theta_2 = \bar{\theta}_2$ as given. If the decrease of θ_1 is sufficiently small, $h_1(\theta_1) > h_2(\bar{\theta}_2)$, $F(\theta_1 \mid \eta_1) < F(\bar{\theta}_2 \mid \eta_1)$ and $\theta_1 < \bar{\theta}_2$, implying that it does not violate the constraints of the problem $\bar{P}^I$, making the value of the objective function:

$$[p_1 F(\theta_1 \mid \eta_1) + p_2][V - h_1(\theta_1)] + h_1(\theta_1) - 1.$$

This converges to $V - 1$, as θ_1 approaches to $\bar{\theta}_1$. Let us consider the derivative of this function with respect of θ_1:

$$p_1 f(\theta_1 \mid \eta_1)(V - h_1(\theta_1)) + p_1[1 - F(\theta_1 \mid \eta_1)]h_1^{'}(\theta_1).$$

By the assumption that $a(\eta_1 \mid \bar{\theta}_1) > 0$ and $\partial a(\eta_1 \mid \theta_1)/\partial \theta_1 \mid_{\theta=\bar{\theta}_1}$ exists, $h_1^{'}(\bar{\theta}_1)$ is bounded above. Evaluated at $\theta_1 = \bar{\theta}_1$, it is $p_1 f(\bar{\theta}_1 \mid \eta_1)(V - h_1(\bar{\theta}_1)) < 0$. It implies that there exists $\theta_1 < \bar{\theta}_1$ which improve the objective function above $V - 1$.

Next we will show that $(\theta_1, \theta_2) = (\bar{\theta}_1, \bar{\theta}_2)$ is the optimal solution of P_E. In order to show it, we show that $\bar{P}_E$ does not have solution. Suppose that it has a solution $(\theta_1^E, \theta_2^E, b^E)$. It is evident that it satisfies the constraints $\theta_1^E \leq \theta_2^E$ and

$$F(\theta_1^E \mid \eta_1) > F(\theta_2^E \mid \eta_2).$$

Since

$$F(\theta_2^E \mid \eta_2)(b^E - \theta_2^E) - F(\theta_2^E \mid \eta_2)(b^E - \theta_2^E) \geq 0$$

from $b^E = \max\{B(\theta_1^E, \theta_2^E), l_1(\theta_1^E), \bar{l}_2(\theta_2^E)\}$. Then the maximum value of $\bar{P}_E$ cannot be larger than

$$p_1 F(\theta_1^E \mid \eta_1)(V - \theta_1^E) + p_2 F(\theta_2^E \mid \eta_2)(V - \theta_2^E).$$

Now let us consider the following problem:

$$\max p_1 F(\theta_1 \mid \eta_1)(V - \theta_1) + p_2 F(\theta_2 \mid \eta_2)(V - \theta_2)$$

subject to $(\theta_1, \theta_2) \in [\underline{\theta}_1, \bar{\theta}_1] \times [\underline{\theta}_2, \bar{\theta}_2]$ satisfying

$$\theta_1 \leq \theta_2$$

and

$$F(\theta_1 \mid \eta_1) \geq F(\theta_2 \mid \eta_2).$$

It is evident that this provides upper bound value for the maximum value of $\bar{P}_E$. Since $F(\theta_1^{SB} \mid \eta_1) < F(\theta_2^{SB} \mid \eta_2)$, $F(\theta_1 \mid \eta_1) \geq F(\theta_2 \mid \eta_2)$ would be binding. Let us define $\hat{\theta}_2(\theta_1)$ as θ_2 which satisfies $F(\theta_1 \mid \eta_1) = F(\theta_2 \mid \eta_2)$ for each θ_1. It is evident that $\hat{\theta}_2(\theta_1) \geq \theta_1$, $\hat{\theta}_2(\bar{\theta}_1) = \bar{\theta}_2$ and

$$\hat{\theta}'(\theta_1) = \frac{f(\theta_1 \mid \eta_1)}{f(\theta_2 \mid \eta_2)} > 0.$$

The problem reduces to the maximization of

$$p_1 F(\theta_1 \mid \eta_1)(V - \theta_1) + p_2 F(\hat{\theta}_2(\theta_1) \mid \eta_2)(V - \hat{\theta}_2(\theta_1))$$

subject to $\theta_1 \in [\underline{\theta}_1, \bar{\theta}_1]$. The derivative of the objective function with respect to θ_1 is

$$f(\theta_1 \mid \eta_1)[V - p_1 h_1(\theta_1) - p_2 h_2(\hat{\theta}_2(\theta_1))].$$

Since $p_1 h_1(\theta_1) + p_2 h_2(\hat{\theta}_2(\theta_1))$ is increasing in θ_1 and $V - p_1 h_1(\bar{\theta}_1) - p_2 h_2(\hat{\theta}_2(\bar{\theta}_1)) > 0$ by the assumption, the problem has the solution $\theta_1 = \bar{\theta}_1$, bringing the maximum value $V - 1$. But this upper bound value $V - 1$ can be approached arbitrarily in the problem $\bar{P}_E$, implying that $\bar{P}_E$ does not have the solution.

This argument implies that the interim collusion has a different solution from the ex-ante collusion and the pure delegation to S is optimal in the interim collusion. It completes the proof of part (i).

Suppose that $F(\theta_1^{SB} \mid \eta_1) \geq F(\theta_2^{SB} \mid \eta_2)$ for any V. It means that $h_2(\theta_2) > h_1(\theta_1)$ for any (θ_1, θ_2) such that $F(\theta_1 \mid \eta_1) < F(\theta_2 \mid \eta_2)$. On the other hand, if $(\theta_1^I, \theta_2^I) \neq (\theta_1^E, \theta_2^E)$, (θ_1^I, θ_2^I) must satisfy $F(\theta_1^I \mid \eta_1) < F(\theta_2^I \mid \eta_2)$, and $h_1(\theta_1^I) = h_2(\theta_2^I)$ if $\theta_2^I \leq \bar{\theta}_2$ and $h_1(\theta_1^I) \geq h_2(\bar{\theta}_2)$ if $\theta_2^I = \bar{\theta}_2$. But this is not possible. Therefore the solution in the interim collusion cannot differ from that in the ex-ante collusion. It completes the proof of part (ii).

Step 2

(iii) of the proposition is evident from (i), since $a(\eta_1 \mid \bar{\theta}) > 0$ and $h_1(\bar{\theta} \mid \eta_1) > h_2(\bar{\theta} \mid \eta_2)$ from Assumption 1. Now consider the partition information structure with θ distributed uniformly and $c \in (0, 2/3)$. Then a pair of the second best thresholds (which satisfy $h_i(\theta_i^{SB}) = V$ for $V \in (h_i(\underline{\theta}_i), h_i(\bar{\theta}_i))$, $\theta_i^{SB} = \underline{\theta}_i$ for $V \leq h_i(\underline{\theta}_i)$ and $\theta_i^{SB} = \bar{\theta}_i$ for $V \geq h_i(\bar{\theta}_i)$)) is

$$\theta_1^{SB} = \min\{V/2, c\}$$

$$\theta_2^{SB} = \max\{c, \min\{(V+2)/2, 1\}\}.$$

Then for any $V > 0$, we have $F(\theta_1^{SB} \mid \eta_1) = \frac{\theta_1^{SB}}{c} \geq \frac{\theta_2^{SB} - c}{1-c} = F(\theta_2^{SB} \mid \eta_2)$ if c is in $(0, 2/3)$. It completes the proof of (ii) of the proposition. ■