Supplement to ‘Ex Ante Collusion and Design of Supervisory Institutions’

Dilip Mookherjee, Alberto Motta and Masatoshi Tsumagari[1]

This Version: August 28, 2017

Contents

[1]Boston University, University of New South Wales and Keio University, respectively

1 Introduction

This material includes some arguments which supplement our paper 'Ex Ante Collusion and Design of Supervisory Institutions'. Some proofs, which are omitted in the paper, are also provided in this note.

2 Justification for EACP Allocations

In this section, we provide a justification for focusing attention on EACP allocations. We use the notion of Perfect Bayesian Equilibrium (PBE) of the subgame (C3) that follows any choice of a grand contract by P.[2] As there are typically multiple PBEs of the continuation game following any given GC offer, we need to specify how these might be selected.

If the mechanism design problem is stated as selection of an allocation by the Principal subject to the constraint that it can be achieved as the outcome of some PBE following a choice of a grand contract, it is presumed that the Principal is free to select continuation beliefs and strategies for noncooperative play of the grand contract following off-equilibrium path rejections of offered side contracts by S to A. It can be shown that in such a setting the problem of collusion can be completely overcome by the Principal, with appropriate selection of off-equilibrium-path continuations.

A heuristic description of how the second-best payoff can be achieved by the Principal as a PBE is as follows. P selects a grand contract and recommends a noncooperative equilibrium of this contract in which (i) conditional on participation by S, noncooperative play results in the second-best allocation; (ii) S is paid nothing; and (iii) if S does not participate, P offers A a 'gilded' contract providing the latter a high payoff in all states. On the equilibrium path S always offers a null side contract. If A rejects any offer of a non-null side-contract, they mutually believe that subsequently S will not participate in the grand contract, and A will receive the gilded contract. This forms a PBE as rejection of any non-null side contract is sequentially rational for A given A's belief that S will exit following any rejection. And exit is sequentially rational for S given his belief that A will reject the side contract and they will subsequently play the grand contract noncooperatively where S will be paid nothing. More formally this is shown in the following statement.

[2] For definition of PBE, see Fudenberg and Tirole (1991).

Proposition 1 *The second-best allocation can be achieved as the outcome of a PBE of the collusion game.*

Proof of Proposition 1: For second best allocation $(u_A^{SB}, u_S^{SB}, q^{SB})$, let us construct the following grand contract which is a revelation mechanism satisfying

$$(X_A(m_A, m_S), X_S(m_A, m_S), q(m_S, m_A); M_S, M_A)$$

where $M_S = \Pi \cup \{e_S\}$ and $M_A = \Theta \cup \{e_A\}$.

(i) $X_S(m_A, m_S) = 0$ for any (m_A, m_S).

(ii) $q(\theta, \eta) = q^{SB}(\theta, \eta)$ and $X_A(\theta, \eta) = \theta q^{SB}(\theta, \eta) + u_A^{SB}(\theta, \eta)$, if $(m_A, m_S) = (\theta, \eta) \in K$, otherwise both are set equal to zero.

(iii) $X_A(e_A, m_S) = q(e_A, m_S) = 0$ for any m_S.

(iv) $(X_A(\theta, e_S), q(\theta, e_S)) = (\hat{X}_A(\theta), \hat{q}(\theta))$, which satisfies the following properties: (a) $\hat{X}_A(\theta) - \theta\hat{q}(\theta) \geq \hat{X}_A(\theta') - \theta\hat{q}(\theta')$ for any $\theta, \theta' \in \Theta$, (b) $\hat{X}_A(\theta) - \theta\hat{q}(\theta) \geq 0$ for any $\theta \in \Theta$ and (c) there exists $\theta' \in \Theta$ such that $\hat{q}(\theta') = q(\theta, \eta)$ and $\hat{X}_A(\theta') > X_A(\theta, \eta)$ for any $(\theta, \eta) \in \Theta \times \Pi$.[3]

For this grand contract, we will check that the second best allocation is achieved in PBE of collusion game. In Bayesian game induced by this grand contract, both $(m_A(\theta, \eta), m_S(\eta)) = (\theta, \eta)$ and $(m_A(\theta, \eta), m_S(\eta)) = (\theta, e_S)$ are non-cooperative equilibria, regardless of S's belief about θ. Let our focus be provided to PBE such that $(m_A(\theta, \eta), m_S(\eta)) = (\theta, \eta)$ is realized in the event that a side-contract (SC) is not offered by S, while that $(m_A(\theta, \eta), m_S(\eta)) = (\theta, e_S)$ is realized in the event that SC is offered by S and is rejected by A. In the latter case, A earns $\hat{X}_A(\theta) - \theta\hat{q}(\theta)$. In order to check that S does not benefit from offering a non-null side-contract, let us consider the following problem:

$$\max E[X_A(\tilde{m}(\theta, \eta)) + X_S(\tilde{m}(\theta, \eta)) - \theta q(\tilde{m}(\theta, \eta)) - \tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to $\tilde{m}(\theta, \eta) \in \Delta(M_A \times M_S)$,

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta) q(\tilde{m}(\theta', \eta))$$

[3] For instance, we can choose $(\hat{X}_A(\theta), \hat{q}(\theta))$ such that (i) $\hat{q}(\theta)$ is continuous and strictly decreasing in θ with $\hat{q}(\underline{\theta}) = \max_{(\theta,\eta)\in\Theta\times\Pi} q(\theta, \eta)$ and $\hat{q}(\bar{\theta}) = \min_{(\theta,\eta)\in\Theta\times\Pi} q(\theta, \eta)$, and (ii) $\hat{X}_A(\theta) = \theta\hat{q}(\theta) + \int_\theta^{\bar{\theta}} \hat{q}(y)dy + R$ for sufficiently large $R > 0$

for any $\theta, \theta^{'} \in \Theta(\eta)$ and

$$\tilde{u}_A(\theta, \eta) \geq \hat{X}_A(\theta) - \theta\hat{q}(\theta)$$

for any (θ, η). By the construction of $(\hat{X}_A(\theta), \hat{q}(\theta))$ in (iv), $\tilde{m}(\theta, \eta) = (\theta, e_S)$ (meaning probability measure with concentration on (θ, e_S)) and $\tilde{u}_A(\theta, \eta) = \hat{X}_A(\theta) - \theta\hat{q}(\theta)$ solve this problem. Then the maximum value is equal to zero. Since A at least receives $\hat{X}_A(\theta) - \theta\hat{q}(\theta)$ in the continuation game for non-null side-contract, this maximum value provides an upper bound of S's payoff in PBE from offering non-null side-contract. It means that S never benefits from offering non-null side-contract. Consequently, there is a PBE of this game in which S never offers any side contract. This implies that S and A play $(m_A(\theta, \eta), m_S(\eta)) = (\theta, \eta)$ and the second-best allocation is achieved, concluding the statement of the proposition. ∎

Collusion is overcome by the Principal here by exploiting a lack of coordination among A and S over continuation beliefs and play of the side contracting game. This denies the essence of collusive activity, which involves coordination by the colluding parties 'behind the Principal's back'. The concept of collusion proofness incorporates this by allowing S and A to collectively coordinate on the choice of side-contracting equilibria that are Pareto-undominated (for the coalition) relative to the given status quo.

Definition 1 *Following selection of a grand contract by P, a PBE(c) is a Perfect Bayesian Equilibrium (PBE) of the subsequent subgame (C3) with the following property. There does not exist some signal realization η for which there is some other Perfect Bayesian Equilibrium (PBE) of (C3) in which (conditional on η) S's payoff is strictly higher and A's payoff not lower for any type $\theta \in \Theta(\eta)$.*

Definition 2 *An allocation (u_A, u_S, q) is EAC feasible if there exists a grand contract and a PBE(c) of the subsequent game which results in this allocation.*

We now show that the PBE(c) refinement corresponds to EACP allocations that satisfy interim participation constraints. Note that the PBE(c) notion allows for collusion to occur (i.e., a non-null side contract to be offered and accepted by some types of A), and also for side-contract offers to be rejected by some types of A, both on and off the equilibrium path. Hence the EACP notion does not rest on arbitrary restrictions on side contract outcomes, e.g., which rule out the possibility of equilibrium-path rejections by A of the side contract offered by S. The problem discussed by Celik and Peters (2011) therefore does not apply to this setting.[4] Moreover, we show the restriction to EACP allocations which correspond to equilibrium outcomes in which collusion does not occur on the equilibrium path, is also without loss of generality.

Proposition 2 *An allocation (u_A, u_S, q) is EAC feasible if and only if it is a EACP allocation satisfying interim participation constraints*

$$E[u_S(\theta,\eta)|\eta] \geq 0 \textit{ for all } \eta \tag{1}$$

$$u_A(\theta,\eta) \geq 0 \textit{ for all } (\theta,\eta) \tag{2}$$

Proof of Proposition 2

Proof of Necessity

Suppose (u_A, u_S, q) is EAC feasible. Then it satisfies interim participation constraints of A and S. Here we show that it is also a EACP allocation. Suppose not. Then there exists $\eta \in \Pi$ such that $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$ does not solve the side-contracting problem $P(\eta)$. Suppose that $(\tilde{m}^*(\theta \mid \eta), \tilde{u}_A^*(\theta,\eta))$ is the solution of $P(\eta)$. Defining

$$\tilde{u}_S^*(\theta,\eta) \equiv \hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}_A^*(\theta,\eta),$$

we have

$$E[\tilde{u}_S^*(\theta,\eta) \mid \eta] > E[u_S(\theta,\eta) \mid \eta]$$

[4]They show in the context of a model of a two-firm cartel that such restrictions can entail a loss of generality. Rejection of a side contract by some types of A can communicate information to S about A's type, affecting subsequent play and resulting payoffs in the noncooperative play of the grand contract. Celik and Peters show that there can be collusive allocations amongst cartel members which can only be supported by side-contract offers which are rejected with positive probability on the equilibrium path. This problem does not arise in our setting as the side contract is offered by S, and the side contract does not have to satisfy interim participation constraints for S. However, the Celik-Peters problem could conceivably arise in a setting where the side contract is offered by a third party. Even in that context, it turns out that the problem can be overcome with a suitable modification of the side contracting game, as explained in the next Section.

and

$$\tilde{u}_A^*(\theta, \eta) \geq u_A(\theta, \eta)$$

for any $\theta \in \Theta(\eta)$. Since (u_A, u_S, q) is achievable in the collusion game, there exists a grand contract GC and an associated PBE(c) which results in this allocation. Let this PBE involve beliefs $b(\eta)$ and non-cooperative equilibrium $c(\eta)$ of GC based on beliefs $b(\eta)$ resulting if A rejects the side contract $SC(\eta)$ offered on the equilibrium path. The payoff accruing to A in this noncooperative equilibrium then cannot exceed $u_A(\theta, \eta)$ in any state (θ, η).

For $\tilde{m}^*(\theta \mid \eta) \in \Delta(K \cup e)$, there exists $\tilde{m}^c(\theta, \eta) \in \Delta(M_A \times M_S)$ such that

$$(\hat{X}(\tilde{m}^*(\theta \mid \eta)), \hat{q}(\tilde{m}^*(\theta \mid \eta))) = (X_A(\tilde{m}^c(\theta, \eta)) + X_S(\tilde{m}^c(\theta, \eta)), q(\tilde{m}^c(\theta, \eta))).$$

Given GC and η, consider the side-contract $SC^c(\eta)$ in which the report to P is selected according to $\tilde{m}^c(\theta', \eta)$ on the basis of A's report of $\theta' \in \Theta(\eta)$, associated with the transfer to A:

$$t_A^c(\theta', \eta) = \tilde{u}_A^*(\theta', \eta) - [X_A(\tilde{m}^c(\theta', \eta)) - \theta' q(\tilde{m}^c(\theta', \eta)))].$$

Now construct a different Perfect Bayesian Equilibrium (PBE) which differs from the previous one only in state η, where on the equilibrium path S offers instead $SC^c(\eta)$, and this is accepted by all types of A. Rejection of this offer results in the same noncooperative equilibrium $c(\eta)$ of the grand contract. What occurs in the continuation of any other side contract offer remains the same as in the previous PBE. To check this is a PBE, note that it is optimal for A to accept $SC^c(\eta)$, and then report truthfully. Moreover, given that this side contract is accepted by all types of A, it is optimal for S to offer it (since offering $SC(\eta)$ was optimal in state η in the previous PBE).

Hence $(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta))$ can be realized as a PBE outcome. Since S is better off without making any type of A worse off, it contradicts the hypothesis that (u_A, u_S, q) is realized as the outcome of a PBE(c).

Proof of Sufficiency

Step 1: Construction of grand contract

Suppose that (u_A, u_S, q) is a EACP allocation satisfying interim participation constraints. We show that there exists a grand contract which achieves (u_A, u_S, q) as a PBE(c) outcome.

The grand contract is constructed as follows:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(e_A, m_S) = X_S(e_A, m_S) = q(e_A, m_S) = X_S(m_A, e_S) = 0$$

for any (m_A, m_S).

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$
- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$, with T sufficiently large
- $(X_A((\theta_A, \eta_A), e_S), q((\theta_A, \eta_A), e_S)) = (\hat{X}(\tilde{m}^*(\theta_A)), \hat{q}(\tilde{m}^*(\theta_A)))$ where $\tilde{m}^*(\theta)$ maximizes $\hat{X}(\tilde{m}) - \theta\hat{q}(\tilde{m})$ subject to $\tilde{m} \in \Delta(K \cup \{e\})$ and the definition of $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$ is provided in Section 3.5 of the paper.

Step 2: Non-cooperative equilibrium

First we argue $(m_A(\theta, \eta), m_S(\eta)) = ((\theta, \eta), \eta)$ is a non-cooperative equilibrium of the grand contract based on prior beliefs (denoted by $b_\phi(\eta)$) for η. EACP and A's participation constraint imply that A always has an incentive to participate and report truthfully: $m_A(\theta, \eta) = (\theta, \eta)$. Since S's interim participation constraint ($E[u_S(\theta, \eta) \mid \eta] \geq 0$) holds, taking A's strategy $m_A(\theta, \eta) = (\theta, \eta)$ as given, S also has an incentive to participate and report truthfully.

This equilibrium results in allocation $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$. By offering a null side-contract, S can always realize the allocation $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$ and achieve interim payoff $E[u_S(\theta, \eta) \mid \eta]$. Therefore S would have an incentive to offer a non-null side-contract only if the deviation results in a higher payoff. We show that there exists a PBE of (C3) following the GC constructed above, in which S's interim payoff from any deviating side contract offer cannot exceed $E[u_S(\theta, \eta) \mid \eta]$.

Consider any deviating side contract offer in state η, and let $b(\eta)$ denote beliefs of S regarding θ which result following rejection of this side contract by A. S and A then play GC noncooperatively with beliefs $b(\eta)$. By construction, A has an incentive to report truthfully and participate in GC irrespective of what S does, i.e., irrespective of the beliefs $b(\eta)$ held by S (as well as irrespective of the particular deviating side contract offered). If T is sufficiently large, it is then a best response for S to report truthfully, conditional on participating. We focus on PBEs satisfying these two properties following rejection by A of any deviating side contract.

In what follows, there are two cases to consider. (a) $E_{b(\eta)}[u_S(\theta, \eta)] \geq 0$, in which case it is a best response for S to participate (and report truthfully) in GC when it is played noncooperatively with beliefs $b(\eta)$. We refer to this as the T case. (b) $E_{b(\eta)}[u_S(\theta, \eta)] < 0$, whereby S exits from GC following rejection of the side contract and attains zero payoff. We refer to this as the E case.

Step 3: Side-contract choice.

Now we argue that without loss of generality, the choice of deviating side contract can be limited to those where in every state (θ, η): either A and S both participate and submit consistent reports $\eta_A = \eta_S$, or where they both exit. That they should submit consistent reports conditional on joint participation, follows if T is sufficiently large. Suppose there is some state in which the side contract prescribes an exit for S alone. Given the construction of the grand contract, for any $(m_A, m_S) = ((\theta, \eta), e_S)$, there exists $\tilde{m}' \in \Delta(M_A \times M_S \backslash \{(\theta, \eta), e_S\})$ such that

$$(X_A(m_A, m_S) + X_S(m_A, m_S), q(m_A, m_S)) = (X_A(\tilde{m}') + X_S(\tilde{m}'), q(\tilde{m}')),$$

given

$$(X_A((\theta, \eta), e_S) + X_S((\theta, \eta), e_S), q((\theta, \eta), e_S)) = (\hat{X}(\tilde{m}^*(\theta_A)), \hat{q}(\tilde{m}^*(\theta_A)))$$

and the definition of $(\hat{X}, \hat{q})$. Therefore $\tilde{m}'$ and $(m_A, m_S) = ((\theta, \eta), e_S)$ generate the same total payment and output target for the coalition. A similar argument ensures that outcomes involving exit for A alone can be eliminated without loss of generality, since $(m_A, m_S) = (e_A, \eta)$ generates the same outcome $X_A = X_S = q = 0$ in the GC as $(m_A, m_S) = (e_A, e_S)$.

Step 4: Continuation payoffs following non-null side-contract

Suppose that S offers some non-null side-contract SC for η, which is described as $(\tilde{m}(\theta, \eta), \tilde{u}_A(\theta, \eta))$ which satisfies

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta) q(\tilde{m}(\theta', \eta))$$

for any $\theta, \theta' \in \Theta(\eta)$ and $\tilde{m}(\theta, \eta) \in \Delta(\hat{M})$. Let $\kappa^*(\theta) \in [0,1]$ denote the probability that $\theta \in \Theta(\eta)$ accepts SC. We focus on PBE's with the property that A reports truthfully to S conditional on accepting the SC. The inequality above ensures that this is optimal for A. In any such PBE, the payoff resulting for S when A accepts the SC equals (in state (θ, η)):

$$X_A(\tilde{m}(\theta, \eta)) + X_S(\tilde{m}(\theta, \eta)) - \theta q(\tilde{m}(\theta, \eta)) - \tilde{u}_A(\theta, \eta).$$

If A rejects SC, A and S play the grand contract non-cooperatively with belief $b^*(\eta)$, which is consistent with Bayes rule as required in a PBE. Sequential rationality of A's participation decision $\kappa^*(\theta)$, given beliefs $b^*(\eta)$ and the non-cooperative equilibrium associated with $b^*(\eta)$, implies the following. In the T-case, $\kappa^*(\theta) = 0$ (or 1 or $\in [0,1]$) if and only if $u_A(\theta, \eta) >$ (or $<$ or $=$) $\tilde{u}_A(\theta, \eta)$. A ends up with payoff

$$\max\{u_A(\theta, \eta), \tilde{u}_A(\theta, \eta)\},$$

and S's interim payoff is

$$E[\kappa^*(\theta)\{X_A(\tilde{m}(\theta, \eta)) + X_S(\tilde{m}(\theta, \eta)) - \theta q(\tilde{m}(\theta, \eta)) - \tilde{u}_A(\theta, \eta)\} + (1-\kappa^*(\theta))u_S(\theta, \eta) \mid \eta].$$

Conversely, in the E-case, $\kappa^*(\theta) = 0$ (or 1 or $\in [0,1]$) if and only if

$$\hat{X}(\tilde{m}^*(\theta_A)) - \theta \hat{q}(\tilde{m}^*(\theta_A)) > (\text{or} < \text{ or } =) \tilde{u}_A(\theta, \eta).$$

A's payoff is

$$\max\{\hat{X}(\tilde{m}^*(\theta_A)) - \theta \hat{q}(\tilde{m}^*(\theta_A)), \tilde{u}_A(\theta, \eta)\}.$$

while S's interim payoff is

$$E[\kappa^*(\theta)\{X_A(\tilde{m}(\theta, \eta)) + X_S(\tilde{m}(\theta, \eta)) - \theta q(\tilde{m}(\theta, \eta)) - \tilde{u}_A(\theta, \eta)\} \mid \eta].$$

Step 5: Upper bound on S's interim payoff in continuation play following non-null side-contract

Here we establish an upper bound of S's interim payoff in PBE of the continuation game for non-null side-contract.

(i) T-Case

Consider the following problem: select $(\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta))$ to

$$\max E[X_A(\hat{m}(\theta,\eta)) + X_S(\hat{m}(\theta,\eta)) - \theta q(\hat{m}(\theta,\eta)) - \hat{u}_A(\theta,\eta) \mid \eta]$$

subject to $\hat{m}(\theta,\eta) \in \Delta(\hat{M})$,

$$\hat{u}_A(\theta,\eta) \geq \hat{u}_A(\theta^{'},\eta) + (\theta^{'} - \theta)q(\hat{m}(\theta^{'},\eta))$$

for any $\theta, \theta^{'} \in \Theta(\eta)$ and

$$\hat{u}_A(\theta,\eta) \geq u_A(\theta,\eta).$$

for any $\theta \in \Theta(\eta)$.

This is equivalent to problem $P(\eta)$ (in our paper) used to characterize EACP allocations. The EACP property implies that $(\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$ solves this problem and the maximum value is $E[u_S(\theta,\eta) \mid \eta]$.

We now show that this is an upper bound on S's interim payoff from the deviating side contract in the T-case. Suppose that non-null side-contract $(\tilde{m}(\theta,\eta), t(\theta,\eta))$ is associated with acceptance probability $\kappa^*(\cdot)$ and the T-case applies. Select $(\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta))$ as follows:

$$\hat{m}(\theta,\eta) = \kappa^*(\theta)\tilde{m}(\theta,\eta) + (1 - \kappa^*(\theta))I(\theta,\eta)$$

and

$$\hat{u}_A(\theta,\eta) = \max\{\tilde{u}_A(\theta,\eta), u_A(\theta,\eta)\}$$

where $I(\theta,\eta)$ is the probability measure concentrated on (θ,η). In this allocation, A earns exactly the same payoffs as in the continuation following offer of side-contract $(\tilde{m}(\theta,\eta), t(\theta,\eta))$. Hence the agent's incentive constraint is satisfied, and so is the participation constraint by construction. Hence the continuation play following offer of side-contract $(\tilde{m}(\theta,\eta), t(\theta,\eta))$ results in an interim payoff for S which cannot exceed $E[u_S(\theta,\eta) \mid \eta]$.

(ii) E-Case

Now consider the following problem: select $(\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta))$ to

$$\max E[X_A(\hat{m}(\theta,\eta)) + X_S(\hat{m}(\theta,\eta)) - \theta q(\hat{m}(\theta,\eta)) - \hat{u}_A(\theta,\eta) \mid \eta]$$

subject to $\hat{m}(\theta,\eta) \in \Delta(\hat{M})$,

$$\hat{u}_A(\theta,\eta) \geq \hat{u}_A(\theta^{'},\eta) + (\theta^{'} - \theta)q(\hat{m}(\theta^{'},\eta))$$

and

$$\hat{u}_A(\theta,\eta) \geq \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta)).$$

In order to derive the solution of this problem, consider the problem of maximizing

$$X_A(\hat{m}) + X_S(\hat{m}) - \theta q(\hat{m})$$

subject to $\hat{m} \in \Delta(\hat{M})$. Denoting its solution by $\hat{m}^*(\theta)$, we have

$$X_A(\hat{m}^*(\theta)) + X_S(\hat{m}^*(\theta)) - \theta q(\hat{m}^*(\theta)) = \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta)),$$

because of the definition of $(\hat{X},\hat{q})$ and $\tilde{m}^*(\theta)$. Therefore in the above problem, an upper bound of objective function is given by

$$E[X_A(\hat{m}^*(\theta)) + X_S(\hat{m}^*(\theta)) - \theta q(\hat{m}^*(\theta)) - \{\hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta))\}|\eta] = 0$$

This upper bound can be achieved by selecting

$$(\hat{m}(\theta,\eta), \tilde{u}_A(\theta,\eta)) = (\hat{m}^*(\theta), \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta))).$$

Since this also satisfies all the constraints of the problem, this is a solution of the problem. It follows that the maximum value is equal to zero.

Next check that this maximum value provides an upper bound on S's payoff in the continuation play following the offer of the deviating side contract $(\tilde{m}(\theta,\eta), t(\theta,\eta))$ in which the E-case arises. Select $(\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta))$ as follows:

$$\hat{m}(\theta,\eta) = \kappa^*(\theta)\tilde{m}(\theta,\eta) + (1-\kappa^*(\theta))\tilde{m}^*(\theta)$$

and

$$\hat{u}_A(\theta,\eta) = \max\{\tilde{u}_A(\theta,\eta), \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta))\}.$$

This generates the same payoffs for A as in the continuation play following the offer of the deviating side contract $(\tilde{m}(\theta,\eta), t(\theta,\eta))$, and is therefore feasible in the maximization problem above. Hence zero is an upper bound to S's interim expected payoff when the E-case applies.

Step 6: PBE in collusion game

We can construct a PBE in the overall collusion game as follows. If S offers null side-contract, he receives $E[u_S(\theta,\eta) \mid \eta]$. If S offers any non-null side-contract, it follows from Step 5 that his subsequent continuation payoff is not larger than $E[u_S(\theta,\eta) \mid \eta]$. Since $E[u_S(\theta,\eta) \mid \eta] \geq 0$, there exists a PBE in which S offers a null side-contract on the equilibrium path, resulting in allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$.

Step 7: Check PBE(c) property

Finally check that PBE constructed in the above argument is also a PBE(c). Otherwise there would exist a PBE resulting in a Pareto superior allocation for the coalition. This would violate the EACP property of the allocation we started with. ∎

Note, however, that while the EACP allocation satisfying interim participation constraints is the outcome of some PBE(c) in some grand contract designed by P, it is possible that there also exist other PBE(c) resulting in distinct allocations (which are also EACP allocations satisfying participation constraints). Hence any given grand contract may be associated with multiple EACP allocations satisfying participation constraints, that are Pareto-noncomparable. Che and Kim (2009) provide a different definition of collusion proofness, by assuming that players revert to noncooperative play with prior beliefs whenever collusion breaks down. This notion can be compatible with the notion of Weak Perfect Bayesian Equilibrium (WPBE), which is a weaker concept than PBE.[5] PBE requires the beliefs to be based on Bayes rule whenever available even in the continuation of the off-equilibrium side-contract. Similar to PBE(c), we can define WPBE(c) as a WPBE with the property that there does not exist some signal realization η for which there is some other WPBE of (C3) in which (conditional on η) S's payoff is strictly higher and A's payoff not lower for any type $\theta \in \Theta(\eta)$. Then if the noncooperative equilibrium payoffs corresponding to prior beliefs are unique, status quo payoffs for negotiation between A and S over the side contract are pinned down, thereby eliminating multiplicity of corresponding WPBE(c) payoffs satisfying their restriction. However, a disadvantage of this definition is that it would be subject to the Celik-Peters (2011) criticism described above. Nevertheless, both definitions give rise to the same characterization of collusion proof allocations.

3 Justification for EACP Allocations When Contracts are Offered by Third Party

To overcome the problem highlighted by Celik and Peters (2011) in the context where the side-contract is designed by a third party, we model side-

[5] For definition of WPBE, see Mas-Colell, Whinston and Green (1995, p.285).

contracts as a two stage game played by S and A. The first stage is a 'participation' stage where they communicate their participation decisions in the side contract, in addition to some auxiliary messages in the event of agreeing to participate. The role of these messages is to allow A to signal information about his type while agreeing to participate, which can help replicate whatever information is communicated by side-contract rejection in a setting where communication concerning participation decisions is dichotomous. A and S observe the messages sent by each other at the end of the first stage. At the second stage, A and S submit type reports, conditional on having agreed to participate at the first stage.

Let (D_A^p, D_S^p) denote the message sets of A and S at the participation stage (or p-stage). $e_A \in D_A^p$ and $e_S \in D_S^p$ are the exit options of A and S respectively. The message sets at this stage may include other auxiliary messages as well.

What occurs at the second stage ('execution' or e-stage) depends on $d^p = (d_A^p, d_S^p)$ chosen at the first stage.

- If $d_A^p \neq e_A$ and $d_S^p \neq e_S$, A and S select $(d_A^e, d_S^e) \in D_A^e(d^p) \times D_S^e(d^p)$ respectively, where the conditional message sets $D_A^e(d^p), D_S^e(d^p)$ are specified by the side contract. The report to P is selected according to $\tilde{m}(d^p, d^e) \in \Delta(M_A \times M_S)$, associated with the transfers to A and S, $t_A(d^p, d^e)$ and $t_S(d^p, d^e)$ respectively. Owing to wealth constraint of the third party, these are constrained to satisfy $t_A(d^p, d^e) + t_S(d^p, d^e) \leq 0$.

- If either $d_A^p = e_A$ or $d_S^p = e_S$, A and S play GC non-cooperatively.

Given GC and η, the third party decides whether to offer a side-contract $SC(\eta)$ or not (i.e., offer a null side-contract NSC). If a non-null side-contract is offered, A and S play a game denoted by $GC \circ SC(\eta)$ with two stages as described above. On the other hand, if the third party offers a null side-contract NSC at the first stage, A and S play GC non-cooperatively based on prior beliefs $b_\phi(\eta)$. The third-party's objective is to maximize $E[\alpha u_A(\theta, \eta) + (1-\alpha) u_S(\theta, \eta) \mid \eta]$ in state η.

The refinement PBE(c) introduced in the paper for the case where the side contract is offered by S, can now be extended as follows.

Definition 3 *Following the selection of a grand contract by P, a PBE(c) is a Perfect Bayesian Equilibrium (PBE) of the subsequent game in which side-contracts are designed by a third party, which has the following property.*

There does not exist some η for which there is a Perfect Bayesian Equilibrium (PBE) of subgame C3 in which (conditional on η) the third-party's payoff is strictly higher, without lowering the payoff of S and any type of A.

Definition 4 *An allocation (u_A, u_S, q) is EAC feasible when side contracts are designed by a third party assigning welfare weight α to A, if there exists a grand contract and a PBE(c) of the subsequent side contract subgame which results in this allocation.*

Proposition 3 *An allocation (u_A, u_S, q) is EAC feasible when side contracts are designed by a third party assigning welfare weight α to A, if and only if it is a EACP(α) allocation satisfying the interim participation constraints $u_A(\theta, \eta) \geq 0$ and $E[u_S(\theta, \eta) \mid \eta] \geq 0$.*

Proof of Proposition 3

Proof of Necessity

For some GC, suppose that allocation (u_A, u_S, q) is achieved in the game with collusion. Suppose the allocation is achieved as the outcome of a PBE(c) of subgame C3 in which a non-null side contract $SC^*(\eta)$ is offered on the equilibrium path in some state η, which is rejected either by some types of A, or by S. We show it can also be achieved as the outcome of a PBE(c) in which a non-null side contract is offered in state η and always accepted by A and S. Let $d_A^p(\theta, \eta)$ and $d_S^p(\eta)$ denote A and S's participation decisions respectively (whether or not they chose the exit option at the first stage). Following rejection by either A or S, they play the grand contract GC based on updated beliefs $b(\cdot \mid d_A^p(\theta, \eta), d_S^p(\eta), \eta)$. Let $d_A^{p*}(\theta, \eta)$ denote A's decision, and $d_S^{p*}(\eta)$ S's participation decisions on the equilibrium path.

Now construct a new side-contract $\tilde{SC}(\eta)$ which differs from $SC^*(\eta)$ by replacing the message space D_A^p for A at the first stage by $D_A^p \times D_A^p$. Similarly S's message space is now $D_S^p \times D_S^p$. The interpretation is that the first component of this message d_A^p is a participation decision, while the second component $\tilde{d}_A^p$ is a 'signal'. This allows a decoupling of the participation decision from sending a signal to the other player which changes beliefs with which they play the grand contract noncooperatively in the event that the side contract is rejected by someone. For example, if A selected $d_A^p = e_A$ in the previous side-contract in order to send a signal about his type θ to S, the same signal can be sent now through the second component of the message, while opting to participate in the choice of the first component (by selecting $d^p_A \neq e_A, \tilde{d}^p_A = e_A$). The first component of the message d^p_A now matters only insofar as it is an exit decision or not; conditional on it not being an exit decision the precise message does not matter. If both decide to participate (i.e., not exit), they move on to the second stage of the game, where the mechanism replicates the allocation resulting on the equilibrium path of the original PBE associated with $SC^*(\eta)$ (i.e., agrees with the second stage mechanism in $SC^*(\eta)$ whenever both agreed to participate in $SC^*(\eta)$, and otherwise assigns the allocation resulting from noncooperative play of GC in the original PBE). If one or both decides not to participate in $\tilde{SC}(\eta)$, they play GC noncooperatively with beliefs based on first stage messages according to $b(\cdot \mid \tilde{d}^p_A(\theta,\eta), \tilde{d}^p_S(\eta), \eta)$. Note that these beliefs do not depend on d^p_A or d^p_S.

It is easily verified that there exists a PBE where the third party offers $\tilde{SC}(\eta)$ in state η, in which A and S always accept the side-contract (i.e., in state (θ,η) they respectively select $d^p_A(\theta,\eta) \neq e_A, d^p_S(\eta) \neq e_S$ while choosing $\tilde{d}^p_A(\theta,\eta)$ equal to $d^p_A(\theta,\eta)$ in the original PBE, and $\tilde{d}^p_S(\eta)$ equal to $d^p_S(\eta)$ in the original PBE). The underlying idea is that since A's first stage report $\tilde{d}^p_A$ now affects beliefs at the second stage in exactly the same way that d^p_A did in the original PBE, it is optimal for A to choose $\tilde{d}^p_A(\theta,\eta)$ equal to $d^p_A(\theta,\eta)$ in the original PBE. Moreover, the first stage d^p_A report now affects only A's participation decision at the second stage, and by construction has no effect on second stage allocations (conditional on participation). So it is optimal for A to decide to participate. The same logic applies to S. Hence the newly constructed strategies and beliefs constitute a PBE. It can also be verified that since the original equilibrium was a PBE(c), so is the newly constructed equilibrium.

Next we show that if allocation (u_A, u_S, q) is realized in a PBE (c) in which the offered side contract is not rejected on the equilibrium path, it must be a EACP(α) allocation. Suppose not: the allocation resulting from some non-null side contract $(\tilde{u}^*_A(\theta,\eta), \tilde{m}^*(\theta,\eta)) \neq (u_A(\theta,\eta), (\theta,\eta))$ solves problem $TP(\eta;\alpha)$ for some η. Define $\tilde{u}^*_S(\theta,\eta) \equiv \hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}^*_A(\theta,\eta)$. It is evident that

$$E[\alpha\tilde{u}^*_A(\theta,\eta) + (1-\alpha)\tilde{u}^*_S(\theta,\eta) \mid \eta] > E[\alpha u_A(\theta,\eta) + (1-\alpha)u_S(\theta,\eta) \mid \eta],$$
$$\tilde{u}^*_A(\theta,\eta) \geq u_A(\theta,\eta)$$

and

$$E[\tilde{u}^*_S(\theta,\eta) \mid \eta] \geq E[u_S(\theta,\eta) \mid \eta].$$

There exists $m^c(\theta, \eta) \in \Delta(M_A \times M_S)$ in GC such that

$$(X_A(m^c(\theta, \eta)) + X_S(m^c(\theta, \eta)), q(m^c(\theta, \eta))) = (\hat{X}(\tilde{m}^*(\theta \mid \eta)), \hat{q}(\tilde{m}^*(\theta \mid \eta))).$$

Now construct a new side-contract $SC(\eta)$ which realizes

$$(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta), \hat{q}(\tilde{m}^*(\theta \mid \eta)))$$

as a PBE outcome, contradicting the hypothesis that (u_A, u_S, q) is realized in a PBE (c). $SC(\eta)$ is specified as follows:

- $D^p \equiv D^{p*}$ where $D^{p*} = (D_A^{p*}, D_S^{p*})$ are A and S's message sets at the participation stage of the original side-contract $SC^*(\eta)$.
- $D_A^e = \Theta(\eta)$ and $D_S^e = \phi$
- A's choice of $d_A^e = \theta \in \Theta(\eta)$ generates the report $m^c(\theta, \eta)$ to P, and side transfers to A and S respectively as follows:

$$t_A(\theta, \eta) = \tilde{u}_A^*(\theta, \eta) - [X_A(m^c(\theta, \eta)) - \theta q(m^c(\theta, \eta))]$$

  and

$$t_S(\theta, \eta) = \tilde{u}_S^*(\theta, \eta) - X_S(m^c(\theta, \eta)).$$

Given any (d_A^p, d_S^p) with $d_A^p \neq e_A$ and $d_S^p \neq e_S$ at the participation stage, it is optimal for A to always select $d_A^e = \theta$, since $\theta' = \theta$ maximizes

$$X_A(m^c(\theta', \eta)) - \theta q(m^c(\theta', \eta)) + t_A(\theta', \eta) = \tilde{u}_A^*(\theta', \eta) + (\theta' - \theta)\hat{q}(\tilde{m}^*(\theta' \mid \eta)).$$

At the participation stage, A is indifferent among any $d_A^p \in D_A^p \backslash \{e_A\}$ as the optimal response to $d_S^p \neq e_S$, since the outcome in the continuation game does not depend on this choice. Select beliefs consequent on non-participation by either A or S in the same way as in the original equilibrium; then participation continues to be optimal for both. In state η, responses to all other side contract offers are unchanged. In all other states $\eta' \neq \eta$, strategies and beliefs are unchanged. Hence this is a PBE resulting in $(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta))$, contradicting the PBE(c) property of the equilibrium resulting in (u_A, u_S, q). This completes the proof of necessity.

Proof of Sufficiency

Take an allocation which is EACP(α) and satisfies interim participation constraints. We show it is achievable as a PBE(c) outcome following choice of the following grand contract GC:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(m_A, m_S) = X_S(m_A, m_S) = q(m_A, m_S) = 0$$

for (m_A, m_S) such that either $m_A = e_A$ or $m_S = e_S$.

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$ for $\eta_A = \eta_S$ and $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (-T, 0)$ for $\eta_A \neq \eta_S$

- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$

where $T > 0$ is sufficiently large. The EACP(α) property implies that $u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$ for any $\theta, \theta' \in \Theta(\eta)$. The interim participation constraints imply that this grand contract has a non-cooperative pure strategy equilibrium

$$(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$$

based on prior beliefs.

For this grand contract, we claim there exists a PBE(c) resulting in $(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$. Let the third party offer a null side contract, following which A and S play truthfully in the GC noncooperatively with prior beliefs. If the third party offers any non-null side contract, all types of A and S reject it and subsequently play truthfully in the noncooperative game with prior beliefs. This is clearly a PBE. That it is a PBE(c) follows from the property that the allocation is EACP(α). ■

4 Proof of Results Used in Proving Propositions 6 and 7

We first prove the following result invoked in the proof of Proposition 6 in the Appendix to the paper.

Result 1 *There exists $z(\cdot \mid \eta^{**}) \in Z(\eta^{**})$ which satisfies the following conditions.*

(B-i) $z(\theta \mid \eta^{**}) = \theta$ *for any* $\theta \notin \Theta_H \cup \Theta_L$

(B-ii) For $\theta \in \Theta_L$, $z(\theta \mid \eta^{**})$ *satisfies (a)* $z(\theta \mid \eta^{**}) \leq \theta$ *with strict inequality for some subinterval of* Θ_L *of positive measure, and (b)* $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) > 0$ *for any* $z \in [z(\theta \mid \eta^{**}), \theta]$.

(B-iii) For $\theta \in \Theta_H$, $z(\theta \mid \eta^{**})$ *satisfies (a)* $z(\theta \mid \eta^{**}) \geq \theta$ *with strict inequality for some some subinterval of* Θ_H *of positive measure, (b)* $z(\theta \mid \eta^{**}) < h(\theta \mid \eta^{**})$ *and (c)* $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) < 0$ *for any* $z \in [\theta, z(\theta \mid \eta^{**})]$.

(B-iv) $E[(z(\theta \mid \eta^{**}) - h(\theta \mid \eta^{**}))q^{NS}(z(\theta \mid \eta^{**})) + \int_{z(\theta|\eta^{**})}^{\bar{\theta}(\eta^{**})} q^{NS}(z)dz \mid \eta^{**}] = 0.$

Proof:

Step A: For any $\eta \in \Pi$ *and any closed interval* $[\theta', \theta''] \subset \Theta(\eta)$ *such that* $\underline{\theta}(\eta) < \theta' < \theta'' < \bar{\theta}(\eta)$, *there exists* $\delta > 0$ *such that* $z(\cdot) \in Z(\eta)$ *for any* $z(\cdot)$ *satisfying the following properties:*

(i) $z(\theta)$ *is increasing and differentiable with* $|z(\theta) - \theta| < \delta$ *and* $|z'(\theta) - 1| < \delta$ *for any* $\theta \in \Theta(\eta)$

(ii) $z(\theta) = \theta$ *for any* $\theta \notin [\theta', \theta'']$.

Proof of Step A

For arbitrary $\eta \in \Pi$ and arbitrary closed interval $[\theta', \theta''] \subset \Theta(\eta)$ such that $\underline{\theta}(\eta) < \theta' < \theta'' < \bar{\theta}(\eta)$, we choose ϵ_1 and ϵ_2 such that

$$\epsilon_1 \equiv \min_{\theta \in [\theta', \theta'']} f(\theta \mid \eta)$$

and

$$\epsilon_2 \equiv \max_{\theta \in [\theta', \theta'']} |f'(\theta \mid \eta)|.$$

From our assumptions that $f(\theta \mid \eta)$ is continuously differentiable and positive on $\Theta(\eta)$, $\epsilon_1 > 0$, and ϵ_2 is positive and bounded above. We choose $\delta > 0$ such that

$$\delta \in (0, \frac{\epsilon_1}{\epsilon_1 + \epsilon_2}).$$

For this δ, it is obvious that there exists $z(\theta)$ which satisfies conditions (i) and (ii) of the statement. Define

$$\Lambda(\theta \mid \eta) \equiv (\theta - z(\theta))f(\theta \mid \eta) + F(\theta \mid \eta).$$

Since $z(\theta)$ is differentiable on $\Theta(\eta)$, $\Lambda(\theta \mid \eta)$ is also so. It is equal to $\Lambda(\theta \mid \eta) = F(\theta \mid \eta)$ on $\theta \notin [\theta', \theta'']$. For $\theta \in [\theta', \theta'']$,

$$\begin{aligned} \frac{\partial \Lambda(\theta \mid \eta)}{\partial \theta} &= (2 - z'(\theta))f(\theta \mid \eta) + (\theta - z(\theta))f'(\theta \mid \eta) > (1-\delta)f(\theta \mid \eta) - \delta|f'(\theta \mid \eta)| \\ &\geq (1-\delta)\epsilon_1 - \delta\epsilon_2. \end{aligned}$$

This is positive by the definition of $(\epsilon_1, \epsilon_2, \delta)$. Then $\Lambda(\theta \mid \eta)$ is increasing in θ on $\Theta(\eta)$ with $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ and $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$. Since $z(\theta)$ is increasing in θ by the definition, it is preserved even by ironing rule. Therefore $z(\cdot) \in Z(\eta)$. ■

Step B

For η^{**} and the closed interval $I = [\theta', \theta''] \subset \Theta(\eta^{**})$ which are selected in Step 1 of the paper's Appendix, we select $\delta > 0$ according to the procedure in Step A. By the continuity of $\frac{F(\theta)}{f(\theta)}$ and $\frac{F(\theta \mid \eta^{**})}{f(\theta \mid \eta^{**})}$ and the closedness of Θ_L and Θ_H, we can select $\epsilon > 0$ such that

$$\lambda < [\frac{F(\theta)}{f(\theta)} - \epsilon] / \frac{F(\theta \mid \eta^{**})}{f(\theta \mid \eta^{**})} \ \text{ for } \theta \in \Theta_L \equiv [\underline{\theta}^L, \bar{\theta}^L]$$

$$\lambda > [\frac{F(\theta)}{f(\theta)} + \epsilon] / \frac{F(\theta \mid \eta^{**})}{f(\theta \mid \eta^{**})} \ \text{ for } \theta \in \Theta_H \equiv [\underline{\theta}^H, \bar{\theta}^H].$$

These conditions are equivalent to

$$H(\theta) - (1-\lambda)\theta - \lambda h(\theta \mid \eta^{**}) > \epsilon \ \text{ for } \theta \in \Theta_L$$

and

$$H(\theta) - (1-\lambda)\theta - \lambda h(\theta \mid \eta^{**}) < -\epsilon \text{ for } \theta \in \Theta_H.$$

By the continuity of $H(\theta) - (1-\lambda)\theta$ for θ and closedness of Θ_L and Θ_H, there exists $\epsilon_L > 0$ and $\epsilon_H > 0$ such that

$$H(\theta) - (1-\lambda)\theta - \epsilon \leq H(z) - (1-\lambda)z$$

for any $z \in [\theta - \epsilon_L, \theta]$ and any $\theta \in \Theta_L$, and

$$H(\theta) - (1-\lambda)\theta + \epsilon \geq H(z) - (1-\lambda)z$$

for any $z \in [\theta, \theta + \epsilon_H]$ and any $\theta \in \Theta_H$. Equivalently, there exists $\epsilon_L > 0$ and $\epsilon_H > 0$ such that

$$H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) > 0 \text{ for any } \theta \in \Theta_L \text{ and any } z \in [\theta - \epsilon_L, \theta]$$

and

$$H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) < 0 \text{ for any } \theta \in \Theta_H \text{ and any } z \in [\theta, \theta + \epsilon_H].$$

Step C

We select $z(\cdot \mid \eta^{**})$ such that

(i) $z(\theta \mid \eta^{**})$ is increasing and differentiable with $|z(\theta \mid \eta^{**}) - \theta| < \min\{\delta, \epsilon_L, \epsilon_H\}$ and $|z_\theta(\theta \mid \eta^{**}) - 1| < \delta$ for any $\theta \in \Theta(\eta^{**})$

(ii) $z(\theta \mid \eta^{**}) = \theta$ for any $\theta \notin \Theta_H \cup \Theta_L$

(iii) For $\theta \in \Theta_L$, $z(\theta \mid \eta^{**}) \leq \theta$ with strict inequality for some subinterval of Θ_L of positive measure.

(iv) For $\theta \in \Theta_H$, $\theta \leq z(\theta \mid \eta^{**})$ with strict inequality for some some subinterval of Θ_H of positive measure, and $z(\theta \mid \eta^{**}) < h(\theta \mid \eta^{**})$.

It is evident that such a $z(\cdot \mid \eta^{**})$ exits. The argument in Step A and B ensures that $z(\cdot \mid \eta^{**})$ is in $\Lambda(\eta^{**})$, and satisfies (B-(ii)) (c) and (B-(iii)) (c). By the construction, it is evident that this satisfies all other conditions.

Step D

Suppose $z(\cdot \mid \eta^{**}) \in Z(\eta^{**})$ which satisfies (B(i)-(iii)). Since

$$(z - h(\theta \mid \eta^{**}))q^{NS}(z) + \int_z^{\bar{\theta}(\eta^{**})} q^{NS}(y)dy$$

is increasing in z for $z < h(\theta \mid \eta^{**})$, and

$$E[(\theta - h(\theta \mid \eta^{**}))q^{NS}(\theta) + \int_\theta^{\bar{\theta}(\eta^{**})} q^{NS}(y)dy \mid \eta^{**}] = 0,$$

the choice of $z(\theta \mid \eta^{**}) \leq \theta$ on Θ_L (or $z(\theta \mid \eta^{**}) \geq \theta$ on Θ_H) reduces (or raises)

$$E[(z(\theta \mid \eta^{**}) - h(\theta \mid \eta^{**}))q^{NS}(z(\theta \mid \eta^{**})) + \int_{z(\theta|\eta^{**})}^{\bar{\theta}(\eta^{**})} q^{NS}(z)dz \mid \eta^{**}]$$

away from zero. For any pair of parameters α_H, α_L lying in $[0, 1]$, define a function $z_{\alpha_L,\alpha_H}(\theta|\eta^{**})$ which equals $(1 - \alpha_L)z(\theta|\eta^{**}) + \alpha_L\theta$ on Θ_L, equals $(1 - \alpha_H)z(\theta|\eta^{**}) + \alpha_H\theta$ on Θ_H and equals θ elsewhere. It is easily checked that any such function also is in $Z(\eta^{**})$ and satisfies conditions (B(i)-(iii)). Define

$$Q(\alpha_L, \alpha_H) \equiv E[(z_{\alpha_L,\alpha_H}(\theta \mid \eta^{**}) - h(\theta \mid \eta^{**}))q^{NS}(z_{\alpha_L,\alpha_H}(\theta \mid \eta^{**})) + \int_{z_{\alpha_L,\alpha_H}(\theta|\eta^{**})}^{\bar{\theta}(\eta^{**})} q^{NS}(z)dz \mid \eta^{**}].$$

Then Q is continuously differentiable, strictly increasing in α_L and strictly decreasing in α_H with $Q(1, 1) = 0$. The Implicit Function Theorem ensures existence of α_L^*, α_H^* both smaller than 1 such that $Q(\alpha_L^*, \alpha_H^*) = 0$. Hence the function $z_{\alpha_L^*,\alpha_H^*}(\theta|\eta^{**})$ is in $Z(\eta^{**})$ and satisfies (B(i)-(iv)). ■

Result 2 *For $z(\cdot \mid \eta)$ constructed in Step 2, consider the following allocation (u_A, u_S, q):*

$$q(\theta, \eta) = q^{NS}(z(\theta \mid \eta))$$

$$u_A(\theta, \eta) = \int_\theta^{\bar{\theta}} q^{NS}(z(y \mid \eta))dy$$

$$u_S(\theta, \eta) = X^{NS}(z(\theta \mid \eta)) - \theta q^{NS}(z(\theta \mid \eta)) - \int_\theta^{\bar{\theta}(\eta)} q^{NS}(z(y \mid \eta))dy - \int_{\bar{\theta}(\eta)}^{\bar{\theta}} q^{NS}(y)dy.$$

where

$$X^{NS}(z(\theta \mid \eta)) \equiv z(\theta \mid \eta) q^{NS}(z(\theta \mid \eta)) + \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(z) dz.$$

Then (u_A, u_S, q) *is a EAC feasible allocation.*

Proof: The construction of $z(\theta \mid \eta)$ implies that $z(\bar{\theta}(\eta) \mid \eta) \leq \bar{\theta}$ for any $\eta \in \Pi$. Hence

$$X^{NS}(z(\theta \mid \eta)) - z(\theta \mid \eta) q^{NS}(z(\theta \mid \eta)) \geq 0$$

for any $(\theta, \eta) \in K$. It is evident that the construction of $z(\theta \mid \eta)$ implies $E[u_S(\theta, \eta) \mid \eta] = 0$.

Since $z(\theta \mid \eta^{**})$ is increasing in θ, there is no pooling region with $\Theta(\pi(\cdot \mid \eta^{**}), \eta^{**}) = \phi$. The coalition incentive constraint is satisfied, since

$$\begin{aligned}
& X(\theta', \eta') - z(\theta \mid \eta) q(\theta', \eta') \\
= \ & X^{NS}(z(\theta' \mid \eta')) - z(\theta \mid \eta) q^{NS}(z(\theta' \mid \eta')) \\
= \ & \int_{z(\theta'|\eta')}^{\bar{\theta}} q^{NS}(z) dz + (z(\theta' \mid \eta') - z(\theta \mid \eta)) q^{NS}(z(\theta' \mid \eta')) \\
\leq \ & \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(z) dz = X(\theta, \eta) - z(\theta \mid \eta) q(\theta, \eta).
\end{aligned}$$

The A's incentive constraint is satisfied, since

$$\begin{aligned}
& u_A(\theta', \eta) + (\theta' - \theta) q(\theta', \eta) \\
= \ & \int_{\theta'}^{\bar{\theta}} q^{NS}(z(y \mid \eta)) dy + (\theta' - \theta) q^{NS}(z(\theta' \mid \eta)) \\
\leq \ & \int_{\theta}^{\bar{\theta}} q^{NS}(z(y \mid \eta)) dy = u_A(\theta, \eta).
\end{aligned}$$

The inequality is obtained from the fact that $q^{NS}(z(\theta \mid \eta))$ is non-increasing in θ. These arguments guarantee that (u_A, u_S, q) is a EAC feasible allocation. ■

Now we prove the following result invoked in the proof of Proposition 7 in the Appendix.

(i) $\hat{h}(\theta \mid \eta^*) > \hat{h}(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta}]$ and $\hat{h}(\underline{\theta} \mid \eta^*) = \hat{h}(\underline{\theta} \mid \eta) = \underline{\theta}$ for any $\eta \neq \eta^*$

(ii) Define $G(h \mid \eta) \equiv \int_{\{\theta \mid \hat{h}(\theta\mid\eta) \leq h\}} f(\theta \mid \eta) d\theta$. Then $G(h \mid \eta^*)$ is a mean-preserving spread of $G(h \mid \eta)$ for any $\eta \neq \eta^*$

Proof of (i): Since $\frac{f(\theta\mid\eta^*)}{f(\theta\mid\eta)}$ is strictly decreasing in θ for any $\eta \neq \eta^*$, $\frac{f(\theta'\mid\eta^*)}{f(\theta\mid\eta^*)} > \frac{f(\theta'\mid\eta)}{f(\theta\mid\eta)}$ for $\theta > \theta'$. $\Theta(\eta) = \Theta(\eta^*) = \Theta$ then implies

$$\frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} = \int_{\underline{\theta}}^{\theta} \frac{f(\theta' \mid \eta^*)}{f(\theta \mid \eta^*)} d\theta' > \int_{\underline{\theta}}^{\theta} \frac{f(\theta' \mid \eta)}{f(\theta \mid \eta)} d\theta' = \frac{F(\theta \mid \eta)}{f(\theta \mid \eta)}.$$

Hence $h(\theta \mid \eta^*) > h(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta}]$ and $h(\underline{\theta} \mid \eta^*) = h(\underline{\theta} \mid \eta) = \underline{\theta}$. The property of the ironing procedure (explained in later section) ensures that $\hat{h}(\theta \mid \eta^*) > \hat{h}(\theta \mid \eta)$ for any $\theta > \underline{\theta}$ and $\hat{h}(\underline{\theta} \mid \eta^*) = \hat{h}(\underline{\theta} \mid \eta) = \underline{\theta}$ for any $\eta \neq \eta^*$. ■

Proof of (ii): Since

$$\int_{\underline{h}}^{\bar{h}} h dG(h \mid \eta) = \int_{\underline{\theta}}^{\bar{\theta}} \hat{h}(\theta \mid \eta) dF(\theta \mid \eta) = \int_{\underline{\theta}}^{\bar{\theta}} h(\theta \mid \eta) dF(\theta \mid \eta) = \bar{\theta}$$

for each η, the two distribution has the same mean. For any convex function $u(h)$ (which is not a linear function),

$$\begin{aligned}
& \int_{\underline{h}}^{\bar{h}} u(h) dG(h \mid \eta^*) = \int_{\underline{\theta}}^{\bar{\theta}} u(h(\theta \mid \eta^*)) f(\theta \mid \eta^*) d\theta \\
> & \int_{\underline{\theta}}^{\bar{\theta}} [u(\hat{h}(\theta \mid \eta)) - u'(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - h(\theta \mid \eta^*))] f(\theta \mid \eta^*) d\theta \\
= & \int_{\underline{\theta}}^{\bar{\theta}} [u(\hat{h}(\theta \mid \eta)) - u'(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - \theta) + \int_{\theta}^{\bar{\theta}} u'(\hat{h}(y \mid \eta)) dy] f(\theta \mid \eta^*) d\theta \\
> & \int_{\underline{\theta}}^{\bar{\theta}} [u(\hat{h}(\theta \mid \eta)) - u'(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - \theta) + \int_{\theta}^{\bar{\theta}} u'(\hat{h}(y \mid \eta)) dy] f(\theta \mid \eta) d\theta \\
= & \int_{\underline{\theta}}^{\bar{\theta}} u(\hat{h}(\theta \mid \eta)) f(\theta \mid \eta) d\theta = \int_{\underline{h}}^{\bar{h}} u(h) dG(h \mid \eta)
\end{aligned}$$

The first inequality uses the convexity of $u(h)$: $u(h) \geq u(h^{'}) - u^{'}(h^{'})(h^{'} - h)$ for any $h, h^{'} \in H$. The second inequality is the result that

$$u(\hat{h}(\theta \mid \eta)) - u^{'}(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - \theta) + \int_{\theta}^{\bar{\theta}} u^{'}(\hat{h}(y \mid \eta))dy$$

is non-increasing in θ and is not constant on Θ, and $F(\theta \mid \eta)$ first order stochastically dominates $F(\theta \mid \eta^{*})$ (because of the MLRP assumption). According to the definition, $G(h \mid \eta^{*})$ is a mean-preserving spread of $G(h \mid \eta)$ for any $\eta \neq \eta^{*}$. ■

5 Optimality of Conditional Delegation

Here we provide the formal proof of the following statement in terms of the optimality of the conditional delegation.

Proposition 4 *Any EAC allocation satisfying interim PCs can also be achieved as a PBE(c) outcome of the subgame (C3) of the modified delegation mechanism, in which P communicates and transacts with S alone on the equilibrium path.*

Proof of Proposition 4:

Step 1: Construction of grand contract

For EACP allocation (u_A, u_S, q) which satisfies interim participation constraints of A and S, define $X(\theta, \eta) \equiv u_A(\theta, \eta) + u_S(\theta, \eta) + \theta q(\theta, \eta)$ for $(\theta, \eta) \in K \equiv \{(\theta, \eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$ and $(X(e), q(e)) = (0, 0)$. Let us construct the following grand contract GC with sufficiently large $T > 0$:

$$(X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S); M_A, M_S)$$

where $M_A = \{\phi\} \cup \tilde{M}_A$ and $M_S = K \cup \{e\} \cup \tilde{M}_S$, associated with $\tilde{M}_A \equiv K \cup \{e_A\}$ and $\tilde{M}_S \equiv \Pi \cup \{e_S\}$, as follows:

- For $(m_A, m_S) \in \tilde{M}_A \times \tilde{M}_S$, $(X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S))$ is equal to that used in the proof of the sufficiency of the above Proposition 2.

- For $(m_A, m_S) = (\phi, (\theta, \eta)), (\phi, e)$,

$$(X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S)) = (0, X(m_S), q(m_S))$$

- For any other $(m_A, m_S) \in M_A \times M_S$,

$$(X_A, X_S, q) = (0, -T, 0)$$

Let us check that this mechanism achieves (u_A, u_S, q) in PBE(c) where P receives the message only from S on the equilibrium path.

Step 2: Non-cooperative equilibrium based on prior beliefs and non-prior beliefs

First we argue $(m_A(\theta, \eta), m_S(\eta)) = ((\theta, \eta), \eta)$ is a non-cooperative equilibrium of the grand contract based on prior beliefs $b_\phi(\eta)$ for η. EACP and A's participation constraint imply that A always has an incentive to participate and report truthfully: $m_A(\theta, \eta) = (\theta, \eta)$. Since S's interim participation constraint ($E[u_S(\theta, \eta) \mid \eta] \geq 0$) holds, taking A's strategy $m_A(\theta, \eta) = (\theta, \eta)$ as given, S also has an incentive to participate and report truthfully. This equilibrium results in allocation $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$.

Suppose that some side contract is offered in state η, and let $b(\eta)$ denote beliefs of S regarding θ which result following rejection of this side contract by A. S and A then play GC noncooperatively with beliefs $b(\eta)$. By construction, A has an incentive to report truthfully and participate in GC irrespective of the beliefs $b(\eta)$ held by S. If T is sufficiently large, it is then a best response for S to report truthfully, conditional on participating. There are two cases to consider. (a) $E_{b(\eta)}[u_S(\theta, \eta)] \geq 0$, in which case it is a best response for S to participate (and report truthfully) in GC when it is played noncooperatively with beliefs $b(\eta)$. Then A receives $u_A(\theta, \eta)$. (b) $E_{b(\eta)}[u_S(\theta, \eta)] < 0$, whereby S exits from GC following rejection of the side contract and attains zero payoff. Then A receives $\hat{X}(\tilde{m}^*(\theta)) - \theta \hat{q}(\tilde{m}^*(\theta))$ (denoted by $\bar{u}_A(\theta)$ hereafter). We focus on a PBE where A receives either $u_A(\theta, \eta)$ or $\hat{u}_A(\theta)$ whenever the collusion breaks down in the continuation of any side-contract.

Step 3: Side-contract choice.

Now we argue that without loss of generality, the choice of side contract can be limited to those where in every state (θ, η): A reports nothing ($m_A = \phi$)

and S reports either (θ, η) or e. It is evident that the side contract never selects reports which generate penalty T to S if T is sufficiently large. Given the construction of the grand contract, for any $(m_A, m_S) \in M_A \times M_S$ such that $X_S \neq -T$, there exists $\tilde{m}' \in \hat{M} \equiv \{(\phi, (\theta, \eta)) \mid (\theta, \eta) \in K\} \cup \{e\}$ such that

$$(X_A(m_A, m_S) + X_S(m_A, m_S), q(m_A, m_S)) = (X_A(\tilde{m}') + X_S(\tilde{m}'), q(\tilde{m}')).$$

It means that we can restrict attention to a side-contract which selects a message from $\hat{M}$ to be sent to P, whenever a non-null side-contract is offered.

Step 4: Upper bound of the S's payoff

Consider the following problem: select $(\hat{m}(\theta, \eta), \hat{u}_A(\theta, \eta))$ to

$$\max E[X(\hat{m}(\theta, \eta)) - \theta q(\hat{m}(\theta, \eta)) - \hat{u}_A(\theta, \eta) \mid \eta]$$

subject to $\hat{m}(\theta, \eta) \in \Delta(\hat{M})$,

$$\hat{u}_A(\theta, \eta) \geq \hat{u}_A(\theta', \eta) + (\theta' - \theta) q(\hat{m}(\theta', \eta))$$

for any $\theta, \theta' \in \Theta(\eta)$ and

$$\hat{u}_A(\theta, \eta) \geq u_A(\theta, \eta).$$

for any $\theta \in \Theta(\eta)$. This is equivalent to problem $P(\eta)$ (in our paper) used to characterize EACP allocations. The EACP property implies that $(\hat{m}(\theta, \eta), \hat{u}_A(\theta, \eta)) = ((\theta, \eta), u_A(\theta, \eta))$ solves this problem and the maximum value is $E[u_S(\theta, \eta) \mid \eta]$.

Next consider the problem: select $(\hat{m}(\theta, \eta), \hat{u}_A(\theta, \eta))$ to

$$\max E[X(\hat{m}(\theta, \eta)) - \theta q(\hat{m}(\theta, \eta)) - \hat{u}_A(\theta, \eta) \mid \eta]$$

subject to $\hat{m}(\theta, \eta) \in \Delta(\hat{M})$,

$$\hat{u}_A(\theta, \eta) \geq \hat{u}_A(\theta', \eta) + (\theta' - \theta) q(\hat{m}(\theta', \eta))$$

and

$$\hat{u}_A(\theta, \eta) \geq \bar{u}_A(\theta).$$

As shown in the proof of the sufficiency of Proposition 2, the solution of this problem is

$$(\hat{m}(\theta, \eta), \tilde{u}_A(\theta, \eta)) = (\hat{m}^*(\theta), \bar{u}_A(\theta)).$$

The maximum value is equal to zero.

Applying the same argument as in the proof of the sufficiency of Proposition 2, we can conclude that $E[u_S(\theta,\eta) \mid \eta]$ is upper bound value of S's payoff in PBE with the property that A receives either $u_A(\theta,\eta)$ or $\hat{u}_A(\theta)$ whenever the collusion breaks down.

Step 5: PBE(c) in collusion game

We can construct a PBE in the overall collusion game as follows. On the equilibrium path, S offers a side-contract $(m(\theta,\eta), t(\theta,\eta))$ such that $m(\theta,\eta) = (\theta,\eta) \in \Delta(\hat{M})$ and $t(\theta,\eta) = u_A(\theta,\eta) + \theta q(\theta,\eta)$, and all types of A accept it and has the truthful report about θ. This is sequential rational for A if A and S play non-cooperative equilibrium based on prior beliefs whenever A rejects it. S receives $E[u_S(\theta,\eta) \mid \eta]$. If S offers any other side-contract (including null side contract), it follows from Step 4 that his subsequent continuation payoff is not larger than $E[u_S(\theta,\eta) \mid \eta]$. Therefore there exists a PBE in which S offers the above side-contract on the equilibrium path, resulting in allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$. Finally check that PBE constructed in the above argument is also a PBE(c). Otherwise there would exist a PBE resulting in a Pareto superior allocation for the coalition. This would violate the EACP property of the allocation we started with. In this PBE, P communicates and transacts with S alone on the equilibrium path. ■

6 Effect of Reallocating Bargaining Power: Proof of Proposition 8

Proof of Proposition 8:

We show that the set of EACP(α) is independent of $\alpha \in [0,1]$. Suppose otherwise that $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ is a EACP(α) allocation, but not a EACP(α') ($\alpha \neq \alpha'$) allocation. It implies that for some η, $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$ is not the solution of $TP(\eta;\alpha')$ defined for $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$. If $(\tilde{m}^*(\theta \mid \eta), u_A^*(\theta,\eta))(\neq ((\theta,\eta), u_A(\theta,\eta)))$ is a solution of $TP(\eta;\alpha')$, it satisfies all constraints of $TP(\eta;\alpha')$ and realizes a higher payoff to the third party than in the choice of $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$:

$$\begin{aligned} & E[(1-\alpha')[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta,\eta)] + \alpha' u_A^*(\theta,\eta) \mid \eta] \\ > \; & E[(1-\alpha')[\hat{X}(\theta,\eta) - \theta\hat{q}(\theta,\eta) - u_A(\theta,\eta)] + \alpha' u_A(\theta,\eta) \mid \eta]. \end{aligned}$$

It also satisfies A and S's participation constraints:

$$u_A^*(\theta,\eta) \geq u_A(\theta,\eta)$$

and

$$E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta,\eta) \mid \eta] \geq E[u_S(\theta,\eta) \mid \eta].$$

On the other hand, since $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$ solves $TP(\eta;\alpha)$,

$$\begin{aligned} & E[(1-\alpha)[\hat{X}(\theta,\eta) - \theta\hat{q}(\theta,\eta) - u_A(\theta,\eta)] + \alpha u_A(\theta,\eta) \mid \eta] \\ \geq\ & E[(1-\alpha)[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(m^*(\theta \mid \eta)) - u_A^*(\theta,\eta)] + \alpha u_A^*(\theta,\eta) \mid \eta] \end{aligned}$$

Let us consider three cases: (i) $\alpha \in (0,1)$, (ii) $\alpha = 1$ and (iii) $\alpha = 0$.

(i) $\alpha \in (0,1)$

The last three inequalities imply

$$u_A^*(\theta,\eta) = u_A(\theta,\eta)$$

and

$$E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(m^*(\theta \mid \eta)) - u_A^*(\theta,\eta) \mid \eta] = E[u_S(\theta,\eta) \mid \eta].$$

But this is not compatible with the first inequality. We obtain a contradiction.

(ii) $\alpha = 1$

With $\alpha = 1$, the above four inequalities imply

$$E[u_A(\theta,\eta) \mid \eta] = E[u_A^*(\theta,\eta) \mid \eta]$$

and

$$E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta,\eta) \mid \eta] > E[u_S(\theta,\eta) \mid \eta].$$

But for sufficiently small $\epsilon > 0$, the choice of

$$(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = (\tilde{m}^*(\theta,\eta), u_A^*(\theta,\eta) + \epsilon)$$

(instead of $((\theta,\eta), u_A(\theta,\eta)))$ in $TP(\eta;\alpha=1)$ generates a higher value of the objection function without violating any constraint. We obtain a contradiction.

(iii) $\alpha = 0$

With $\alpha = 0$, the four inequalities imply

$$E[u_S(\theta,\eta) \mid \eta] = E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta,\eta) \mid \eta]$$

and

$$E[u_A^*(\theta,\eta) \mid \eta] > E[u_A(\theta,\eta) \mid \eta].$$

Since $u_A^*(\theta,\eta) \geq u_A(\theta,\eta)$ for any θ, there is a subset of θ with the positive measure such that $u_A^*(\theta,\eta) > u_A(\theta,\eta)$. Consider a modified problem of $TP(\eta;\alpha = 0)$ such that the constraint $\tilde{u}_A(\theta,\eta) \geq u_A(\theta,\eta)$ is replaced by $\tilde{u}_A(\theta,\eta) \geq u_A^*(\theta,\eta)$ in $TP(\eta;\alpha = 0)$. Since the optimal solution $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) = ((\theta,\eta), u_A(\theta,\eta))$ in $TP(\eta;\alpha = 0)$ violates the constraint, the maximum value of the objective function in the modified problem would become lower. On the other hand, $(\tilde{m}^*(\theta \mid \eta), u_A^*(\theta,\eta))$ satisfies all the constraints of the modified problem, and brings

$$E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta,\eta) \mid \eta].$$

The argument implies

$$E[u_S(\theta,\eta) \mid \eta] > E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta,\eta) \mid \eta].$$

We obtain a contradiction. ■

7 Ironing Rule and Related Results

Here we summarize the ironing procedure and its related properties which are frequently used throughout the paper. We specify an ironing rule to construct $\hat{\pi}(x)$ from two functions $\pi(x)$ and $G(x)$, and explain some properties about $\hat{\pi}(x)$. According to Myerson (1981) and Baron and Myerson (1982), the ironing rule is described as follows.

Definition 1 *Suppose that $\pi(x)$ and $G(x)$ defined on $[\underline{x}, \bar{x}]$ have the following properties:*

(i) $\pi(x-) \geq \pi(x+)$ for any $x \in [\underline{x}, \bar{x}]$.

(ii) $G(x)$ is distribution function with $G(\underline{x}) = 0$ and $G(\bar{x}) = 1$. $G(x)$ is strictly increasing and continuously differentiable on $[\underline{x}, \bar{x}]$.

Then $\hat{\pi}(x) \equiv \hat{\pi}(x \mid \pi(\cdot), G(\cdot))$ *is constructed from* $\pi(x)$ *and* $G(x)$ *as follows.*

(i) $\Pi(\phi) = \int_0^{\phi} \pi(h(y))dy$ *where* $h(\phi)$ *satisfies* $G(h(\phi)) = \phi$ *for* $\phi \in [0, 1]$.

(ii) $\underline{\Pi}(\phi)$ *is maximum convex function so that* $\Pi(\phi) \geq \underline{\Pi}(\phi)$.

(iii) $\hat{\pi}(x)$ *satisfies (i)* $\hat{\pi}(x) = \underline{\Pi}'(G(x))$ *whenever the derivative* $\underline{\Pi}'(G(x))$ *is defined,*[6] *and (ii)* $\hat{\pi}(x) = \underline{\Pi}'(G(x-))$ *for any* $x \in (\underline{x}, \bar{x}]$.

We provide two lemmata, which show some properties used in the paper.

Lemma 1 $\hat{\pi}(x) = \hat{\pi}(x \mid \pi(\cdot), G(\cdot))$ *constructed from* $\pi(x)$ *and* $G(x)$ *satisfies:*

(i) $\hat{\pi}(x)$ *is continuous and non-decreasing in* x. *If* $\pi(x)$ *is non-decreasing in* x, $\hat{\pi}(x) = \pi(x)$.

(ii) $\int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x)$ *if* $q(x)$ *is constant for each interval of* x *such that* $\Pi(G(x)) > \underline{\Pi}(G(x))$ *(or* $\hat{\pi}(x)$ *takes constant value).*

(iii) If $\pi(x) > x$ *on* $(\underline{x}, \bar{x}]$, $\hat{\pi}(x) > \hat{\pi}_{\alpha}(x)$ *on* $(\underline{x}, \bar{x}]$ *for* $\pi_{\alpha}(x) \equiv (1-\alpha)\pi(x) + \alpha x$ *with* $\alpha \in (0, 1]$.

(iv) $\hat{\pi}(\underline{x}) \leq \pi(\underline{x})$ *and* $\hat{\pi}(\bar{x}) \geq \pi(\bar{x})$. *If there exists an increasing* $v(x)$ *so that* $v(x) < \pi(x)$ *for any* $x > \underline{x}$, $v(x) < \hat{\pi}(x)$ *for any* $x > \underline{x}$ *and if there exists an increasing* $v(x)$ *so that* $v(x) > \pi(x)$ *for any* $x > \underline{x}$, $v(x) > \hat{\pi}(x)$ *for any* $x > \underline{x}$.

(v) Suppose that $q^*(x)$ *is the solution of the following problem:*

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)]dG(x)$$

subject to $q(x)$ *is non-increasing. Then* $q^*(x)$ *solves*

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

Then

$$\int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \pi(x)q^*(x)]dG(x) = \int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \hat{\pi}(x)q^*(x)]dG(x).$$

[6] Since $\underline{\Pi}(\phi)$ is convex, it is almost everywhere differentiable.

Proof of Lemma 1

The proof of (i)

Since $\underline{\Pi}(\phi)$ is convex and $G(x)$ is increasing, $\hat{\pi}(x)$ is non-decreasing. Suppose that there exists x so that $\hat{\pi}(x) < \hat{\pi}(x+)$. It means that $\underline{\Pi}^{'}(G(x-)) < \underline{\Pi}^{'}(G(x+))$. Then $\underline{\Pi}(G(x)) = \Pi(G(x))$, since otherwise you can find a higher convex function than $\underline{\Pi}(\phi)$. This implies that

$$\pi(x-) = \Pi^{'}(G(x-)) \leq \underline{\Pi}^{'}(G(x-)) < \underline{\Pi}^{'}(G(x+)) \leq \Pi^{'}(G(x+)) = \pi(x+)$$

This is contradiction since we assume that $\pi(x-) \geq \pi(x+)$. Therefore $\hat{\pi}(x)$ is continuous.

Suppose that $\pi(x)$ is non-decreasing in x. With $\Pi(\phi) = \int_0^{\phi} \pi(h(y))dy$, $\Pi^{'}(\phi) = \pi(h(\phi))$. Then $\Pi(\phi)$ is convex and $\Pi(\phi) = \underline{\Pi}(\phi)$, implying $\pi(x) = \hat{\pi}(x)$.

The proof of (ii)

Define I by

$$I \equiv \{x \in [\underline{x}, \bar{x}] \mid \Pi(G(x)) > \underline{\Pi}(G(x))\}.$$

For any $x \in I$, there exists $d(x)$ and $u(x)$ such as

$$\Pi(G(x^{'})) > \underline{\Pi}(G(x^{'}))$$

on $x^{'} \in (d(x), u(x))$, $\Pi(G(d(x))) = \underline{\Pi}(G(d(x)))$ and $\Pi(G(u(x))) = \underline{\Pi}(G(u(x)))$. Then $\underline{\Pi}(\phi^{'})$ is a linear function of $\phi^{'}$ on $[G(d(x)), G(u(x))]$ and $\hat{\pi}(x^{'})$ is constant on $x^{'} \in [d(x), u(x)]$. Then since $q(x^{'})$ is constant on $x^{'} \in [d(x), u(x)]$,

$$\int_{[d(x),u(x)]} q(x^{'})d\Pi(G(x^{'})) = \int_{[d(x),u(x)]} q(x^{'})d\underline{\Pi}(G(x^{'})).$$

Therefore it implies that

$$\int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) = \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)).$$

Since $\underline{\Pi}(\phi)$ is convex, it is almost everywhere differentiable with $\underline{\Pi}^{'}(G(x)) = \hat{\pi}(x)$ almost everywhere. This means that

$$\int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) = \int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x).$$

It is concluded that

$$\int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x).$$

The proof of (iii)

Since the linear combination of two convex functions is convex, $(1-\alpha)\underline{\Pi}(\phi)+\alpha\int_0^\phi h(y)dy$ is convex function. Defining $\Pi_\alpha(\phi)$ by

$$\Pi_\alpha(\phi) \equiv \int_0^\phi \pi_\alpha(h(y))dy = (1-\alpha)\Pi(\phi) + \alpha\int_0^\phi h(y)dy.$$

Since

$$\Pi_\alpha(\phi) \geq (1-\alpha)\underline{\Pi}(\phi) + \alpha\int_0^\phi h(y)dy,$$

$\underline{\Pi}_\alpha(\phi)$, which is the maximum convex function such that $\Pi_\alpha(\phi) \geq \underline{\Pi}_\alpha(\phi)$, satisfies

$$\Pi_\alpha(\phi) \geq \underline{\Pi}_\alpha(\phi) \geq (1-\alpha)\underline{\Pi}(\phi) + \alpha\int_0^\phi h(y)dy.$$

Here our proof is composed of the analysis of two cases: (a) the region of x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$ and (b) the region of x such that $\Pi(G(x)) = \underline{\Pi}(G(x))$.

(a) For arbitrary x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, there exists $d(x)$ and $u(x)$ such as

$$\Pi(G(x^{'})) > \underline{\Pi}(G(x^{'}))$$

on $x^{'} \in (d(x), u(x))$, $\Pi(G(d(x))) = \underline{\Pi}(G(d(x)))$ and $\Pi(G(u(x))) = \underline{\Pi}(G(u(x)))$. At $\phi = G(d(x))$ and $\phi = G(u(x))$,

$$\Pi_\alpha(\phi) = (1-\alpha)\underline{\Pi}(\phi) + \alpha\int_0^\phi h(y)dy.$$

It implies that

$$\underline{\Pi}_\alpha(\phi) = (1-\alpha)\underline{\Pi}(\phi) + \alpha\int_0^\phi h(y)dy$$

at $\phi = G(d(x))$ and $\phi = G(u(x))$. Then since (i) of this lemma implies that $\underline{\Pi}_\alpha^{'}(\phi)$ and $\underline{\Pi}(\phi)$ are differentiable with respect to ϕ for any $\phi \in [0,1]$, the derivatives of both sides of the above equation with respect to ϕ, if evaluated at $G(u(x))$, have the following relationship:

$$\underline{\Pi}'_{\alpha}(G(u(x))) \leq (1-\alpha)\underline{\Pi}'(G(u(x))) + \alpha u(x) = (1-\alpha)\hat{\pi}(u(x)) + \alpha u(x).$$

Since $\hat{\pi}(u(x)) = \pi(u(x)) > u(x)$ (by $u(x) > \underline{x}$) and $\hat{\pi}_{\alpha}(u(x)) = \underline{\Pi}'_{\alpha}(G(u(x)))$,

$$\hat{\pi}_{\alpha}(u(x)) < \hat{\pi}(u(x))$$

for any $\alpha \in (0,1]$. For any $x' \in (d(x), u(x))$, $\hat{\pi}(x') = \hat{\pi}(u(x))$ and $\hat{\pi}_{\alpha}(x') \leq \hat{\pi}_{\alpha}(u(x))$ (since $\hat{\pi}_{\alpha}(x)$ is non-decreasing in x). Therefore

$$\hat{\pi}_{\alpha}(x') < \hat{\pi}(x')$$

for any $x' \in (d(x), u(x))$.

(b) For any $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$,

$$\Pi_{\alpha}(G(x)) = (1-\alpha)\underline{\Pi}(G(x)) + \alpha \int_0^{G(x)} h(y)dy.$$

It implies

$$\underline{\Pi}_{\alpha}(G(x)) = (1-\alpha)\underline{\Pi}(G(x)) + \alpha \int_0^{G(x)} h(y)dy$$

and

$$\hat{\pi}_{\alpha}(x) = \underline{\Pi}'_{\alpha}(G(x)) = (1-\alpha)\hat{\pi}(x) + \alpha x < \hat{\pi}(x)$$

for any $\alpha \in (0,1]$, since $\hat{\pi}(x) = \pi(x) > x$ for $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$.

The argument in (a) and (b) implies the statement of (iii).

The proof of (iv)

(a) $\hat{\pi}(\underline{x}) \leq \pi(\underline{x})$ and $\hat{\pi}(\bar{x}) \geq \pi(\bar{x})$ are obtained from $\Pi'(\phi = 0) \geq \underline{\Pi}'(\phi = 0)$, $\Pi'(\phi = 1) \leq \underline{\Pi}'(\phi = 1)$ and $\Pi'(G(x)) = \pi(x)$.
(b) The case of $v(x) < \pi(x)$: For $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$, $\hat{\pi}(x) = \pi(x) > v(x)$. For $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, and for $u(x)$ that is defined in the proof of (iii), $\hat{\pi}(x) = \Pi'(G(u(x))) = \pi(u(x)) > v(u(x)) \geq v(x)$. It implies $\hat{\pi}(x) > v(x)$ for any $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$. Therefore $\hat{\pi}(x) > v(x)$ for any $x > \underline{x}$.
(c) The case of $v(x) > \pi(x)$: For $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$, $\hat{\pi}(x) = \pi(x) < v(x)$. For $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, and for $d(x)$ that is defined in the proof of (iii), $\hat{\pi}(x) = \Pi'(G(d(x))) = \pi(d(x)) \leq v(d(x)) < v(x)$. It implies $\hat{\pi}(x) < v(x)$ for any $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$. Therefore $\hat{\pi}(x) < v(x)$ for any $x > \underline{x}$.

The proof of (v)

Step 1:

For any non-increasing $q(x)$,

$$\int_{\underline{x}}^{\bar{x}} \pi(x)q(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) \geq \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) = \int_{\underline{x}}^{\bar{x}} \hat{\pi}(x)q(x)dG(x)$$

Proof of Step 1

Since $\Pi(G(x))$ and $\underline{\Pi}(G(x))$ are continuous, applying the integration by parts,

$$\int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) + \int_{\underline{x}}^{\bar{x}} \Pi(G(x))dq(x) = \Pi(1)q(\bar{x}) - \Pi(0)q(\underline{x})$$

and

$$\int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) + \int_{\underline{x}}^{\bar{x}} \underline{\Pi}(G(x))dq(x) = \underline{\Pi}(1)q(\bar{x}) - \underline{\Pi}(0)q(\underline{x}).$$

With $\Pi(1) = \underline{\Pi}(1)$ and $\Pi(0) = \underline{\Pi}(0)$,

$$\begin{aligned} & \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) - \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) \\ = & \int_{\underline{x}}^{\bar{x}} (\underline{\Pi}(G(x)) - \Pi(G(x)))dq(x) \geq 0 \end{aligned}$$

Step 2:

$$\int_{[\underline{x},\bar{x}]} [V(q^{**}(x)) - \pi(x)q^{**}(x)]dG(x) = \int_{[\underline{x},\bar{x}]} [V(q^{**}(x)) - \hat{\pi}(x)q^{**}(x)]dG(x)$$

for $q^{**}(x) \in \arg\max_q V(q) - \hat{\pi}(x)q$.

Proof of Step 2:

By the definition, $q^{**}(x)$ is constant for each interval of x where $\hat{\pi}(x)$ is constant. Then by (ii) of the lemma,

$$\int_{\underline{x}}^{\bar{x}} \pi(x)q^{**}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} \hat{\pi}(x)q^{**}(x)dG(x).$$

This completes the proof of Step 2.

Step 3:

By Step 1, for any non-decreasing $q(x)$,

$$\int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)]dG(x) \leq \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

By Step 2, if $q^*(x)$ is the solution of

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)]dG(x)$$

subject to $q(x)$ is non-increasing, then $q^*(x)$ solves

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

Then

$$\int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \pi(x)q^*(x)]dG(x) = \int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \hat{\pi}(x)q^*(x)]dG(x).$$

It completes the proof of (v). ■

Lemma 2 *$\hat{h}(\theta \mid \eta)$ is non-increasing and continuous in θ on $\Theta(\eta)$ with $\hat{h}(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$ and $\hat{h}(\theta \mid \eta) > \theta$ for $\theta > \underline{\theta}(\eta)$.*

Proof of Lemma 2

Since $h(\theta \mid \eta)$ is continuous, Lemma 1(i) implies that $\hat{h}(\theta \mid \eta)$ is continuous and non-decreasing in θ. Since $\theta < h(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$, Lemma 1(iv) implies that $\theta < \hat{h}(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$. By the continuity of $\hat{h}(\theta \mid \eta)$, $\underline{\theta}(\eta) \leq \hat{h}(\underline{\theta}(\eta) \mid \eta)$. Lemma 1(iv) also implies $\hat{h}(\underline{\theta}(\eta) \mid \eta) \leq h(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$. Therefore $\hat{h}(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$. ■

References

Baron, B., and R. B. Myerson (1982), "Regulating a Monopolist with Unknown Costs", *Econometrica*, 50(4), 911-930.

Celik, G. and M. Peters (2011), “Equilibrium Rejection of a Mechanism”, *Games and Economic Behavior*, 73(2), 375-387.

Che, Y. K. and J. Kim (2009), “Optimal Collusion-Proof Auctions”, *Journal of Economic Theory*, 144(2), 565-603.

Faure-Grimaud, A., J. J. Laffont and D. Martimort (2003), “Collusion, Delegation and Supervision with Soft Information”, *Review of Economic Studies*, 70(2), 253-279.

Fudenberg, D., and J. Tirole (1991), “Perfect Bayesian Equilibrium and Sequential Equilibrium,” *Journal of Economic Theory*, 53(2), 236-260.

Mas-Colell, A., M. D. Whinston, and J. Green (1995), “Microeconomic Theory”, Oxford University Press, New York.

Myerson, R. (1981), “Optimal Auction Design,” *Mathematics of Operations Research* 6, 58–73.