Supplement to 'Ex Ante Collusion and Design of Supervisory Institutions'

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1 Introduction

This material includes some arguments which supplement our paper 'Ex Ante Collusion and Design of Supervisory Institutions'. Some proofs, which are omitted in the paper, are also provided in this note.

2 WPBE(wc) vs PBE(wc)

In the paper, our justification of WCP was based on the adoption of Weak Perfect Bayesian Equilibrium (WPBE) as an equilibrium concept. This can be provided also when we replace WPBE by the stronger notion of Perfect Bayesian Equilibrium (PBE) in the sense of Fudenberg and Tirole (1991). PBE differs from WPBE in imposing some restrictions on off-equilibrium-path beliefs. In our context, there are two additional restrictions. One is that following any offer of a non-equilibrium side-contract, the subsequent continuation game must be played with initial beliefs that are the prior beliefs, owing to the "no-signaling-what-you-don't know" principle. Second, if the side-contract is rejected by some types of A in an equilibrium of this continuation game, S's belief about θ must be updated according to Bayes rule. PBE(wc) is defined in exactly the same way as WPBE(wc) in Definition 5 except that WPBE is replaced by PBE. With this stronger equilibrium concept, the proof of Proposition 3 in the paper is modified as follows.

While extending the necessity part is straightforward, the sufficiency part is more involved owing to the need to impose Bayes rule on beliefs following rejection of offered side contracts. In the proof provided in the text, a WPBE was constructed to achieve a WCP allocation in which rejection of any deviating side contract resulted in noncooperative play of the grand contract with prior beliefs. Such beliefs are not consistent with the requirement of a PBE. Hence a more elaborate argument is needed to establish the same result for PBE.

Proof of Proposition 3 using PBE

Proof of Necessity

Proposition 3 in our paper establishes that any allocation achieved as an outcome of WPBE(wc) must be a WCP allocation satisfying participation constraints. Therefore it suffices to show that any PBE(wc) of WC3 is a WPBE(wc).

By definition, every PBE is a WPBE. If the claim is false, there is a PBE(wc) which is not a WPBE(wc). Then it must be the case that there exists some distinct WPBE which results in an interim Pareto superior allocation for the coalition. Using arguments in the paper, this alternative allocation can equivalently be attained as the outcome of a WPBE in which a null side contract is offered, and S and A play the GC noncooperatively with prior beliefs. The latter WPBE is also a PBE since there is no scope for updating of beliefs. Hence the Pareto dominating allocation can be attained as the outcome of a PBE, contradicting the hypothesis that we started with a PBE(wc) allocation.

Proof of Sufficiency

Step 1: Construction of grand contract

Suppose that (u_A, u_S, q) is a WCP allocation satisfying interim participation constraints. We show that there exists a grand contract which achieves (u_A, u_S, q) as a PBE(wc) outcome.

The grand contract is constructed as follows:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(e_A, m_S) = X_S(e_A, m_S) = q(e_A, m_S) = X_S(m_A, e_S) = 0$$

for any (m_A, m_S).

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$
- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$, with T sufficiently large
- $(X_A((\theta_A, \eta_A), e_S), q((\theta_A, \eta_A), e_S)) = (\hat{X}(\tilde{m}^*(\theta_A)), \hat{q}(\tilde{m}^*(\theta_A)))$ where $\tilde{m}^*(\theta)$ maximizes $\hat{X}(\tilde{m}) - \theta\hat{q}(\tilde{m})$ subject to $\tilde{m} \in \Delta(K \cup \{e\})$ and the definition of $(\hat{X}(\tilde{m}), \hat{q}(\tilde{m}))$ is provided in Section 3.5 of the paper.

Step 2: Non-cooperative equilibrium

First we argue $(m_A(\theta,\eta), m_S(\eta)) = ((\theta,\eta),\eta)$ is a non-cooperative equilibrium of the grand contract based on prior beliefs $p_\phi(\eta)$ for η. WCP and A's participation constraint imply that A always has an incentive to participate and report truthfully: $m_A(\theta,\eta) = (\theta,\eta)$. Since S's interim participation constraint ($E[u_S(\theta,\eta) \mid \eta] \geq 0$) holds, taking A's strategy $m_A(\theta,\eta) = (\theta,\eta)$ as given, S also has an incentive to participate and report truthfully.

This equilibrium results in allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$. By offering a null side-contract, S can always realize the allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ and achieve interim payoff $E[u_S(\theta,\eta) \mid \eta]$. Therefore S would have an incentive to offer a non-null side-contract only if the deviation results in a higher payoff. We show that there exists a PBE of WC3 following the GC constructed above, in which S's interim payoff from any deviating side contract offer cannot exceed $E[u_S(\theta,\eta) \mid \eta]$.

Consider any deviating side contract offer in state η, and let $p(\eta)$ denote beliefs of S regarding θ which result following rejection of this side contract by A. S and A then play GC noncooperatively with beliefs $p(\eta)$. By construction, A has an incentive to report truthfully and participate in GC irrespective of what S does, i.e., irrespective of the beliefs $p(\eta)$ held by S (as well as irrespective of the particular deviating side contract offered). If T is sufficiently large, it is then a best response for S to report truthfully, conditional on participating. We focus on PBEs satisfying these two properties following rejection by A of any deviating side contract.

In what follows, there are two cases to consider. (a) $E_{p(\eta)}[u_S(\theta,\eta)] \geq 0$, in which case it is a best response for S to participate (and report truthfully) in GC when it is played noncooperatively with beliefs $p(\eta)$. We refer to this as the T case. (b) $E_{p(\eta)}[u_S(\theta,\eta)] < 0$, whereby S exits from GC following rejection of the side contract and attains zero payoff. We refer to this as the E case.

Step 3: Side-contract choice.

Now we argue that without loss of generality, the choice of deviating side contract can be limited to those where in every state θ,η: either A and S both participate and submit consistent reports $\eta_A = \eta_S$, or where they both exit. That they should submit consistent reports conditional on joint participation, follows if T is sufficiently large. Suppose there is some state in which the side contract prescribes an exit for S alone. Given the construction of the grand contract, for any $(m_A, m_S) = ((\theta, \eta), e_S)$, there exists $\tilde{m}' \in \Delta(M_A \times M_S \backslash \{(\theta, \eta), e_S\})$ such that

$$(X_A(m_A, m_S) + X_S(m_A, m_S), q(m_A, m_S)) = (X_A(\tilde{m}') + X_S(\tilde{m}'), q(\tilde{m}')),$$

given

$$(X_A((\theta, \eta), e_S) + X_S((\theta, \eta), e_S), q((\theta, \eta), e_S)) = (\hat{X}(\tilde{m}^*(\theta_A)), \hat{q}(\tilde{m}^*(\theta_A)))$$

and the definition of $(\hat{X}, \hat{q})$. Therefore $\tilde{m}'$ and $(m_A, m_S) = ((\theta, \eta), e_S)$ generate the same total payment and output target for the coalition. A similar argument ensures that outcomes involving exit for A alone can be eliminated without loss of generality, since $(m_A, m_S) = (e_A, \eta)$ generates the same outcome $X_A = X_S = q = 0$ in the GC as $(m_A, m_S) = (e_A, e_S)$.

Step 4: Continuation payoffs following non-null side-contract

Suppose that S offers some non-null side-contract SC for η, which is described as $(\tilde{m}(\theta, \eta), \tilde{u}_A(\theta, \eta))$ which satisfies

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta) q(\tilde{m}(\theta', \eta))$$

for any $\theta, \theta' \in \Theta(\eta)$ and $\tilde{m}(\theta, \eta) \in \Delta(\hat{M})$. Let $\kappa^*(\theta) \in [0, 1]$ denote the probability that $\theta \in \Theta(\eta)$ accepts SC. We focus on PBE's with the property that A reports truthfully to S conditional on accepting the SC. The inequality above ensures that this is optimal for A. In any such PBE, the payoff resulting for S when A accepts the SC equals (in state θ, η):

$$X_A(\tilde{m}(\theta, \eta)) + X_S(\tilde{m}(\theta, \eta)) - \theta q(\tilde{m}(\theta, \eta)) - \tilde{u}_A(\theta, \eta).$$

If A rejects SC, A and S play the grand contract non-cooperatively with belief $p^*(\eta)$, which is consistent with Bayes rule as required in a PBE. Sequential rationality of A's participation decision $\kappa^*(\theta)$, given beliefs $p^*(\eta)$ and the non-cooperative equilibrium associated with $p^*(\eta)$, implies the following. In the T-case, $\kappa^*(\theta) = 0$ (or 1 or $\in [0, 1]$) if and only if $u_A(\theta, \eta) >$ (or $<$ or $=$) $\tilde{u}_A(\theta, \eta)$. A ends up with payoff

$$\max\{u_A(\theta, \eta), \tilde{u}_A(\theta, \eta)\},$$

and S's interim payoff is

$$E[\kappa^*(\theta)\{X_A(\tilde{m}(\theta,\eta))+X_S(\tilde{m}(\theta,\eta))-\theta q(\tilde{m}(\theta,\eta))-\tilde{u}_A(\theta,\eta)\}+(1-\kappa^*(\theta))u_S(\theta,\eta) \mid \eta].$$

Conversely, in the E-case, $\kappa^*(\theta)=0$ (or 1 or $\in[0,1]$) if and only if

$$\hat{X}(\tilde{m}^*(\theta_A))-\theta\hat{q}(\tilde{m}^*(\theta_A)) > (\text{or} < \text{ or } =)\tilde{u}_A(\theta,\eta).$$

A's payoff is

$$\max\{\hat{X}(\tilde{m}^*(\theta_A))-\theta\hat{q}(\tilde{m}^*(\theta_A)),\tilde{u}_A(\theta,\eta)\}.$$

while S's interim payoff is

$$E[\kappa^*(\theta)\{X_A(\tilde{m}(\theta,\eta))+X_S(\tilde{m}(\theta,\eta))-\theta q(\tilde{m}(\theta,\eta))-\tilde{u}_A(\theta,\eta)\} \mid \eta].$$

Step 5: Upper bound on S's interim payoff in continuation play following non-null side-contract

Here we establish an upper bound of S's interim payoff in PBE of the continuation game for non-null side-contract.

(i) T-Case

Consider the following problem: select $\hat{m}(\theta,\eta),\hat{u}_A(\theta,\eta)$ to

$$\max E[X_A(\hat{m}(\theta,\eta))+X_S(\hat{m}(\theta,\eta)-\theta q(\hat{m}(\theta,\eta))-\hat{u}_A(\theta,\eta) \mid \eta]$$

subject to $\hat{m}(\theta,\eta)\in\Delta(\hat{M})$,

$$\hat{u}_A(\theta,\eta)\geq\hat{u}_A(\theta',\eta)+(\theta'-\theta)q(\hat{m}(\theta',\eta))$$

for any $\theta,\theta'\in\Theta(\eta)$ and

$$\hat{u}_A(\theta,\eta)\geq u_A(\theta,\eta).$$

for any $\theta\in\Theta(\eta)$.

This is equivalent to problem $P(\eta)$ (in our paper) used to characterize WCP allocations. The WCP property implies that $(\hat{m}(\theta,\eta),\hat{u}_A(\theta,\eta)) = ((\theta,\eta),u_A(\theta,\eta))$ solves this problem and the maximum value is $E[u_S(\theta,\eta) \mid \eta]$.

We now show that this is an upper bound on S's interim payoff from the deviating side contract in the T-case. Suppose that non-null side-contract

$(\tilde{m}(\theta,\eta), t(\theta,\eta))$ is associated with acceptance probability $\kappa^*(\cdot)$ and the T-case applies. Select $(\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta))$ as follows:

$$\hat{m}(\theta,\eta) = \kappa^*(\theta)\tilde{m}(\theta,\eta) + (1-\kappa^*(\theta))I(\theta,\eta)$$

and

$$\hat{u}_A(\theta,\eta) = \max\{\tilde{u}_A(\theta,\eta), u_A(\theta,\eta)\}$$

where $I(\theta,\eta)$ is the probability measure concentrated on (θ,η). In this allocation, A earns exactly the same payoffs as in the continuation following offer of side-contract $(\tilde{m}(\theta,\eta), t(\theta,\eta))$. Hence the agent's incentive constraint is satisfied, and so is the participation constraint by construction. Hence the continuation play following offer of side-contract $(\tilde{m}(\theta,\eta), t(\theta,\eta))$ results in an interim payoff for S which cannot exceed $E[u_S(\theta,\eta) \mid \eta]$.

(ii) E-Case

Now consider the following problem: select $\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta)$ to

$$\max E[X_A(\hat{m}(\theta,\eta)) + X_S(\hat{m}(\theta,\eta)) - \theta q(\hat{m}(\theta,\eta)) - \hat{u}_A(\theta,\eta) \mid \eta]$$

subject to $\hat{m}(\theta,\eta) \in \Delta(\hat{M})$,

$$\hat{u}_A(\theta,\eta) \geq \hat{u}_A(\theta',\eta) + (\theta' - \theta)q(\hat{m}(\theta',\eta))$$

and

$$\hat{u}_A(\theta,\eta) \geq \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta)).$$

In order to derive the solution of this problem, consider the problem of maximizing

$$X_A(\hat{m}) + X_S(\hat{m}) - \theta q(\hat{m})$$

subject to $\hat{m} \in \Delta(\hat{M})$. Denoting its solution by $\hat{m}^*(\theta)$, we have

$$X_A(\hat{m}^*(\theta)) + X_S(\hat{m}^*(\theta)) - \theta q(\hat{m}^*(\theta)) = \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta)),$$

because of the definition of $(\hat{X}, \hat{q})$ and $\tilde{m}^*(\theta)$. Therefore in the above problem, an upper bound of objective function is given by

$$E[X_A(\hat{m}^*(\theta)) + X_S(\hat{m}^*(\theta)) - \theta q(\hat{m}^*(\theta)) - \{\hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta))\}|\eta] = 0$$

This upper bound can be achieved by selecting

$$(\hat{m}(\theta,\eta), \tilde{u}_A(\theta,\eta)) = (\hat{m}^*(\theta), \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta))).$$

Since this also satisfies all the constraints of the problem, this is a solution of the problem. It follows that the maximum value is equal to zero.

Next check that this maximum value provides an upper bound on S's payoff in the continuation play following the offer of the deviating side contract $\tilde{m}(\theta,\eta), t(\theta,\eta)$ in which the E-case arises. Select $(\hat{m}(\theta,\eta), \hat{u}_A(\theta,\eta))$ as follows:

$$\hat{m}(\theta,\eta) = \kappa^*(\theta)\tilde{m}(\theta,\eta) + (1-\kappa^*(\theta))\tilde{m}^*(\theta)$$

and

$$\hat{u}_A(\theta,\eta) = \max\{\tilde{u}_A(\theta,\eta), \hat{X}(\tilde{m}^*(\theta)) - \theta\hat{q}(\tilde{m}^*(\theta))\}.$$

This generates the same payoffs for A as in the continuation play following the offer of the deviating side contract $\tilde{m}(\theta,\eta), t(\theta,\eta)$, and is therefore feasible in the maximization problem above. Hence zero is an upper bound to S's interim expected payoff when the E-case applies.

Step 6: PBE in weak collusion game

We can construct a PBE in the overall weak collusion game as follows. If S offers null side-contract, he receives $E[u_S(\theta,\eta) \mid \eta]$. If S offers any non-null side-contract, it follows from Step 5 that his subsequent continuation payoff is not larger than $E[u_S(\theta,\eta) \mid \eta]$. Since $E[u_S(\theta,\eta) \mid \eta] \geq 0$, there exists a PBE in which S offers a null side-contract on the equilibrium path, resulting in allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$.

Step 7: Check PBE(wc) property

Finally check that PBE constructed in the above argument is also a PBE(wc). Otherwise there would exist a PBE resulting in a Pareto superior allocation for the coalition. This would violate the WCP property of the allocation we started with. ■

3 Justification for WCP Allocations When Contracts are Offered by Third Party

To address the problem highlighted by Celik and Peters (2011), the side-contract is modelled as a two stage game played by S and A. The first stage is a 'participation' stage where they communicate their participation decisions in the side contract, in addition to some auxiliary messages in the event of agreeing to participate. The role of these messages is to allow A to signal information about his type while agreeing to participate, which can help replicate whatever information is communicated by side-contract rejection in a setting where communication concerning participation decisions is dichotomous. A and S observe the messages sent by each other at the end of the first stage. At the second stage, A and S submit type reports, conditional on having agreed to participate at the first stage.

Let (D^p_A, D^p_S) denote the message sets of A and S at the participation stage (or p-stage). $e_A \in D^p_A$ and $e_S \in D^p_S$ are the exit options of A and S respectively. The message sets at this stage may include other auxiliary messages as well.

What occurs at the second stage ('execution' or e-stage) depends on $d^p = (d^p_A, d^p_S)$ chosen at the first stage.

- If $d^p_A \neq e_A$ and $d^p_S \neq e_S$, A and S select $(d^e_A, d^e_S) \in D^e_A(d^p) \times D^e_S(d^p)$ respectively, where the conditional message sets $D^e_A(d^p), D^e_S(d^p)$ are specified by the side contract. The report to P is selected according to $\tilde{m}(d^p, d^e) \in \Delta(M_A \times M_S)$, associated with the transfers to A and S, $t_A(d^p, d^e)$ and $t_S(d^p, d^e)$ respectively. Owing to wealth constraint of the third party, these are constrained to satisfy $t_A(d^p, d^e) + t_S(d^p, d^e) \leq 0$.

- If either $d^p_A = e_A$ or $d^p_S = e_S$, A and S play GC non-cooperatively.

Given GC and η, the third party decides whether to offer a side-contract $SC(\eta)$ or not (i.e., offer a null side-contract NSC). If a non-null side-contract is offered, A and S play a game denoted by $GC \circ SC(\eta)$ with two stages as described above. On the other hand, if the third party offers a null side-contract NSC at the first stage, A and S play GC non-cooperatively based on prior beliefs $p_\phi(\eta)$. The third-party's objective is to maximize $E[\alpha u_A(\theta, \eta) + (1-\alpha) u_S(\theta, \eta) \mid \eta]$ in state η.

The refinement WPBE(wc) introduced in the paper for the case where the side contract is offered by S, can now be extended as follows.

Definition 1 *Following the selection of a grand contract by P, a WPBE(wc) is a Weak Perfect Bayesian Equilibrium (WPBE) of the subsequent game in which side-contracts are designed by a third party, which has the following property. There does not exist some η for which there is a Weak Perfect Bayesian Equilibrium (WPBE) of subgame WC3 in which (conditional on η) the third-party's payoff is strictly higher, without lowering the payoff of S and any type of A.*

Definition 2 *An allocation* (u_A, u_S, q) *is achievable in the weak collusion game with side contracts designed by a third party assigning welfare weight* α *to A, if there exists a grand contract and a WPBE(wc) of the subsequent side contract subgame which results in this allocation.*

Proposition 1 *An allocation* (u_A, u_S, q) *is achievable in the weak collusion game with side contracts designed by a third party assigning welfare weight* α *to A, if and only if it is a WCP(*α*) allocation satisfying the interim participation constraints* $u_A(\theta, \eta) \geq 0$ *and* $E[u_S(\theta, \eta) \mid \eta] \geq 0$.

Proof of Proposition 1

Proof of Necessity

For some GC, suppose that allocation (u_A, u_S, q) is achieved in the game with weak collusion. Suppose the allocation is achieved as the outcome of a WPBE(wc) of subgame WC3 in which a non-null side contract $SC^*(\eta)$ is offered on the equilibrium path in some state η, which is rejected either by some types of A, or by S. We show it can also be achieved as the outcome of a WPBE(wc) in which a non-null side contract is offered in state η and always accepted by A and S. Let $d_A^p(\theta, \eta)$ and $d_S^p(\eta)$ denote A and S's participation decisions respectively (whether or not they chose the exit option at the first stage). Following rejection by either A or S, they play the grand contract GC based on updated beliefs $p(\cdot \mid d_A^p(\theta, \eta), d_S^p(\eta), \eta)$. Let $d_A^{p*}(\theta, \eta)$ denote A's decision, and $d_S^{p*}(\eta)$ S's participation decisions on the equilibrium path.

Now construct a new side-contract $\tilde{SC}(\eta)$ which differs from $SC^*(\eta)$ by replacing the message space D_A^p for A at the first stage by $D_A^p \times D_A^p$. Similarly S's message space is now $D_S^p \times D_S^p$. The interpretation is that the first component of this message d_A^p is a participation decision, while the second component $\tilde{d}_A^p$ is a 'signal'. This allows a decoupling of the participation decision from sending a signal to the other player which changes beliefs with which they play the grand contract noncooperatively in the event that the side contract is rejected by someone. For example, if A selected $d_A^p = e_A$ in the previous side-contract in order to send a signal about his type θ to S, the same signal can be sent now through the second component of the message, while opting to participate in the choice of the first component (by selecting $d_A^p \neq e_A, \tilde{d}_A^p = e_A$). The first component of the message d_A^p now matters only insofar as it is an exit decision or not; conditional on it not being an exit decision the precise message does not matter. If both decide to participate (i.e., not exit), they move on to the second stage of the game, where the mechanism replicates the allocation resulting on the equilibrium path of the original WPBE associated with $SC^*(\eta)$ (i.e., agrees with the second stage mechanism in $SC^*(\eta)$ whenever both agreed to participate in $SC^*(\eta)$, and otherwise assigns the allocation resulting from noncooperative play of GC in the original WPBE). If one or both decides not to participate in $\tilde{SC}(\eta)$, they play GC noncooperatively with beliefs based on first stage messages according to $p(\cdot \mid \tilde{d}^p_A(\theta, \eta), \tilde{d}^p_S(\eta), \eta)$. Note that these beliefs do not depend on d^p_A or d^p_S.

It is easily verified that there exists a WPBE where the third party offers $\tilde{SC}(\eta)$ in state η, in which A and S always accept the side-contract (i.e., in state θ, η they respectively select $d^p_A(\theta, \eta) \neq e_A, d^p_S(\eta) \neq e_S$ while choosing $\tilde{d}^p_A(\theta, \eta)$ equal to $d^p_A(\theta, \eta)$ in the original WPBE, and $\tilde{d}^p_S(\eta)$ equal to $d^p_S(\eta)$ in the original WPBE). The underlying idea is that since A's first stage report $\tilde{d}^p_A$ now affects beliefs at the second stage in exactly the same way that d^p_A did in the original WPBE, it is optimal for A to choose $\tilde{d}^p_A(\theta, \eta)$ equal to $d^p_A(\theta, \eta)$ in the original WPBE. Moreover, the first stage d^p_A report now affects only A's participation decision at the second stage, and by construction has no effect on second stage allocations (conditional on participation). So it is optimal for A to decide to participate. The same logic applies to S. Hence the newly constructed strategies and beliefs constitute a WPBE. It can also be verified that since the original equilibrium was a WPBE(wc), so is the newly constructed equilibrium.

Next we show that if allocation (u_A, u_S, q) is realized in a WPBE (wc) in which the offered side contract is not rejected on the equilibrium path, it must be a WCP(α) allocation. Suppose not: the allocation resulting from some non-null side contract $(\tilde{u}^*_A(\theta, \eta), \tilde{m}^*(\theta, \eta)) \neq (u_A(\theta, \eta), (\theta, \eta))$ solves problem $TP(\eta; \alpha)$ for some η. Define $\tilde{u}^*_S(\theta, \eta) \equiv \hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta \hat{q}(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}^*_A(\theta, \eta)$. It is evident that

$$E[\alpha \tilde{u}^*_A(\theta, \eta) + (1-\alpha)\tilde{u}^*_S(\theta, \eta) \mid \eta] > E[\alpha u_A(\theta, \eta) + (1-\alpha) u_S(\theta, \eta) \mid \eta],$$

$$\tilde{u}^*_A(\theta, \eta) \geq u_A(\theta, \eta)$$

and

$$E[\tilde{u}^*_S(\theta, \eta) \mid \eta] \geq E[u_S(\theta, \eta) \mid \eta].$$

There exists $m^c(\theta, \eta) \in \Delta(M_A \times M_S)$ in GC such that

$$(X_A(m^c(\theta, \eta)) + X_S(m^c(\theta, \eta)), q(m^c(\theta, \eta))) = (\hat{X}(\tilde{m}^*(\theta \mid \eta)), \hat{q}(\tilde{m}^*(\theta \mid \eta))).$$

Now construct a new side-contract $SC(\eta)$ which realizes

$$(\tilde{u}_A^*(\theta,\eta), \tilde{u}_S^*(\theta,\eta), \hat{q}(\tilde{m}^*(\theta \mid \eta)))$$

as a WPBE outcome, contradicting the hypothesis that (u_A, u_S, q) is realized in a WPBE (wc). $SC(\eta)$ is specified as follows:

- $D^p \equiv D^{p*}$ where $D^{p*} = (D_A^{p*}, D_S^{p*})$ are A and S's message sets at the participation stage of the original side-contract $SC^*(\eta)$.
- $D_A^e = \Theta(\eta)$ and $D_S^e = \phi$
- A's choice of $d_A^e = \theta \in \Theta(\eta)$ generates the report $m^c(\theta,\eta)$ to P, and side transfers to A and S respectively as follows:

$$t_A(\theta,\eta) = \tilde{u}_A^*(\theta,\eta) - [X_A(m^c(\theta,\eta)) - \theta q(m^c(\theta,\eta))]$$

  and

$$t_S(\theta,\eta) = \tilde{u}_S^*(\theta,\eta) - X_S(m^c(\theta,\eta)).$$

Given any (d_A^p, d_S^p) with $d_A^p \neq e_A$ and $d_S^p \neq e_S$ at the participation stage, it is optimal for A to always select $d_A^e = \theta$, since $\theta^{'} = \theta$ maximizes

$$X_A(m^c(\theta^{'},\eta)) - \theta q(m^c(\theta^{'},\eta)) + t_A(\theta^{'},\eta) = \tilde{u}_A^*(\theta^{'},\eta) + (\theta^{'} - \theta)\hat{q}(\tilde{m}^*(\theta^{'} \mid \eta)).$$

At the participation stage, A is indifferent among any $d_A^p \in D_A^p \backslash \{e_A\}$ as the optimal response to $d_S^p \neq e_S$, since the outcome in the continuation game does not depend on this choice. Select beliefs consequent on non-participation by either A or S in the same way as in the original equilibrium; then participation continues to be optimal for both. In state η, responses to all other side contract offers are unchanged. In all other states $\eta' \neq \eta$, strategies and beliefs are unchanged. Hence this is a WPBE resulting in $(\tilde{u}_A^*(\theta,\eta), \tilde{u}_S^*(\theta,\eta))$, contradicting the WPBE (wc) property of the equilibrium resulting in (u_A, u_S, q). This completes the proof of necessity.

Proof of Sufficiency

Take an allocation which is WCP(α) and satisfies interim participation constraints. We show it is achievable as a WPBE(wc) outcome following choice of the following grand contract GC:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \Pi \cup \{e_S\}$$

$$X_A(m_A, m_S) = X_S(m_A, m_S) = q(m_A, m_S) = 0$$

for (m_A, m_S) such that either $m_A = e_A$ or $m_S = e_S$.

- $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (u_A(\theta_A, \eta_S) + \theta_A q(\theta_A, \eta_S), q(\theta_A, \eta_S))$ for $\eta_A = \eta_S$ and $(X_A((\theta_A, \eta_A), \eta_S), q((\theta_A, \eta_A), \eta_S)) = (-T, 0)$ for $\eta_A \neq \eta_S$

- $X_S((\theta_A, \eta_A), \eta_S) = u_S(\theta_A, \eta_A)$ for $\eta_S = \eta_A$ and $X_S((\theta_A, \eta_A), \eta_S) = -T$ for $\eta_S \neq \eta_A$

where $T > 0$ is sufficiently large. The WCP(α) property implies that $u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$ for any $\theta, \theta' \in \Theta(\eta)$. The interim participation constraints imply that this grand contract has a non-cooperative pure strategy equilibrium

$$(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$$

based on prior beliefs.

For this grand contract, we claim there exists a WPBE(wc) resulting in $(m_A^*(\theta, \eta), m_S^*(\eta)) = ((\theta, \eta), \eta)$. Let the third party offer a null side contract, following which A and S play truthfully in the GC noncooperatively with prior beliefs. If the third party offers any non-null side contract, all types of A and S reject it and subsequently play truthfully in the noncooperative game with prior beliefs as long as either A or S rejects it. This is clearly a WPBE. That it is a WPBE(wc) follows from the property that the allocation is WCP(α). ■

This result also holds with PBE(wc) instead of WPBE(wc). The sufficiency argument extends straightforwardly: the constructed WPBE of WC3 following the same GC is also a PBE as it satisfies the belief restrictions imposed by this notion. The WCP(α) property then implies it is a PBE(wc). The first part of the necessity argument which augments first stage message spaces and constructs an equivalent equilibrium is unaffected when WPBE is replaced by PBE. The second part of the necessity argument also extends given that any PBE(wc) is also WPBE(wc).

4 Possibility of Second Best without Collusion

Here we show that the second best allocation is uniquely achievable in the absence of collusion between A and S. Any feasible allocation (u_A, u_S, q) has to satisfy the incentive constraint of A and the interim participation constraints of A and S:

$$u_A(\theta, \eta) \geq u_A(\theta^{'}, \eta) + (\theta^{'} - \theta)q(\theta^{'}, \eta) \tag{1}$$

for any $\theta, \theta^{'} \in \Theta(\eta)$ and any $\eta \in \Pi$

$$u_A(\theta, \eta) \geq 0 \tag{2}$$

for any $(\theta, \eta) \in K$

$$E[u_S(\theta, \eta) \mid \eta] \geq 0 \tag{3}$$

for any $\eta \in \Pi$. We obtain the following result by using the grand contract similar to one constructed in Faure-Grimaud, Laffont and Martimort (2003).

Proposition 2 *Given any allocation (u_A, u_S, q) satisfying (1), (2) and (3), there exists a grand contract such that (u_A, u_S, q) is realized in a unique non-cooperative equilibrium.*

Proof of Proposition 2:

Step 1: Construction of function D

From our assumption about informativeness of η, there exists a subset of Θ with positive measure such that $a(\eta \mid \theta) \neq a(\eta^{'} \mid \theta)$ for any $\eta, \eta^{'} \in \Pi$ $(\eta \neq \eta^{'})$. Then we can select two intervals of θ with the positive probability measure, Θ_1 and Θ_2, so that $f(\theta \mid \eta) > f(\theta \mid \eta^{'})$ for $\theta \in \Theta_1$ and $f(\theta \mid \eta) < f(\theta \mid \eta^{'})$ for $\theta \in \Theta_2$. Then for any $((\theta, \eta^{'}), \eta)$ $(\eta \neq \eta^{'})$, there exists a function $D((\theta, \eta^{'}), \eta)$ such that

$$E[D((\theta, \eta^{'}), \eta) \mid \eta] > 0 > E[D((\theta, \eta^{'}), \eta) \mid \eta^{'}].$$

For instance let us select a function $d((\theta, \eta^{'}), \eta)$ such that it is positive on Θ_1, negative on Θ_2 and zero elsewhere. $D((\theta, \eta^{'}), \eta) \equiv d((\theta, \eta^{'}), \eta) - a$ satisfies the above inequality with the appropriate choice of $a \in \Re$.

Step 2: Extension of allocation

For allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ defined on $K \equiv \{(\theta,\eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$ satisfying (1), (2) and (3), we construct $(\tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta), \tilde{q}(\theta,\eta))$ on $\Theta \times \Pi$, which is an extension of $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ over the augmented domain, such that

(a) $(\tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta), \tilde{q}(\theta,\eta)) = (u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ for $(\theta,\eta) \in K$

(b) $(\tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta), \tilde{q}(\theta,\eta)) = (u_A(\underline{\theta}(\eta),\eta)+(\underline{\theta}(\eta)-\theta)q(\underline{\theta}(\eta),\eta), u_S(\underline{\theta}(\eta),\eta), q(\underline{\theta}(\eta),\eta))$ for any $(\theta,\eta) \in K$ such that $\theta \in [\underline{\theta}, \underline{\theta}(\eta))$

(c) $(\tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta), \tilde{q}(\theta,\eta)) = (u_A(\bar{\theta}(\eta),\eta)+(\bar{\theta}(\eta)-\theta)q(\bar{\theta}(\eta),\eta), u_S(\bar{\theta}(\eta),\eta), q(\bar{\theta}(\eta),\eta))$ for any $(\theta,\eta) \in K$ such that $\theta \in (\bar{\theta}(\eta), \bar{\theta}]$ and $u_A(\bar{\theta}(\eta),\eta) + (\bar{\theta}(\eta) - \theta)q(\bar{\theta}(\eta),\eta) \geq 0$

(d) $(\tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta), \tilde{q}(\theta,\eta)) = (0,0,0)$ for any $(\theta,\eta) \in K$ such that $\theta \in (\bar{\theta}(\eta), \bar{\theta}]$ and $u_A(\bar{\theta}(\eta),\eta) + (\bar{\theta}(\eta) - \theta)q(\bar{\theta}(\eta),\eta) < 0$.

$(\tilde{u}_A(\theta,\eta), \tilde{q}(\theta,\eta))$ satisfies (1) and (2) on $\Theta \times \Pi$, since

$$\tilde{u}_A(\theta,\eta) \geq 0$$

for any $(\theta,\eta) \in \Theta \times \Pi$ and

$$\tilde{u}_A(\theta,\eta) \geq \tilde{u}_A(\theta',\eta) + (\theta' - \theta)\tilde{q}(\theta',\eta)$$

for any $\theta, \theta' \in \Theta$ and any $\eta \in \Pi$.

Step 3: Construction of grand contract

Now construct a grand contract

$$(X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S); M_A, M_S)$$

where $M_A = \Theta \times \Pi \cup \{e_A\}$ and $M_S = \Pi \cup \{e_S\}$ as follows:

(i) For $(m_A, m_S) = ((\theta,\eta'),\eta)$, $X_A((\theta,\eta'),\eta) = \tilde{u}_A(\theta,\eta) + \theta\tilde{q}(\theta,\eta) - T(\eta,\eta')$ with $T(\eta,\eta') = 0$ for $\eta = \eta'$ and $T(\eta,\eta') = T > 0$ for $\eta \neq \eta'$ where T is sufficiently large.

(ii) For $m_A = e_A$, $(X_A(e_A, m_S), X_S(e_A, m_S), q(e_A, m_S)) = (0,0,0)$ for any $m_S \in M_S$.

(iii) For $(m_A, m_S) = ((\theta,\eta),\eta)$, $X_S((\theta,\eta),\eta) = \tilde{u}_S(\theta,\eta)$. For $m_S = e_S$, $X_S(m_A, e_S) = 0$ for any $m_A \in M_A$.

(iv) For $(m_A, m_S) = ((\theta,\eta^{'}),\eta)$ $(\eta \neq \eta^{'})$,

$$X_S((\theta,\eta^{'}),\eta) = D((\theta,\eta^{'}),\eta)$$

if $(\eta,\eta^{'})$ satisfies $E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] = E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta] \leq 0$, and

$$X_S((\theta,\eta^{'}),\eta) = \tilde{u}_S(\theta,\eta^{'}) + D((\theta,\eta^{'}),\eta).$$

if $(\eta,\eta^{'})$ satisfies $E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] = E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta] > 0$.

(v) For $(m_A, m_S) = ((\theta,\eta), e_S)$,

$$(X_A((\theta,\eta),e_S), q((\theta,\eta),e_S)) = (X_A(m^*(\theta)) + X_S(m^*(\theta)), q(m^*(\theta)))$$

for any $(\theta,\eta) \in \Theta \times \Pi$ where

$$m^*(\theta) \in \max_{m \in \{(e_A,e_S)\} \cup M_A \times M_S \setminus e_S} X_A(m) + X_S(m) - \theta q(m).$$

Step 4: Some properties of the grand contract

First we argue some implications from (iv) and our construction of $D((\theta,\eta^{'}),\eta)$. From (iv), for $\eta \neq \eta^{'}$, if $E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] \leq 0$,

$$E[X_S((\theta,\eta^{'}),\eta) \mid \eta] = E[D((\theta,\eta^{'}),\eta) \mid \eta] > 0 \geq E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] = E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta]$$

and if $E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] > 0$,

$$\begin{aligned} & E[X_S((\theta,\eta^{'}),\eta) \mid \eta] = E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta] + E[D((\theta,\eta^{'}),\eta) \mid \eta] \\ > \; & E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta] = E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] > 0. \end{aligned}$$

These are summarized into

$$E[X_S((\theta,\eta^{'}),\eta) \mid \eta] > \max\{0, E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta]\}.$$

On the other hand, (3) implies

$$E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta^{'}] = E[u_S(\theta,\eta^{'}) \mid \eta^{'}] \geq 0.$$

Similarly from (iv), if $E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] \leq 0$,

$$E[X_S((\theta,\eta^{'}),\eta) \mid \eta^{'}] = E[D((\theta,\eta^{'}),\eta) \mid \eta^{'}] < 0 \leq E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta^{'}]$$

and if $E[\tilde{u}_S(\theta,\eta^{'}) \mid \eta] > 0$,

$$E[X_S((\theta,\eta^{'}),\eta) \mid \eta^{'}] = E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta^{'}]+E[D((\theta,\eta^{'}),\eta) \mid \eta^{'}] < E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta^{'}].$$

The above two inequalities are summarized into

$$E[X_S((\theta,\eta^{'}),\eta^{'}) \mid \eta^{'}] > E[X_S((\theta,\eta^{'}),\eta) \mid \eta^{'}].$$

These arguments imply that if A reports truthfully, S has an incentive to participate in the grand contract and report η truthfully, irrespective of A's report about η.[2]

Next we examine some implications of (v). By definition of $m^*(\theta)$ in (v), $(X_A((\theta,\eta),e_S), q((\theta,\eta),e_S))$ satisfies

$$X_A((\theta,\eta),e_S) - \theta q((\theta,\eta),e_S)) \geq X_A((\theta^{'},\eta),e_S) - \theta q((\theta^{'},\eta),e_S)$$

and

$$X_A((\theta,\eta),e_S) - \theta q((\theta,\eta),e_S)) \geq 0$$

for any $(\theta,\eta) \in \Theta \times \Pi$. Therefore even when S does not participate on the grand contract, A always has an incentive to participate to the grand contract and report θ truthfully.[3]

Step 5: Non-cooperative equilibrium

Now let us check that the following strategies constitute a non-cooperative equilibrium of the grand contract. All types of A always have incentives to participate to the grand contract and report θ truthfully (associated with the same report of η as S has). On the other hand, if A reports θ truthfully, S also has an incentive to participate in the grand contract and report η truthfully, regardless of A's report about η. With S reporting η truthfully, A has an incentive to report η truthfully in order to avoid a large penalty T. Therefore there exists an equilibrium $m_A(\theta,\eta) = (\theta,\eta)$ and $m_S(\eta) = \eta$ such that all types of A and S participate to the grand contract and have the truthful report. Moreover it is evident that this is the unique equilibrium. ■

[2]Note that S's indifference between the participation and the non-participation can be broken with small adjustment of $u_S(\theta,\eta)$.

[3]Note that the A's indifference between the participation and the non-participation can be broken with small adjustment of $X_A((\theta,\eta),e_S)$.

5 Possibility of Second Best without Refinement Restriction

Here we provide a proof regarding the possibility of second best allocation in the case that we do not impose any refinement criterion on equilibrium concept, besides WPBE. If the mechanism design problem is stated as selection of an allocation by the principal subject to the constraint that it can be achieved as the outcome of some WPBE following a choice of a grand contract, it is presumed that the principal is free to select continuation beliefs and strategies for noncooperative play of the grand contract following off-equilibrium path rejections of offered side contracts by S to A.

Proposition 3 *The second-best allocation is achievable in a WPBE of the weak collusion game.*

Proof of Proposition 3: For second best allocation $(u_A^{SB}, u_S^{SB}, q^{SB})$, let us construct the following grand contract which is a revelation mechanism satisfying

$$(X_A(m_A, m_S), X_S(m_A, m_S), q(m_S, m_A); M_S, M_A)$$

where $M_S = \Pi \cup \{e_S\}$ and $M_A = \Theta \cup \{e_A\}$.

(i) $X_S(m_A, m_S) = 0$ for any (m_A, m_S).

(ii) $q(\theta, \eta) = q^{SB}(\theta, \eta)$ and $X_A(\theta, \eta) = \theta q^{SB}(\theta, \eta) + u_A^{SB}(\theta, \eta)$, if $(m_A, m_S) = (\theta, \eta) \in K$, otherwise both are set equal to zero.

(iii) $X_A(e_A, m_S) = q(e_A, m_S) = 0$ for any m_S.

(iv) $(X_A(\theta, e_S), q(\theta, e_S)) = (\hat{X}_A(\theta), \hat{q}(\theta))$, which satisfies the following properties: (a) $\hat{X}_A(\theta) - \theta\hat{q}(\theta) \geq \hat{X}_A(\theta^{'}) - \theta\hat{q}(\theta^{'})$ for any $\theta, \theta^{'} \in \Theta$, (b) $\hat{X}_A(\theta) - \theta\hat{q}(\theta) \geq 0$ for any $\theta \in \Theta$ and (c) there exists $\theta^{'} \in \Theta$ such that $\hat{q}(\theta^{'}) = q(\theta, \eta)$ and $\hat{X}_A(\theta^{'}) > X_A(\theta, \eta)$ for any $(\theta, \eta) \in \Theta \times \Pi$.[4]

[4]For instance, we can choose $(\hat{X}_A(\theta), \hat{q}(\theta))$ such that (i) $\hat{q}(\theta)$ is continuous and strictly decreasing in θ with $\hat{q}(\underline{\theta}) = \max_{(\theta,\eta)\in\Theta\times\Pi} q(\theta, \eta)$ and $\hat{q}(\bar{\theta}) = \min_{(\theta,\eta)\in\Theta\times\Pi} q(\theta, \eta)$, and (ii) $\hat{X}_A(\theta) = \theta\hat{q}(\theta) + \int_{\theta}^{\bar{\theta}} \hat{q}(y)dy + R$ for sufficiently large $R > 0$

For this grand contract, we will check that the second best allocation is achieved in WPBE of collusion game. In Bayesian game induced by this grand contract, both $(m_A(\theta,\eta), m_S(\eta)) = (\theta,\eta)$ and $(m_A(\theta,\eta), m_S(\eta)) = (\theta, e_S)$ are non-cooperative equilibria, regardless of S's belief about θ. Let our focus be provided to WPBE such that $(m_A(\theta,\eta), m_S(\eta)) = (\theta,\eta)$ is realized in the event that a side-contract (SC) is not offered by S, while that $(m_A(\theta,\eta), m_S(\eta)) = (\theta, e_S)$ is realized in the event that SC is offered by S and is rejected by A. In the latter case, A earns $\hat{X}_A(\theta) - \theta\hat{q}(\theta)$. In order to check that S does not benefit from offering a non-null side-contract, let us consider the following problem:

$$\max E[X_A(\tilde{m}(\theta,\eta)) + X_S(\tilde{m}(\theta,\eta)) - \theta q(\tilde{m}(\theta,\eta)) - \tilde{u}_A(\theta,\eta) \mid \eta]$$

subject to $\tilde{m}(\theta,\eta) \in \Delta(M_A \times M_S)$,

$$\tilde{u}_A(\theta,\eta) \geq \tilde{u}_A(\theta',\eta) + (\theta' - \theta)q(\tilde{m}(\theta',\eta))$$

for any $\theta, \theta' \in \Theta(\eta)$ and

$$\tilde{u}_A(\theta,\eta) \geq \hat{X}_A(\theta) - \theta\hat{q}(\theta)$$

for any (θ,η). By the construction of $(\hat{X}_A(\theta), \hat{q}(\theta))$ in (iv), $\tilde{m}(\theta,\eta) = (\theta, e_S)$ (meaning probability measure with concentration on (θ, e_S)) and $\tilde{u}_A(\theta,\eta) = \hat{X}_A(\theta) - \theta\hat{q}(\theta)$ solve this problem. Then the maximum value is equal to zero. Since A at least receives $\hat{X}_A(\theta) - \theta\hat{q}(\theta)$ in the continuation game for non-null side-contract, this maximum value provides an upper bound of S's payoff in WPBE from offering non-null side-contract. It means that S never benefits from offering non-null side-contract. Consequently, there is a WPBE of this game in which S never offers any side contract. This implies that S and A play $(m_A(\theta,\eta), m_S(\eta)) = (\theta,\eta)$ and the second-best allocation is achieved, concluding the statement of the proposition. ■

6 Suboptimality of Pure Delegation: Proof for More General Model

In the Appendix to the paper, we provided a proof of the suboptimality of the pure delegation (in Proposition 1) which is specific to the case of strictly concave V satisfying Inada conditions. With small modification of the proof, we can show that this result holds in more general environments that include the case of linear V or an indivisible good. Here we impose only the following weak assumption. Suppose that $V(q)$ is differentiable, increasing and concave with $V(0) = 0$, and the domain of q is Q where Q is a closed set with $0 \in Q$ and $q \geq 0$ for any $q \in Q$. $q^*(h)$ is defined as

$$q^*(h) \in \arg\max_{q \in Q} V(q) - hq.$$

Assumption 1 *$q^*(h)$ exists for any $h \geq \underline{\theta}$ and there exists $h > \underline{\theta}$ such that $q^*(h) > 0$.*

This is a reasonable assumption, since if $q^*(h) = 0$ for any $h > \underline{\theta}$, P never benefit from hiring A, even when θ is observed by P. This assumption also implies $\Pi_{NS} = E[V(q^{NS}(\theta)) - H(\theta)q^{NS}(\theta)] > 0$ where $q^{NS}(\theta)$ maximizes $V(q) - H(\theta)q$, since $q^{NS}(\theta) > 0$ for θ sufficiently close to $\underline{\theta}$.

This assumption is satisfied when $V(q)$ is strictly concave, satisfying Inada conditions and $Q = R_+$, or $V(q) = Vq$ with $Q = [0, 1]$. Then we obtain the following statement.

Proposition 4 $\Pi_{DS} < \Pi_{NS}$*: delegated supervision is worse for the Principal compared to hiring no supervisor.*

Proof of Proposition 4:

At the first step, note that the optimal side contract problem for S in DS involves an outside option for A which is identically zero. This reduces to a standard problem of contracting with a single agent with cost $\hat{h}(\theta|\eta)$, which is obtained by applying the ironing rule to $h(\theta|\eta)$ and distribution $F(\theta|\eta)$. P's prior over this supplier's cost is given by distribution function

$$G(h) \equiv \Pr((\theta, \eta) \mid \hat{h}(\theta \mid \eta) \leq h)$$

for $h \geq \underline{\theta}$ and $G(h) = 0$ for $h < \underline{\theta}$. Let $G(h \mid \eta)$ denote the cumulative distribution function of $h = \hat{h}(\theta \mid \eta)$ conditional on η:

$$G(h \mid \eta) \equiv \Pr(\theta \mid \hat{h}(\theta \mid \eta) \leq h, \eta)$$

for $h \geq \hat{h}(\underline{\theta}(\eta) \mid \eta) (= \underline{\theta}(\eta))$ and $G(h \mid \eta) = 0$ for $h < \underline{\theta}(\eta)$. Then $G(h) = \Sigma_{\eta \in \Pi} p(\eta) G(h \mid \eta)$. Since $\hat{h}(\theta \mid \eta)$ is continuous on $\Theta(\eta)$, $G(h \mid \eta)$ is strictly increasing in h on $[\underline{\theta}(\eta), \hat{h}(\bar{\theta}(\eta) \mid \eta)]$. However, $G(h \mid \eta)$ may fail to be left-continuous.

Hence P's problem in DS reduces to

$$\max E_h[V(q(h)) - X(h)]$$

subject to

$$X(h) - hq(h) \geq X(h') - hq(h')$$

for any $h, h' \in [\underline{\theta}, \bar{h}]$ and

$$X(h) - hq(h) \geq 0$$

for any $h \in [\underline{\theta}, \bar{h}]$ where the distribution function of h is $G(h)$ and $\bar{h} \equiv \max_{\eta \in \Pi} \hat{h}(\bar{\theta}(\eta) \mid \eta)$. The corresponding problem in NS is

$$\max E_\theta[V(q(\theta)) - X(\theta)]$$

subject to

$$X(\theta) - \theta q(\theta) \geq X(\theta') - \theta q(\theta')$$

for any $\theta, \theta' \in \Theta$ and

$$X(\theta) - \theta q(\theta) \geq 0$$

for any $\theta \in \Theta$. The two problems differ only in the underlying cost distributions of P: $G(h)$ in the case of DS and $F(\theta)$ in the case of NS. Since $\theta < \hat{h}(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$,

$$G(h \mid \eta) \equiv \Pr(\theta \mid \hat{h}(\theta \mid \eta) \leq h, \eta) < \Pr(\theta \mid \theta \leq h, \eta) = F(h \mid \eta)$$

for $h \in (\underline{\theta}(\eta), \hat{h}(\bar{\theta}(\eta) \mid \eta))$, implying

$$G(h) = \Sigma_{\eta \in \Pi} p(\eta) G(h \mid \eta) < \Sigma_{\eta \in \Pi} p(\eta) F(h \mid \eta) = F(h)$$

for any $h \in (\underline{\theta}, \bar{h})$. Therefore the distribution of h in DS (strictly) dominates that of θ in NS in the first order stochastic sense. Since $\hat{h}(\bar{\theta}(\eta) \mid \eta) > \bar{\theta}$ for η such that $\bar{\theta}(\eta) = \bar{\theta}$, $\bar{h} > \bar{\theta}$.

The optimal payoffs in DS and NS are given respectively by

$$\Pi_{DS} = \max_{q(h) \in Q} \int_{\underline{h}}^{\bar{h}} [V(q(h)) - hq(h) - \int_{h}^{\bar{h}} q(y) dy] dG(h)$$

subject to $q(h)$ is non-increasing in h, and

$$\Pi_{NS} = \max_{q(h)\in Q} \int_{\underline{\theta}}^{\bar{\theta}} [V(q(h)) - \theta q(h) - \int_{h}^{\bar{\theta}} q(h)dy]dF(h)$$

subject to $q(h)$ is non-increasing in h. Let $q^{DS}(h)$ and $q^{NS}(h)$ be optimal output schedule in DS and NS respectively.

First suppose that $q^{DS}(h)$ is constant on $(\underline{h}, \bar{h})$, which may occur depending on $V(q)$ and Q. With $q^{DS}(h) = q$,

$$\int_{\underline{h}}^{\bar{h}} [V(q(h)) - hq(h) - \int_{h}^{\bar{h}} q(y)dy]dG(h) = V(q) - \bar{h}q.$$

Therefore $\Pi_{DS} = V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})$. If $q^*(\bar{h}) = 0$, $\Pi_{DS} = 0 < \Pi_{NS}$. Suppose that $q^*(\bar{h}) > 0$. Then $q(h) = q^*(\bar{h})$ for $h \in [\underline{\theta}, \bar{\theta}]$ is implementable in NS, bringing the P's payoff $V(q^*(\bar{h})) - \bar{\theta}q^*(\bar{h})$. With $\bar{h} > \bar{\theta}$, it is large than Π_{DS}.

Next suppose that $q^{DS}(h)$ is not constant on $(\underline{h}, \bar{h})$, implying $\Pi_{DS} > V(q^*(\bar{h})) - \bar{h}q^*(\bar{h})$. First we show that $q^{DS}(h) \leq q^*(h)$ for any $h \in [\underline{h}, \bar{h}]$. Suppose that there exists some interval over which $q^{DS}(h) > q^*(h)$. Then we can replace the portion of $q^{DS}(h)$ with $q^{DS}(h) > q^*(h)$ by $q^*(h)$, without violating the constraint that $q(h)$ is non-increasing. It raises the value of the objective function, since $V(q^{DS}(h)) - hq^{DS}(h) \leq V(q^*(h)) - hq^*(h)$ for h where $q^{DS}(h)$ is replaced by $q^*(h)$, and $\int_{h}^{\bar{h}} q(y)dy$ decreases with this replacement. We obtain a contradiction, implying that $q^{DS}(h) \leq q^*(h)$ almost everywhere on $[\underline{h}, \bar{h}]$. We can select $q^*(h)$ which is left-continuous without loss of generality. Then since both $q^*(h)$ and $q^{DS}(h)$ are non-increasing, $q^{DS}(h) \leq q^*(h)$ must hold for any $h \in [\underline{h}, \bar{h}]$.

Define

$$\Phi(h) \equiv V(q^{DS}(h)) - hq^{DS}(h) - \int_{h}^{\bar{h}} q^{DS}(y)dy.$$

We claim that $\Phi(h)$ is left-continuous and bounded. First we show that without loss of generality our attention can be restricted to the case that $q^{DS}(h)$ is left-continuous. Otherwise, there exists $h' \in (\underline{h}, \bar{h})$ such that $q^{DS}(h'-) > q^{DS}(h')$. Now consider $\tilde{q}^{DS}(h)$ (which is left-continuous at h') such that $\tilde{q}^{DS}(h') = q^{DS}(h'-)$ and $\tilde{q}^{DS}(h) = q^{DS}(h)$ for any $h \neq h'$. This is possible because Q is closed set. Defining $\tilde{\Phi}(h) \equiv V(\tilde{q}^{DS}(h)) - h\tilde{q}^{DS}(h) - \int_{h}^{\bar{h}} \tilde{q}^{DS}(y)dy$, observe that $\tilde{\Phi}(h) = \Phi(h)$ for $h \neq h'$ and $\tilde{\Phi}(h) \geq \Phi(h)$ when $h = h'$. Then

$$\begin{aligned} & \int_{[\underline{h},\bar{h}]} \tilde{\Phi}(h)dG(h) = \int_{[\underline{h},\bar{h}]\setminus h'} \tilde{\Phi}(h)dG(h) + \tilde{\Phi}(h')[G(h'+) - G(h'-)] \\ \geq & \int_{[\underline{h},\bar{h}]\setminus h'} \tilde{\Phi}(h)dG(h) + \Phi(h')[G(h'+) - G(h'-)] = \int_{[\underline{h},\bar{h}]} \Phi(h)dG(h). \end{aligned}$$

This implies in turn that our attention is restricted to left-continuous $\Phi(h)$ without loss of generality. $\Phi(h)$ is also bounded, since $0 \leq q^{DS}(h) \leq q^*(h)$ for any h and $q^*(h)$ is bounded from Assumption 1.

Next we claim that $\Phi(h)$ is non-increasing in h and is not constant on $(\underline{h}, \bar{h})$. To show the former, note that for any h, we have

$$\begin{aligned} & \lim_{\epsilon\to 0+} \frac{\Phi(h+\epsilon) - \Phi(h)}{\epsilon} \\ = & \lim_{\epsilon\to 0+} (1/\epsilon)[V(q^{DS}(h+\epsilon)) - (h+\epsilon)q^{DS}(h+\epsilon) - \int_{h+\epsilon}^{\bar{h}} q^{DS}(y)dy \\ - & [V(q^{DS}(h)) - hq^{DS}(h) - \int_{h}^{\bar{h}} q^{DS}(y)dy]] \\ = & [V'(\hat{q}(h)) - h] \lim_{\epsilon\to 0+} \frac{q^{DS}(h+\epsilon) - q^{DS}(h)}{\epsilon} \\ - & q^{DS}(h+) + \lim_{\epsilon\to 0+} (1/\epsilon) \int_{h}^{h+\epsilon} q^{DS}(y)dy \\ = & [V'(\hat{q}(h)) - h] \lim_{\epsilon\to 0+} \frac{q^{DS}(h+\epsilon) - q^{DS}(h)}{\epsilon} \end{aligned}$$

for some $\hat{q}(h) \in [q^{DS}(h+), q^{DS}(h)]$. If $V'(q^*(h)) \geq h$, this is non-positive since $V'(\hat{q}(h)) \geq V'(q^{DS}(h)) \geq V'(q^*(h)) \geq h$ (from $q^{DS}(h) \leq q^*(h)$ and the concavity of $V(q)$) and $\lim_{\epsilon\to 0+} \frac{q^{DS}(h+\epsilon)-q^{DS}(h)}{\epsilon} \leq 0$. This is zero, if $V'(q^*(h)) < h$, since it implies $q^*(h) = 0$ and $q^{DS}(h') = 0$ for any $h' \geq h$, resulting in $\lim_{\epsilon\to 0+} \frac{q^{DS}(h+\epsilon)-q^{DS}(h)}{\epsilon} = 0$. Because of left-continuity of $\Phi(h)$, it implies that $\Phi(h)$ is non-increasing in h.

In order to show that $\Phi(h)$ is not constant on $(\underline{h}, \bar{h})$, suppose otherwise that $\Phi(h)$ is constant. Then

$$\Phi(h) = \Phi(\bar{h}-) = V(q^{DS}(\bar{h}-)) - \bar{h}q^{DS}(\bar{h}-),$$

which is equal to Π_{DS}. This contradicts that

$$\Pi_{DS} > V(q^*(\bar{h})) - \bar{h}q^*(\bar{h}) \geq V(q^*(\bar{h}-)) - \bar{h}q^*(\bar{h}-).$$

Now consider the contracting problem in NS with cost distribution $F(h)$. Since $q^{DS}(h)$ is non-increasing in h, it is feasible for P to select this output schedule in NS. Hence $\Pi_{NS} \geq \int_{\underline{h}}^{\bar{h}} \Phi(h)dF(h)$. Therefore if $\int_{\underline{h}}^{\bar{h}} \Phi(h)dF(h) > \int_{\underline{h}}^{\bar{h}} \Phi(h)dG(h) = \Pi_{DS}$, it follows that $\Pi_{NS} > \Pi_{DS}$. Since $G(h)$ is right-continuous and $\Phi(h)$ is left-continuous and bounded, we can integrate by parts:

$$\int_{\underline{h}}^{\bar{h}} \Phi(h)dG(h) + \int_{\underline{h}}^{\bar{h}} G(h)d\Phi(h) = \Phi(\bar{h})G(\bar{h}) - \Phi(\underline{h})G(\underline{h}) = \Phi(\bar{h}).$$

Similarly for $F(h)$ which is continuous,

$$\int_{\underline{h}}^{\bar{h}} \Phi(h)dF(h) + \int_{\underline{h}}^{\bar{h}} F(h)d\Phi(h) = \Phi(\bar{h})F(\bar{h}) - \Phi(\underline{h})F(\underline{h}) = \Phi(\bar{h}).$$

Hence

$$\int_{\underline{h}}^{\bar{h}} \Phi(h)dF(h) - \int_{\underline{h}}^{\bar{h}} \Phi(h)dG(h) = \int_{\underline{h}}^{\bar{h}} [G(h) - F(h)]d\Phi(h).$$

By the property of $\Phi(h)$ and $F(h) > G(h)$ for $h \in (\underline{h}, \bar{h})$, this is positive. ■

7 Optimality of Conditional Delegation

Here we provide the formal proof of Proposition 6 in the paper, and also some arguments about the reverse pattern of modified delegation where P communicates only with A on the equilibrium path.

Proof of Proposition 6

Step 1: Construction of grand contract

For WCP allocation (u_A, u_S, q) which satisfies interim participation constraints of A and S, define $X(\theta, \eta) \equiv u_A(\theta, \eta) + u_S(\theta, \eta) + \theta q(\theta, \eta)$ for $(\theta, \eta) \in$

$K \equiv \{(\theta,\eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$. Let us construct the following grand contract GC with sufficiently large $T > 0$:

$$(X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S); M_A, M_S)$$

where $M_A = \{\phi\} \cup \tilde{M}_A$ and $M_S = K \cup \{e\} \cup \tilde{M}_S$, associated with $\tilde{M}_A \equiv K \cup \{e_A\}$ and $\tilde{M}_S \equiv \Pi \cup \{e_S\}$, as follows:

- For $(m_A, m_S) = ((\theta,\eta),\eta) \in \tilde{M}_A \times \tilde{M}_S$,

$$\begin{aligned} & (X_A((\theta,\eta),\eta), X_S((\theta,\eta),\eta), q((\theta,\eta),\eta)) \\ = \;& (u_A(\theta,\eta) + \theta q(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta)) \end{aligned}$$

- For $(m_A, m_S) = ((\theta,\eta),\eta') \in \tilde{M}_A \times \tilde{M}_S$ with $\eta \neq \eta'$,

$$(X_A, X_S, q) = (-T, -T, 0)$$

- For $(m_A, m_S) \in \tilde{M}_A \times \tilde{M}_S$ with at least one of either $m_A = e_A$ or $m_S = e_S$,

$$(X_A, X_S, q) = (0, 0, 0)$$

- For $(m_A, m_S) = (\phi, (\theta,\eta))$,

$$(X_A(\phi,(\theta,\eta)), X_S(\phi,(\theta,\eta)), q(\phi,(\theta,\eta))) = (0, X(\theta,\eta), q(\theta,\eta)).$$

- For $(m_A, m_S) = (\phi, e)$,

$$(X_A(\phi,e), X_S(\phi,e), q(\phi,e) = (0,0,0).$$

- For any other $(m_A, m_S) \in M_A \times M_S$,

$$(X_A, X_S, q) = (-T, -T, 0)$$

Let us check that this mechanism achieves (u_A, u_S, q) in WPBE(wc) where P receives the message only from S on the equilibrium path.

Step 2: Non-cooperative equilibrium based on prior beliefs

Give η and S's prior belief about θ, when A takes $m_A(\theta,\eta)=(\theta,\eta)$, if S takes $m_S(\eta)=e_S$, his payoff is equal to zero, and if S takes either either $m_S(\eta)=\eta'$ with $\eta'\neq\eta$ or $m_S(\eta)\in K\cup\{e\}$, his payoff is $-T$. S's payoff $E[u_S(\theta,\eta)\mid\eta]\geq 0$ is maximized at $m_S(\eta)=\eta$. For $m_S(\eta)=\eta$, if A who observes (θ,η) takes $m_A(\theta,\eta)=(\theta',\eta)$, the A's payoff is $u_A(\theta',\eta)+(\theta'-\theta)q(\theta',\eta)$, which is maximized at $\theta'=\theta$, bringing $u_A(\theta,\eta)$ to A. On the other hand, the A's payoff is $-T$ for $m_A=\phi$ or (θ',η') with $\eta'\neq\eta$, and 0 for e_A. Since $u_A(\theta,\eta)\geq 0$, the A's payoff is maximized at $m_A(\theta,\eta)=(\theta,\eta)$. Therefore $(m_A(\theta,\eta),m_S(\eta))=((\theta,\eta),\eta)$ is a non-cooperative equilibrium of the grand contract based on prior beliefs.

Step 3: Optimal side-contract

Let us focus on WPBE where the non-cooperative equilibrium based on prior beliefs (described above) is always realized in the event that any non-null side-contract is rejected by A. The problem to solve the optimal side-contract is set up as follows:

$$\max E[X_A(\tilde{m}(\theta,\eta))+X_S(\tilde{m}(\theta,\eta))-\theta q(\tilde{m}(\theta,\eta))-\tilde{u}_A(\theta,\eta)\mid\eta]$$

subject to $\tilde{m}(\theta,\eta)\in\Delta(M_A\times M_S)$,

$$\tilde{u}_A(\theta,\eta)\geq\tilde{u}_A(\theta',\eta)+(\theta'-\theta)q(\tilde{m}(\theta,\eta))$$

$$\tilde{u}_A(\theta,\eta)\geq u_A(\theta,\eta).$$

Without loss of generality, our attention is restricted to $\tilde{m}(\theta,\eta)$ which places positive probability only on $m_S\in K\cup\{e\}$ and $m_A=\phi$, since for any $(m_A,m_S)\in\tilde{M}_A\times\tilde{M}_S$, there exists $m_S'\in K\cup\{e\}$ such that

$$(X_A(m_A,m_S)+X_S(m_A,m_S),q(m_A,m_S))=(X_A(\phi,m_S')+X_S(\phi,m_S'),q(\phi,m_S'))$$

and for any other (m_A,m_S), the total payment is a large negative number. By the definition of allocation $(u_A(\theta,\eta),u_S(\theta,\eta),q(\theta,\eta))$ which satisfies WCP,

$$(\tilde{m}(\theta,\eta),\tilde{u}_A(\theta,\eta))=((\phi,(\theta,\eta)),u_A(\theta,\eta))$$

solves this problem. It implies that there exists a WPBE such that (i) S offers non-null side-contract on the equilibrium path, (ii) all types of A accept it and reports θ truthfully, (iii) (θ,η) is reported by S to P and S receives $X(\theta,\eta)$ from P, and (iv) $t(\theta,\eta)=u_A(\theta,\eta)-[X_A(\theta,\eta)-\theta q(\theta,\eta)]$ is transferred from

S to A. In this WPBE, only S sends the report to P on the equilibrium path, resulting in allocation $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$. It is also a WPBE(wc), since by the property of WCP, there is no room to makes S better off while keeping A's payoff at least $u_A(\theta,\eta)$. ■

The following is the statement about the reverse pattern of modified delegation where P communicates only with A on the equilibrium path.

Proposition 5 *Any allocation which is achievable in weak collusion game is achieved as a WPBE(wc) outcome of the mechanism where P communicates and transacts with A alone on the equilibrium path.*

Proof of Proposition 5: The proof is the same as that of Proposition 6. It differs only in the construction of a grand contract. The message sets are modified to $M_A = K \cup \{e\} \cup \tilde{M}_A$ and $M_S = \{\phi\} \cup \tilde{M}_S$ such that A instead of S takes the report of (θ,η) or e. A scheme of payments and output is also replaced by

$$(X_A((\theta,\eta),\phi), X_S((\theta,\eta),\phi), q((\theta,\eta),\phi)) = (X(\theta,\eta), 0, q(\theta,\eta))$$

where A receives the payment from P conditional on the report of $(m_A, m_S) = ((\theta,\eta),\phi)$. It is evident from the proof of Proposition 6 that $(m_A(\theta,\eta), m_S(\eta)) = ((\theta,\eta),\eta)$ is still a non-cooperative equilibrium of the grand contract based on the prior belief. With the presumption of S's perfect bargaining power, the problem to solve the optimal side-contract reduces to the same one as in Step 3 of the proof of Proposition 6. It induces the same WPBE except that P communicates and transacts only with A on the equilibrium path. ■

8 Effect of Re-Allocation of Bargaining Power

Proposition 7 in the paper states the independence of the set of $WCP(\alpha)$ allocation of α. Here we provide its formal proof:

Proof of Proposition 7

We show that the set of implementable allocation is independent of α. Suppose otherwise that $(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$ is WCP allocation in α, but not in α' $(\alpha \neq \alpha')$. It implies that for some η, $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta,\eta)) =$

$((\theta,\eta),u_A(\theta,\eta))$ is not the solution of $TP(\eta;\alpha^{'})$ defined for $(u_A(\theta,\eta),u_S(\theta,\eta),q(\theta,\eta))$. If $(\tilde{m}^*(\theta\mid\eta),u_A^*(\theta,\eta))(\neq((\theta,\eta),u_A(\theta,\eta)))$ is a solution of $TP(\eta;\alpha^{'})$, it satisfies all constraints of $TP(\eta;\alpha^{'})$ and realizes a higher payoff to the third party than in the choice of $(\tilde{m}(\theta\mid\eta),\tilde{u}_A(\theta,\eta))=((\theta,\eta),u_A(\theta,\eta))$:

$$\begin{aligned}&E[(1-\alpha^{'})[\hat{X}(\tilde{m}^*(\theta\mid\eta))-\theta\hat{q}(\tilde{m}^*(\theta\mid\eta))-u_A^*(\theta,\eta)]+\alpha^{'}u_A^*(\theta,\eta)\mid\eta]\\ >\ &E[(1-\alpha^{'})[\hat{X}(\theta,\eta)-\theta\hat{q}(\theta,\eta)-u_A(\theta,\eta)]+\alpha^{'}u_A(\theta,\eta)\mid\eta].\end{aligned}$$

It also satisfies A and S's participation constraints:

$$u_A^*(\theta,\eta)\geq u_A(\theta,\eta)$$

and

$$E[\hat{X}(\tilde{m}^*(\theta\mid\eta))-\theta\hat{q}(\tilde{m}^*(\theta\mid\eta))-u_A^*(\theta,\eta)\mid\eta]\geq E[u_S(\theta,\eta)\mid\eta].$$

On the other hand, since $(\tilde{m}(\theta\mid\eta),\tilde{u}_A(\theta,\eta))=((\theta,\eta),u_A(\theta,\eta))$ solves $TP(\eta;\alpha)$,

$$\begin{aligned}&E[(1-\alpha)[\hat{X}(\theta,\eta)-\theta\hat{q}(\theta,\eta)-u_A(\theta,\eta)]+\alpha u_A(\theta,\eta)\mid\eta]\\ \geq\ &E[(1-\alpha)[\hat{X}(\tilde{m}^*(\theta\mid\eta))-\theta\hat{q}(m^*(\theta\mid\eta))-u_A^*(\theta,\eta)]+\alpha u_A^*(\theta,\eta)\mid\eta]\end{aligned}$$

Let us consider three cases: (i) $\alpha\in(0,1)$, (ii) $\alpha=1$ and (iii) $\alpha=0$.

(i) $\alpha\in(0,1)$

The last three inequalities imply

$$u_A^*(\theta,\eta)=u_A(\theta,\eta)$$

and

$$E[\hat{X}(\tilde{m}^*(\theta\mid\eta))-\theta\hat{q}(m^*(\theta\mid\eta))-u_A^*(\theta,\eta)\mid\eta]=E[u_S(\theta,\eta)\mid\eta].$$

But this is not compatible with the first inequality. We obtain a contradiction.

(ii) $\alpha=1$

With $\alpha = 1$, the above four inequalities imply

$$E[u_A(\theta, \eta) \mid \eta] = E[u_A^*(\theta, \eta) \mid \eta]$$

and

$$E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta, \eta) \mid \eta] > E[u_S(\theta, \eta) \mid \eta].$$

But for sufficiently small $\epsilon > 0$, the choice of

$$(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta)) = (\tilde{m}^*(\theta, \eta), u_A^*(\theta, \eta) + \epsilon)$$

(instead of $((\theta, \eta), u_A(\theta, \eta)))$ in $TP(\eta; \alpha = 1)$ generates a higher value of the objection function without violating any constraint. We obtain a contradiction.

(iii) $\alpha = 0$

With $\alpha = 0$, the four inequalities imply

$$E[u_S(\theta, \eta) \mid \eta] = E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta, \eta) \mid \eta]$$

and

$$E[u_A^*(\theta, \eta) \mid \eta] > E[u_A(\theta, \eta) \mid \eta].$$

Since $u_A^*(\theta, \eta) \geq u_A(\theta, \eta)$ for any θ, there is a subset of θ with the positive measure such that $u_A^*(\theta, \eta) > u_A(\theta, \eta)$. Consider a modified problem of $TP(\eta; \alpha = 0)$ such that the constraint $\tilde{u}_A(\theta, \eta) \geq u_A(\theta, \eta)$ is replaced by $\tilde{u}_A(\theta, \eta) \geq u_A^*(\theta, \eta)$ in $TP(\eta; \alpha = 0)$. Since the optimal solution $(\tilde{m}(\theta \mid \eta), \tilde{u}_A(\theta, \eta)) = ((\theta, \eta), u_A(\theta, \eta))$ in $TP(\eta; \alpha = 0)$ violates the constraint, the maximum value of the objective function in the modified problem would become lower. On the other hand, $(\tilde{m}^*(\theta \mid \eta), u_A^*(\theta, \eta))$ satisfies all the constraints of the modified problem, and brings

$$E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta, \eta) \mid \eta].$$

The argument implies

$$E[u_S(\theta, \eta) \mid \eta] > E[\hat{X}(\tilde{m}^*(\theta \mid \eta)) - \theta\hat{q}(\tilde{m}^*(\theta \mid \eta)) - u_A^*(\theta, \eta) \mid \eta].$$

We obtain a contradiction. ■

9 Coalitional Incentive and Participation Constraint in Indivisible Good Case

Here we provide the characterization of the payment schedules which satisfy the coalitional incentive and participation constraints in the model of Section 6 with an indivisible good. For a given pair of thresholds (θ_1, θ_2), let K_0 and K_1 be the set of (θ, η) such that $q(\theta, \eta) = 0$ and $q(\theta, \eta) = 1$ respectively, i.e.

$$K_0 \equiv \{(\theta, \eta_i) \mid \theta > \theta_i\}.$$

and

$$K_1 \equiv \{(\theta, \eta_i) \mid \theta \leq \theta_i\}.$$

For the moment we suppose that both K_0 and K_1 are not empty. The coalitional incentive constraint implies that the total payment X to the coalition depends only on the output level. Therefore there exist X_0 and X_1 such that $X(\theta, \eta) = X_0$ for any $(\theta, \eta) \in K_0$ and $X(\theta, \eta) = X_1$ for any $(\theta, \eta) \in K_1$. The coalitional incentive constraint reduces to the existence of $z(\cdot \mid \eta) \in Z(\eta)$ for $\eta \in \{\eta_1, \eta_2\}$ such that

$$X_1 - z(\theta \mid \eta) \geq X_0$$

for any $(\theta, \eta) \in K_1$ and

$$X_0 \geq X_1 - z(\theta \mid \eta)$$

for any $(\theta, \eta) \in K_0$. With $b \equiv X_1 - X_0$, the payment schedule is interpreted as a pair of b (as a bonus for high output) and X_0 (as a basic salary). Since $z(\theta \mid \eta)$ is continuous for θ, taking thresholds (θ_1, θ_2) (with nonempty K_0 and K_1) as given, the coalitional incentive constraint implies that b satisfy the following conditions:

- If $\theta_i \in (\underline{\theta}_i, \bar{\theta}_i)$, $b = z(\theta_i \mid \eta_i)$ for some $z(\cdot \mid \eta_i) \in Z(\eta_i)$.
- If $\theta_i = \underline{\theta}_i$, $b \leq z(\underline{\theta}_i \mid \eta_i)$ for some $z(\cdot \mid \eta_i) \in Z(\eta_i)$.
- If $\theta_i = \bar{\theta}_i$, $b \geq z(\bar{\theta}_i \mid \eta_i)$ for some $z(\cdot \mid \eta_i) \in Z(\eta_i)$.

The following lemma is useful for simplifying the characterization of the coalitional incentive constraint.

Lemma 1 *Under the assumption that $l_i(\theta)$ and $h_i(\theta)$ are increasing in θ, for any $(b, \tilde{\theta})$ such that $b \in [l_i(\tilde{\theta}), h_i(\tilde{\theta})]$, there exists $z(\cdot \mid \eta) \in Z(\eta)$ such that $z(\tilde{\theta} \mid \eta) = b$ and $z(\cdot \mid \eta)$ is not transformed by ironing procedure at $\tilde{\theta}$.*

Proof of Lemma 1

For any $(b, \tilde{\theta})$ such that $b \in [l_i(\tilde{\theta}), h_i(\tilde{\theta})]$, we can select $\Lambda(\theta \mid \eta_i) = \alpha F(\theta \mid \eta_i)$ for $\theta < \bar{\theta}_i$ and $\Lambda(\bar{\theta}_i \mid \eta) = 1$ with α such that $b = \tilde{\theta} + (1-\alpha)\frac{F(\tilde{\theta}|\eta_i)}{f(\tilde{\theta}|\eta_i)}$. ■

From this lemma, we can always construct $z(\cdot \mid \eta_i) \in Z(\eta_i)$ such that $b = z(\theta_i \mid \eta_i)$ and $z(\cdot \mid \eta_i)$ is not ironed at θ_i if and only if $b \in [l_i(\theta_i), h_i(\theta_i)]$. Then we can state that (θ_1, θ_2, z) satisfies the coalitional incentive constraint if and only if for any $i \in \{1, 2\}$,

(i) If $\theta_i \in (\underline{\theta}_i, \bar{\theta}_i)$, $l_i(\theta_i) \leq b \leq h_i(\theta_i)$

(ii) If $\theta_i = \underline{\theta}_i$, $b \leq \underline{\theta}_i$

(iii) If $\theta_i = \bar{\theta}_i$, $b \geq \bar{\theta}_i$

Next consider implications of the coalitional participation constraint, which has to be satisfied in the ex-ante collusion case, but not in the interim collusion case. It implies $X_1 - z(\theta \mid \eta) \geq 0$ for any $(\theta, \eta) \in K_1$ and $X_0 \geq 0$ for any $(\theta, \eta) \in K_0$. Since the coalitional incentive constraint implies $X_1 - z(\theta \mid \eta) \geq X_0$ for any $(\theta, \eta) \in K_1$, the condition reduces to $X_0 \geq 0$.

Note that the above argument is based on non-empty K_0 and K_1. Here consider the case that one of either K_0 or K_1 is empty. Then we do not need to consider the coalitional incentive constraint, since the total payment takes constant value for any state. If K_0 is empty, the output is always equal to 1. Then the total payment to the coalition would always take constant value X_1 and the coalitional participation constraint reduces to $X_1 \geq 1$, since there exists $z(\cdot \mid \eta) \in Z(\eta)$ such that $X_1 - z(\theta \mid \eta) \geq 0$ for any (θ, η) if and only if $X_1 \geq 1$. If K_1 is empty, the output is always equal to 0, making the total payment equal to X_0 in all states. Then the coalitional participation constraint $X_0 \geq 0$ needs to be satisfied.

10 Ex-Post Participation Constraint of S

The paper argues the case that the participation constraint for S is required to hold ex post rather than interim, in footnote 35. Here we provide some results about S's value for both cases of divisible good and indivisible good.

10.1 Divisible Good Case

Suppose that S has the option to exit from both side-contract and grand-contract at the ex-post stage. Then achievable allocation must satisfy ex-post participation constraint: $u_S(\theta, \eta) \geq 0$ for any $(\theta, \eta) \in K$ instead of $E[u_S(\theta, \eta) \mid \eta] \geq 0$, in addition to WCP and A's participation constraint. Here we examine whether or not S's value is generated through small variation from optimal allocation in NS. Since $u_S(\theta, \eta) = 0$ in the optimal NS, we need to generate a variation with a positive payoff $u_S(\theta, \eta) > 0$ for some (θ, η).

Before the formal statement, let us begin with providing some intuitive arguments. In order to find a variation which is profitable to P, let us consider the following problem for a given η:

$$\max E[V(q^{NS}(z(\theta \mid \eta))) - X^{NS}(z(\theta \mid \eta)) \mid \eta]$$

subject to $z(\cdot \mid \eta) \in Z(\eta)$ and

$$u_S(\theta, \eta) = X^{NS}(z(\theta \mid \eta)) - \theta q^{NS}(z(\theta \mid \eta)) - \int_{\theta}^{\bar{\theta}} q^{NS}(z(y \mid \eta))dy \geq 0$$

for any $\theta \in \Theta(\eta)$.[5] Since $z(\theta \mid \eta) = \theta$ satisfies all constraints and brings the optimal payoff in NS, our investigation is provided in whether or not this problem has the solution $z(\cdot \mid \eta)$ which differs from $z(\theta \mid \eta) = \theta$. If this is true, S is still valuable in an organization with stronger constraint $u_S(\theta, \eta) \geq 0$.

In order to investigate our question, it is useful to consider the following revised maximization problem.

$$\max E[V(q^{NS}(z(\theta \mid \eta))) - X^{NS}(z(\theta \mid \eta)) \mid \eta]$$

subject to $z(\cdot \mid \eta) \in Z(\eta)$ and

$$u_S(\underline{\theta}(\eta), \eta) = X^{NS}(z(\underline{\theta}(\eta) \mid \eta)) - \underline{\theta}(\eta) q^{NS}(z(\underline{\theta}(\eta) \mid \eta)) - \int_{\underline{\theta}(\eta)}^{\bar{\theta}} q^{NS}(z(y \mid \eta))dy = 0.$$

[5] $(X^{NS}(z), q^{NS}(z))$ is defined as $X^{NS}(z) \equiv z q^{NS}(z) + \int_z^{\bar{\theta}} q^{NS}(y)dy$ and $q^{NS}(z)$ maximizes $V(q) - H(z)q$ for $z \in \Theta$, $q^{NS}(z) \equiv q^{NS}(\underline{\theta})$ for $z < \underline{\theta}$ and $q^{NS}(z) \equiv q^{NS}(\bar{\theta})$ for $z > \bar{\theta}$.

The constraint is satisfied with $z(\cdot \mid \eta)$ with $z(\theta \mid \eta) = \theta$, bringing the value equal to the optimal payoff in NS. For $\lambda \geq 0$, define the following Lagrangean for the revised problem:

$$\begin{aligned} & \mathcal{L} \equiv E[V(q^{NS}(z(\theta \mid \eta))) - X^{NS}(z(\theta \mid \eta)) \mid \eta] + \lambda[X^{NS}(z(\underline{\theta}(\eta) \mid \eta)) \\ - \quad & \underline{\theta}(\eta) q^{NS}(z(\underline{\theta}(\eta) \mid \eta)) - \int_{\underline{\theta}(\eta)}^{\bar{\theta}} q^{NS}(z(y \mid \eta)) dy] \end{aligned}$$

$\mathcal{L}$ can be rewritten to

$$\begin{aligned} & E[V(q^{NS}(z(\theta \mid \eta))) - X^{NS}(z(\theta \mid \eta)) - \frac{\lambda}{f(\theta \mid \eta)} q^{NS}(z(\theta \mid \eta)) \mid \eta] \\ + \quad & \lambda[(z(\underline{\theta}(\eta) \mid \eta) - \underline{\theta}(\eta)) q^{NS}(z(\underline{\theta}(\eta) \mid \eta)) + \int_{z(\underline{\theta}(\eta)|\eta)}^{\bar{\theta}(\eta)} q^{NS}(z) dz] \end{aligned}$$

Taking some θ as given, the marginal effect in the change of $z(\theta \mid \eta)$ evaluated at θ is shown by

$$\begin{aligned} & \frac{\partial[V(q^{NS}(z)) - X^{NS}(z) - \frac{\lambda}{f(\theta|\eta)} q^{NS}(z)]}{\partial z} \Big|_{z=\theta} \\ = \quad & [V'(q^{NS}(\theta)) - \theta - \frac{\lambda}{f(\theta \mid \eta)}] q^{NS'}(\theta) \\ = \quad & [\frac{F(\theta)}{f(\theta)} - \frac{\lambda}{f(\theta \mid \eta)}] q^{NS'}(\theta) = \frac{1}{a(\eta \mid \theta) f(\theta)} [F(\theta) a(\eta \mid \theta) - p(\eta)\lambda] q^{NS'}(\theta). \end{aligned}$$

If there exists $(\theta', \theta'') \subset \Theta(\eta)$ so that $F(\theta)a(\eta \mid \theta)$ is decreasing in θ on (θ', θ''), there are $\lambda > 0$ and $\hat{\theta}$ so that $F(\theta)a(\eta \mid \theta) - p(\eta)\lambda > 0$ on $(\theta', \hat{\theta})$ and $F(\theta)a(\eta \mid \theta) - p(\eta)\lambda < 0$ on $(\hat{\theta}, \theta'')$. Then there must exist an output schedule with $q^{NS}(z(\theta \mid \eta)) > q^{NS}(\theta)$ on $(\theta', \hat{\theta})$ and $q^{NS}(z(\theta \mid \eta)) < q^{NS}(\theta)$ on $(\hat{\theta}, \theta'')$ which improves $\mathcal{L}$ over the optimal payoff in NS. With $u_S(\underline{\theta}(\eta), \eta) = 0$, this selection of output schedule does not violate the constraint in the original problem: $u_S(\theta, \eta) \geq 0$, since $u_S(\theta, \eta)$ is nondecreasing in $(\theta', \hat{\theta})$, nonincreasing in $(\hat{\theta}, \theta'')$ and $u_S(\bar{\theta}, \eta) = 0$ with $z(\bar{\theta}, \eta) = \bar{\theta}$. This argument suggests that the following formal statement holds.

Proposition 6 *If there exists η and an interval of θ so that $F(\theta)a(\eta \mid \theta)$ is decreasing in θ, there exists a WCP allocation satisfying $u_S(\theta, \eta) \geq 0$ and $u_A(\theta, \eta) \geq 0$ for any $(\theta, \eta) \in K$ so that the P's payoff is higher than that in the optimal allocation in NS.*

Proof of Proposition 6

From conditions of Proposition 6, there exists η^* and $\lambda > 0$ so that for the closed intervals Θ^L and Θ^H,

$$\lambda < F(\theta)a(\eta^* \mid \theta) \ \text{ for } \theta \in \Theta^L \equiv [\underline{\theta}^L, \bar{\theta}^L] \subset (\underline{\theta}(\eta), \bar{\theta}(\eta))$$

$$\lambda > F(\theta)a(\eta^* \mid \theta) \ \text{ for } \theta \in \Theta^H \equiv [\underline{\theta}^H, \bar{\theta}^H] \subset (\underline{\theta}(\eta), \bar{\theta}(\eta))$$

with $\bar{\theta}^L < \underline{\theta}^H$. These conditions are equivalent to

$$H(\theta) - \theta - \frac{\lambda}{a(\eta^* \mid \theta)f(\theta)} > 0 \ \text{ for } \theta \in \Theta^L$$

and

$$H(\theta) - \theta - \frac{\lambda}{a(\eta^* \mid \theta)f(\theta)} < 0 \ \text{ for } \theta \in \Theta^H.$$

Define $q^{NS}(z)$ for $z \in [\underline{\theta}, \bar{\theta}]$ so that $V'(q^{NS}(z)) = H(z)$. $q^{NS}(z)$ is strictly decreasing in z from our assumption of $V(q)$ and $H(z)$.

With notations defined above, let us select $z(\theta \mid \eta)$ which satisfies the following conditions.

(i) For $\eta \neq \eta^*$, $z(\theta \mid \eta) = \theta$.

(ii) For $\theta \notin \Theta^H \cup \Theta^L$, $z(\theta \mid \eta^*) = \theta$.

(iii) For $\theta \in \Theta^H$, $z(\theta \mid \eta^*) \geq \theta$ with strict inequality for some portions with positive measure, and

$$H(z) - z - \frac{\lambda}{a(\eta^* \mid \theta)f(\theta)} < 0 \ \text{ for any } z \in [\theta, z(\theta \mid \eta^*)]$$

(iv) For $\theta \in \Theta^L$, $z(\theta \mid \eta^*) \leq \theta$ with strict inequality for some portions with positive measure, and

$$H(z) - z - \frac{\lambda}{a(\eta^* \mid \theta)f(\theta)} > 0 \ \text{ for any } z \in [z(\theta \mid \eta^*), \theta]$$

(v) $z(\cdot \mid \eta) \in Z(\eta)$.

(vi) $\theta q^{NS}(z(\theta \mid \eta^*)) + \int_\theta^{\bar\theta} q^{NS}(z(y \mid \eta^*))dy \leq z(\theta \mid \eta^*)q^{NS}(z(\theta \mid \eta^*)) + \int_{z(\theta|\eta^*)}^{\bar\theta} q^{NS}(z)dz$ for any $\theta \in [\underline{\theta}(\eta^*), \bar\theta(\eta^*)]$ and

$$\int_{\underline{\theta}(\eta^*)}^{\bar\theta(\eta^*)} q^{NS}(z(y \mid \eta^*))dy = \int_{\underline{\theta}(\eta^*)}^{\bar\theta(\eta^*)} q^{NS}(z)dz \tag{4}$$

Step 1

Suppose that there exists $z(\theta \mid \eta)$ which satisfies (i)-(vi). Let us specify the allocation $(u_A(\theta,\eta), X^{NS}(z(\theta \mid \eta)), q^{NS}(z(\theta \mid \eta)))$ with

$$u_A(\theta,\eta) = \int_\theta^{\bar\theta} q^{NS}(z(y \mid \eta))dy$$

$$X^{NS}(z(\theta \mid \eta)) = z(\theta \mid \eta)q^{NS}(z(\theta \mid \eta)) + \int_{z(\theta|\eta)}^{\bar\theta} q^{NS}(z)dz$$

From (i) and (vi), for any η,

$$\begin{aligned} u_S(\theta,\eta) &= z(\theta \mid \eta)q^{NS}(z(\theta \mid \eta)) + \int_{z(\theta|\eta)}^{\bar\theta} q^{NS}(z)dz \\ &- \quad [\theta q^{NS}(z(\theta \mid \eta)) + \int_\theta^{\bar\theta} q^{NS}(z(y \mid \eta))dy] \geq 0. \end{aligned}$$

It means that this allocation is WCP allocation satisfying ex-post participation conditions of A and S. This allocation induces the P's payoff

$$E[V(q^{NS}(z(\theta \mid \eta))) - z(\theta \mid \eta)q^{NS}(z(\theta \mid \eta)) - \int_{z(\theta|\eta)}^{\bar\theta} q^{NS}(z)dz].$$

From (4), this is equal to

$$
\begin{aligned}
& E[V(q^{NS}(z(\theta \mid \eta))) - z(\theta \mid \eta) q^{NS}(z(\theta \mid \eta)) - \int_{z(\theta|\eta)}^{\bar{\theta}} q^{NS}(z) dz] \\
+ \quad & \lambda[\int_{\underline{\theta}(\eta^*)}^{\bar{\theta}} q^{NS}(z) dz - \int_{\underline{\theta}(\eta^*)}^{\bar{\theta}} q^{NS}(z(y, \eta^*)) dy] \\
= \quad & p(\eta^*) E[V(q^{NS}(z(\theta, \eta^*))) - \{z(\theta, \eta^*) + \frac{\lambda}{f(\theta) a(\eta^* \mid \theta)}\} q^{NS}(z(\theta, \eta^*)) \\
- \quad & \int_{z(\theta, \eta^*)}^{\bar{\theta}} q^{NS}(z) dz \mid \eta^*] \\
+ \quad & (1 - p(\eta^*)) E[V(q^{NS}(\theta)) - \{\theta q^{NS}(\theta) + \int_{\theta}^{\bar{\theta}} q^{NS}(z) dz\}] \\
+ \quad & \lambda \int_{\underline{\theta}(\eta^*)}^{\bar{\theta}} q^{NS}(z) dz.
\end{aligned}
$$

On the other hand, the P's optimal payoff in NS is,

$$
\begin{aligned}
& E[V(q^{NS}(\theta)) - \theta q^{NS}(\theta) - \int_{\theta}^{\bar{\theta}} q^{NS}(z) dz] \\
= \quad & p(\eta^*) E[V(q^{NS}(\theta)) - \{\theta + \frac{\lambda}{f(\theta) a(\eta^* \mid \theta)}\} q^{NS}(\theta) - \int_{\theta}^{\bar{\theta}} q^{NS}(z) dz \mid \eta^*] \\
+ \quad & (1 - p(\eta^*)) E[V(q^{NS}(\theta)) - \{\theta q^{NS}(\theta) + \int_{\theta}^{\bar{\theta}} q^{NS}(z) dz\} \mid \eta \neq \eta^*] \\
+ \quad & \lambda \int_{\underline{\theta}(\eta^*)}^{\bar{\theta}(\eta^*)} q^{NS}(z) dz
\end{aligned}
$$

The difference between two payoffs is

$$\begin{aligned}
& p(\eta^*)E[V(q^{NS}(z(\theta \mid \eta^*))) - \{z(\theta \mid \eta^*) + \frac{\lambda}{f(\theta)a(\eta^* \mid \theta)}\}q^{NS}(z(\theta \mid \eta^*)) \\
- & \int_{z(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
- & p(\eta^*)E[V(q^{NS}(\theta)) - \{\theta + \frac{\lambda}{f(\theta)a(\eta^* \mid \theta)}\}q^{NS}(\theta) - \int_{\theta}^{\bar{\theta}} q^{NS}(z)dz \mid \eta^*] \\
= & p(\eta^*)E[\int_{\theta}^{z(\theta|\eta^*)} \{V'(q^{NS}(z)) - z - \frac{\lambda}{f(\theta)a(\eta^* \mid \theta)}\}q^{NS'}(z)dz \mid \eta^*] \\
= & p(\eta^*)E[\int_{\theta}^{z(\theta|\eta^*)} \{H(z) - z - \frac{\lambda}{f(\theta)a(\eta^* \mid \theta)}\}q^{NS'}(z)dz \mid \eta^*]
\end{aligned}$$

From condition (iii) and (iv), $q^{NS'}(z) < 0$ implies that this is positive, implying that P benefits from this allocation.

Step 2

Finally we construct $z(\theta \mid \eta)$ which satisfies (i)-(vi). The argument in the proof of Proposition 4 (in the paper) can be exactly applicable here with the replacement of $H(\theta) - (1-\lambda)\theta - \lambda h(\theta \mid \eta^*)$ with $H(\theta) - \theta - \frac{\lambda}{f(\theta)a(\eta^*|\theta)}$. For $z(\theta \mid \eta^*)$ which satisfies (i)-(v) and any pair of parameters α_H, α_L lying in $[0,1]$, define a function $z_{\alpha_L,\alpha_H}(\theta|\eta^*)$ which equals $(1-\alpha_L)z(\theta|\eta^*) + \alpha_L\theta$ on Θ^L, equals $(1-\alpha_H)z(\theta|\eta^*) + \alpha_H\theta$ on Θ^H and equals θ elsewhere. Define

$$\begin{aligned}
& Q(\theta, \alpha_L, \alpha_H) \equiv z_{\alpha_L,\alpha_H}(\theta \mid \eta^*)q^{NS}(z_{\alpha_L,\alpha_H}(\theta \mid \eta^*)) + \int_{z_{\alpha_L,\alpha_H}(\theta|\eta^*)}^{\bar{\theta}} q^{NS}(z)dz \\
- & [\theta q^{NS}(z_{\alpha_L,\alpha_H}(\theta \mid \eta^*)) + \int_{\theta}^{\bar{\theta}} q^{NS}(z_{\alpha_L,\alpha_H}(y \mid \eta^*))dy]
\end{aligned}$$

Since

$$\frac{\partial Q(\theta, \alpha_L, \alpha_H)}{\partial \theta} = (z_{\alpha_L,\alpha_H}(\theta \mid \eta^*) - \theta)q^{NS'}(z_{\alpha_L,\alpha_H}(\theta \mid \eta^*))\frac{\partial z_{\alpha_L,\alpha_H}(\theta \mid \eta^*)}{\partial \theta},$$

$\frac{\partial Q(\theta,\alpha_L,\alpha_H)}{\partial \theta} \leq 0$ on Θ^H and $\frac{\partial Q(\theta,\alpha_L,\alpha_H)}{\partial \theta} \geq 0$ on Θ^L and it is zero elsewhere. $z_{\alpha_L,\alpha_H}(\theta \mid \eta^*) = \theta$ for $\theta > \bar{\theta}^H$ implies that $Q(\theta, \alpha_L, \alpha_H) = 0$ for this region of θ. Therefore if $Q(\underline{\theta}^L, \alpha_L, \alpha_H) = 0$ implies (vi). $Q(\underline{\theta}^L, \alpha_L, \alpha_H)$ is continuously differentiable, strictly increasing in α_L and strictly decreasing in α_H. $Q(1,1) = 0$. The Implicit Function Theorem ensures existence of α_L^*, α_H^* both smaller than 1 such that $Q(\underline{\theta}^L, \alpha_L^*, \alpha_H^*) = 0$. Hence the function $z_{\alpha_L^*, \alpha_H^*}(\theta|\eta^*)$ satisfies (i)-(vi). ■

10.2 Indivisible Good Case

With the case of indivisible good, we examine the case that S's ex-post participation constraint must be satisfied, or $u_S(\theta, \eta) \geq 0$ for any (θ, η). Since $u_S(\theta, \eta_i) = X_0 + b - \theta_i$ for $\theta \leq \theta_i$ and $u_S(\theta, \eta_i) = X_0$ for $\theta \geq \theta_i$, it implies

$$X_0 \geq \max\{\theta_1 - b, \theta_2 - b, 0\}.$$

Evidently this constraint becomes binding in the optimal allocation. Then the principal's problem is represented by

$$\max[V - b][p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)] - \max\{\theta_1 - b, \theta_2 - b, 0\}$$

subject to

$$b \in Z(\theta_1, \theta_2).$$

We obtain the following statement.

Proposition 7 *P cannot benefit from S with an ex-post participation constraint.*

Proof of Proposition 7

Suppose $\theta_1 \leq \theta_2$. Then the objective function reduces to

$$[V - b][p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)] - \max\{\theta_2 - b, 0\}.$$

This is non-decreasing in b for $b < \theta_2$ and decreasing in b for $b > \theta_2$, and takes a maximum value at $b = \theta_2$. Therefore

$$\begin{aligned} & [V - b][p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)] - \max\{\theta_2 - b, 0\} \\ \leq \;& [V - \theta_2][p_1 F(\theta_1 \mid \eta_1) + p_2 F(\theta_2 \mid \eta_2)] \end{aligned}$$

If $V > \theta_2$, the right hand side is not larger than $[V - \theta_2]F(\theta_2)$, which is not larger than Π_{NS}. If $V \leq \theta_2$, the right hand side is non-positive, which is not larger than Π_{NS}. We have the same argument for $\theta_1 > \theta_2$. It concludes that P does not benefit from hiring S. ■

11 Ironing Rule and Related Results

Here we summarize the ironing procedure and its related properties which are frequently used throughout the paper. We specify an ironing rule to construct $\hat{\pi}(x)$ from two functions $\pi(x)$ and $G(x)$, and explain some properties about $\hat{\pi}(x)$. According to Myerson (1981) and Baron and Myerson (1982), the ironing rule is described as follows.

Definition 1 *Suppose that $\pi(x)$ and $G(x)$ defined on $[\underline{x}, \bar{x}]$ have the following properties:*

(i) $\pi(x-) \geq \pi(x+)$ for any $x \in [\underline{x}, \bar{x}]$.

(ii) $G(x)$ is distribution function with $G(\underline{x}) = 0$ and $G(\bar{x}) = 1$. $G(x)$ is strictly increasing and continuously differentiable on $[\underline{x}, \bar{x}]$.

Then $\hat{\pi}(x) \equiv \hat{\pi}(x \mid \pi(\cdot), G(\cdot))$ is constructed from $\pi(x)$ and $G(x)$ as follows.

(i) $\Pi(\phi) = \int_0^{\phi} \pi(h(y))dy$ where $h(\phi)$ satisfies $G(h(\phi)) = \phi$ for $\phi \in [0, 1]$.

(ii) $\underline{\Pi}(\phi)$ is maximum convex function so that $\Pi(\phi) \geq \underline{\Pi}(\phi)$.

(iii) $\hat{\pi}(x)$ satisfies (i) $\hat{\pi}(x) = \underline{\Pi}^{'}(G(x))$ whenever the derivative $\underline{\Pi}^{'}(G(x))$ is defined,[6] *and (ii) $\hat{\pi}(x) = \underline{\Pi}^{'}(G(x-))$ for any $x \in (\underline{x}, \bar{x}]$.*

We provide two lemmata, which show some properties used in the paper.

Lemma 2 *$\hat{\pi}(x) = \hat{\pi}(x \mid \pi(\cdot), G(\cdot))$ constructed from $\pi(x)$ and $G(x)$ satisfies:*

(i) $\hat{\pi}(x)$ is continuous and non-decreasing in x. If $\pi(x)$ is non-decreasing in x, $\hat{\pi}(x) = \pi(x)$.

(ii) $\int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x)$ if $q(x)$ is constant for each interval of x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$ (or $\hat{\pi}(x)$ takes constant value).

(iii) If $\pi(x) > x$ on $(\underline{x}, \bar{x}]$, $\hat{\pi}(x) > \hat{\pi}_{\alpha}(x)$ on $(\underline{x}, \bar{x}]$ for $\pi_{\alpha}(x) \equiv (1-\alpha)\pi(x) + \alpha x$ with $\alpha \in (0, 1]$.

(iv) $\hat{\pi}(\underline{x}) \leq \pi(\underline{x})$ and $\hat{\pi}(\bar{x}) \geq \pi(\bar{x})$. If there exists an increasing $v(x)$ so that $v(x) < \pi(x)$ for any $x > \underline{x}$, $v(x) < \hat{\pi}(x)$ for any $x > \underline{x}$ and if there exists an increasing $v(x)$ so that $v(x) > \pi(x)$ for any $x > \underline{x}$, $v(x) > \hat{\pi}(x)$ for any $x > \underline{x}$.

[6] Since $\underline{\Pi}(\phi)$ is convex, it is almost everywhere differentiable.

(v) Suppose that $q^*(x)$ is the solution of the following problem:

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)]dG(x)$$

subject to $q(x)$ is non-increasing. Then $q^*(x)$ solves

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

Then

$$\int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \pi(x)q^*(x)]dG(x) = \int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \hat{\pi}(x)q^*(x)]dG(x).$$

Proof of Lemma 2

The proof of (i)

Since $\underline{\Pi}(\phi)$ is convex and $G(x)$ is increasing, $\hat{\pi}(x)$ is non-decreasing. Suppose that there exists x so that $\hat{\pi}(x) < \hat{\pi}(x+)$. It means that $\underline{\Pi}'(G(x-)) < \underline{\Pi}'(G(x+))$. Then $\underline{\Pi}(G(x)) = \Pi(G(x))$, since otherwise you can find a higher convex function than $\underline{\Pi}(\phi)$. This implies that

$$\pi(x-) = \Pi'(G(x-)) \leq \underline{\Pi}'(G(x-)) < \underline{\Pi}'(G(x+)) \leq \Pi'(G(x+)) = \pi(x+)$$

This is contradiction since we assume that $\pi(x-) \geq \pi(x+)$. Therefore $\hat{\pi}(x)$ is continuous.

Suppose that $\pi(x)$ is non-decreasing in x. With $\Pi(\phi) = \int_0^\phi \pi(h(y))dy$, $\Pi'(\phi) = \pi(h(\phi))$. Then $\Pi(\phi)$ is convex and $\Pi(\phi) = \underline{\Pi}(\phi)$, implying $\pi(x) = \hat{\pi}(x)$.

The proof of (ii)

Define I by

$$I \equiv \{x \in [\underline{x}, \bar{x}] \mid \Pi(G(x)) > \underline{\Pi}(G(x))\}.$$

For any $x \in I$, there exists $d(x)$ and $u(x)$ such as

$$\Pi(G(x')) > \underline{\Pi}(G(x'))$$

on $x^{'} \in (d(x), u(x))$, $\Pi(G(d(x))) = \underline{\Pi}(G(d(x)))$ and $\Pi(G(u(x))) = \underline{\Pi}(G(u(x)))$. Then $\underline{\Pi}(\phi^{'})$ is a linear function of $\phi^{'}$ on $[G(d(x)), G(u(x))]$ and $\hat{\pi}(x^{'})$ is constant on $x^{'} \in [d(x), u(x)]$. Then since $q(x^{'})$ is constant on $x^{'} \in [d(x), u(x)]$,

$$\int_{[d(x),u(x)]} q(x^{'})d\Pi(G(x^{'})) = \int_{[d(x),u(x)]} q(x^{'})d\underline{\Pi}(G(x^{'})).$$

Therefore it implies that

$$\int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) = \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)).$$

Since $\underline{\Pi}(\phi)$ is convex, it is almost everywhere differentiable with $\underline{\Pi}^{'}(G(x)) = \hat{\pi}(x)$ almost everywhere. This means that

$$\int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) = \int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x).$$

It is concluded that

$$\int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x).$$

The proof of (iii)

Since the linear combination of two convex functions is convex, $(1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^{\phi} h(y)dy$ is convex function. Defining $\Pi_{\alpha}(\phi)$ by

$$\Pi_{\alpha}(\phi) \equiv \int_0^{\phi} \pi_{\alpha}(h(y))dy = (1-\alpha)\Pi(\phi) + \alpha \int_0^{\phi} h(y)dy.$$

Since

$$\Pi_{\alpha}(\phi) \geq (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^{\phi} h(y)dy,$$

$\underline{\Pi}_{\alpha}(\phi)$, which is the maximum convex function such that $\Pi_{\alpha}(\phi) \geq \underline{\Pi}_{\alpha}(\phi)$, satisfies

$$\Pi_{\alpha}(\phi) \geq \underline{\Pi}_{\alpha}(\phi) \geq (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^{\phi} h(y)dy.$$

Here our proof is composed of the analysis of two cases: (a) the region of x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$ and (b) the region of x such that $\Pi(G(x)) = \underline{\Pi}(G(x))$.

(a) For arbitrary x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, there exists $d(x)$ and $u(x)$ such as

$$\Pi(G(x^{'})) > \underline{\Pi}(G(x^{'}))$$

on $x^{'} \in (d(x), u(x))$, $\Pi(G(d(x))) = \underline{\Pi}(G(d(x)))$ and $\Pi(G(u(x))) = \underline{\Pi}(G(u(x)))$. At $\phi = G(d(x))$ and $\phi = G(u(x))$,

$$\Pi_{\alpha}(\phi) = (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^{\phi} h(y)dy.$$

It implies that

$$\underline{\Pi}_{\alpha}(\phi) = (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^{\phi} h(y)dy$$

at $\phi = G(d(x))$ and $\phi = G(u(x))$. Then since (i) of this lemma implies that $\underline{\Pi}^{'}_{\alpha}(\phi)$ and $\underline{\Pi}(\phi)$ are differentiable with respect to ϕ for any $\phi \in [0,1]$, the derivatives of both sides of the above equation with respect to ϕ, if evaluated at $G(u(x))$, have the following relationship:

$$\underline{\Pi}^{'}_{\alpha}(G(u(x))) \leq (1-\alpha)\underline{\Pi}^{'}(G(u(x))) + \alpha u(x) = (1-\alpha)\hat{\pi}(u(x)) + \alpha u(x).$$

Since $\hat{\pi}(u(x)) = \pi(u(x)) > u(x)$ (by $u(x) > \underline{x}$) and $\hat{\pi}_{\alpha}(u(x)) = \underline{\Pi}^{'}_{\alpha}(G(u(x)))$,

$$\hat{\pi}_{\alpha}(u(x)) < \hat{\pi}(u(x))$$

for any $\alpha \in (0,1]$. For any $x^{'} \in (d(x), u(x))$, $\hat{\pi}(x^{'}) = \hat{\pi}(u(x))$ and $\hat{\pi}_{\alpha}(x^{'}) \leq \hat{\pi}_{\alpha}(u(x))$ (since $\hat{\pi}_{\alpha}(x)$ is non-decreasing in x). Therefore

$$\hat{\pi}_{\alpha}(x^{'}) < \hat{\pi}(x^{'})$$

for any $x^{'} \in (d(x), u(x))$.

(b) For any $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$,

$$\Pi_{\alpha}(G(x)) = (1-\alpha)\underline{\Pi}(G(x)) + \alpha \int_0^{G(x)} h(y)dy.$$

It implies

$$\underline{\Pi}_{\alpha}(G(x)) = (1-\alpha)\underline{\Pi}(G(x)) + \alpha \int_0^{G(x)} h(y)dy$$

and

$$\hat{\pi}_{\alpha}(x) = \underline{\Pi}^{'}_{\alpha}(G(x)) = (1-\alpha)\hat{\pi}(x) + \alpha x < \hat{\pi}(x)$$

for any $\alpha \in (0,1]$, since $\hat{\pi}(x) = \pi(x) > x$ for $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$.

The argument in (a) and (b) implies the statement of (iii).

The proof of (iv)

(a) $\hat{\pi}(\underline{x}) \leq \pi(\underline{x})$ and $\hat{\pi}(\bar{x}) \geq \pi(\bar{x})$ are obtained from $\Pi^{'}(\phi = 0) \geq \underline{\Pi}^{'}(\phi = 0)$, $\Pi^{'}(\phi = 1) \leq \underline{\Pi}^{'}(\phi = 1)$ and $\Pi^{'}(G(x)) = \pi(x)$.
(b) The case of $v(x) < \pi(x)$: For $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$, $\hat{\pi}(x) = \pi(x) > v(x)$. For $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, and for $u(x)$ that is defined in the proof of (iii), $\hat{\pi}(x) = \Pi^{'}(G(u(x))) = \pi(u(x)) > v(u(x)) \geq v(x)$. It implies $\hat{\pi}(x) > v(x)$ for any $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$. Therefore $\hat{\pi}(x) > v(x)$ for any $x > \underline{x}$.
(c) The case of $v(x) > \pi(x)$: For $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$, $\hat{\pi}(x) = \pi(x) < v(x)$. For $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, and for $d(x)$ that is defined in the proof of (iii), $\hat{\pi}(x) = \Pi^{'}(G(d(x))) = \pi(d(x)) \leq v(d(x)) < v(x)$. It implies $\hat{\pi}(x) < v(x)$ for any $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$. Therefore $\hat{\pi}(x) < v(x)$ for any $x > \underline{x}$.

The proof of (v)

Step 1:

For any non-increasing $q(x)$,

$$\int_{\underline{x}}^{\bar{x}} \pi(x)q(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) \geq \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) = \int_{\underline{x}}^{\bar{x}} \hat{\pi}(x)q(x)dG(x)$$

Proof of Step 1

Since $\Pi(G(x))$ and $\underline{\Pi}(G(x))$ are continuous, applying the integration by parts,

$$\int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) + \int_{\underline{x}}^{\bar{x}} \Pi(G(x))dq(x) = \Pi(1)q(\bar{x}) - \Pi(0)q(\underline{x})$$

and

$$\int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) + \int_{\underline{x}}^{\bar{x}} \underline{\Pi}(G(x))dq(x) = \underline{\Pi}(1)q(\bar{x}) - \underline{\Pi}(0)q(\underline{x}).$$

With $\Pi(1) = \underline{\Pi}(1)$ and $\Pi(0) = \underline{\Pi}(0)$,

$$\begin{aligned} & \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) - \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) \\ = & \int_{\underline{x}}^{\bar{x}} (\underline{\Pi}(G(x)) - \Pi(G(x)))dq(x) \geq 0 \end{aligned}$$

Step 2:

$$\int_{[\underline{x},\bar{x}]} [V(q^{**}(x)) - \pi(x)q^{**}(x)]dG(x) = \int_{[\underline{x},\bar{x}]} [V(q^{**}(x)) - \hat{\pi}(x)q^{**}(x)]dG(x)$$

for $q^{**}(x) \in \arg\max_q V(q) - \hat{\pi}(x)q$.

Proof of Step 2:

By the definition, $q^{**}(x)$ is constant for each interval of x where $\hat{\pi}(x)$ is constant. Then by (ii) of the lemma,

$$\int_{\underline{x}}^{\bar{x}} \pi(x)q^{**}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} \hat{\pi}(x)q^{**}(x)dG(x).$$

This completes the proof of Step 2.

Step 3:

By Step 1, for any non-decreasing $q(x)$,

$$\int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)]dG(x) \leq \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

By Step 2, if $q^*(x)$ is the solution of

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)]dG(x)$$

subject to $q(x)$ is non-increasing, then $q^*(x)$ solves

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

Then

$$\int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \pi(x)q^*(x)]dG(x) = \int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \hat{\pi}(x)q^*(x)]dG(x).$$

It completes the proof of (v). ■

Lemma 3 $\hat{h}(\theta \mid \eta)$ *is non-increasing and continuous in* θ *on* $\Theta(\eta)$ *with* $\hat{h}(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$ *and* $\hat{h}(\theta \mid \eta) > \theta$ *for* $\theta > \underline{\theta}(\eta)$.

Proof of Lemma 3

Since $h(\theta \mid \eta)$ is continuous, Lemma 2(i) implies that $\hat{h}(\theta \mid \eta)$ is continuous and non-decreasing in θ. Since $\theta < h(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$, Lemma 2(iv) implies that $\theta < \hat{h}(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$. By the continuity of $\hat{h}(\theta \mid \eta)$, $\underline{\theta}(\eta) \leq \hat{h}(\underline{\theta}(\eta) \mid \eta)$. Lemma 2(iv) also implies $\hat{h}(\underline{\theta}(\eta) \mid \eta) \leq h(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$. Therefore $\hat{h}(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$. ∎

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