Hierarchical Control Rights and Strong Collusion[1]

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Abstract

A hallmark of hierarchies is that superiors exercise greater authority over appointment of subordinates than the other way around. We provide a rationale for this in a model with strong collusion between a (less well-informed) supervisor and (informed) agent, which allows each to commit to threats to punish the other for refusing to collude. Providing greater *ex ante* authority to the supervisor is necessary for the Principal to exploit bargaining frictions within the coalition. By contrast, in contexts of weak collusion where such commitments are not possible, or where there is no collusion, the allocation of *ex ante* authority is irrelevant.

KEYWORDS: mechanism design, supervision, collusion, bargaining power

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1 Introduction

A hallmark of a hierarchical organization is asymmetry of control rights between superiors and subordinates. For instance, managers exercise considerable control over worker hiring decisions; workers typically have little or no say in appointment of their managers. Auditors exercise authority over who to audit; regulators decide which private utility will be awarded a franchise and the contract they are offered. While governance institutions or legal rules may constrain some aspects of the control that superiors exercise over subordinates, by and large the design of a hierarchy seems intended to tilt control rights in favor of the former. Indeed, the very notion of a hierarchy connotes an organization of agents with layers of vertical authority. Most models of hierarchies takes such a vertical organization as given, wherein managers at any layer appoint and design contracts for their direct subordinates at the next layer. There are relatively few theoretical explanations of the rationale for such vertical top-down assignment of authority, by comparing their consequences with alternative assignments — either 'bottom up' organizations where supervisees have the authority to appoint and offer contracts to their supervisors, or teams with egalitarian sharing of power among members. Understanding the trade-offs involved could also shed some light on normative policy questions such as the implications of raising bargaining power of agents, e.g., via greater regulatory competition (where regulated firms can choose between different regulators or regulatory jurisdictions), auditing practices where firms can appoint their own auditors, or allowing CEOs to appoint Directors to company boards.

In this paper we offer a rationale for granting hierarchical authority: to control consequences of collusion between supervisors and supervisees. Evidence for such collusion has recently been forthcoming in many contexts, e.g., between outside Directors and CEOs (Hallock (1997), Hwang and Kim (2009), Fracassi and Tate (2012), Kramarz and Thesmar (2013), Schmidt (2015)), between management and workers (Bertrand and Mullainathan (1999, 2003), Atanassov and Kim (2009), Cronqvist et al. (2009)), 'revolving doors' between credit-rating agencies and firms (de Haan et al. (2015), Cornaggia et al. (2016)) and between auditors and their clients (Lennox (2005), Lennox and Park (2007), Firth et al. (2012)).

We pursue an approach initiated by Tirole (1986), Kofman and Lawarree (1993) and Laffont and Tirole (1993) which models collusion as hidden side-contracts that are exogenously enforceable, subject to asymmetric information within the coalition. We focus on *strong* collusion, where colluding parties can commit to (off-equilibrium-path) threats they will carry out if the other party refuses to participate in the collusion. This is in contrast to the more standard assumption of *weak collusion*, where such commitments are not possible. Strong collusion enlarges the scope of coalitional deviations, by limiting the control that the Principal can effectively exercise on outcomes of coalitional bargaining. We also assume that the structure of asymmetric information within the organization is hierarchical: the Supervisor is better informed than the Principal, while the supervisee (hereafter referred to as the Agent) is better informed than the Supervisor. Our main result is that organizational design to limit consequences of strong collusion in the presence of one-sided asymmetric information requires countervailing allocation of formal authority. Under weak collusion (or absence of collusion) the allocation of control rights between the Supervisor and Agent turns out to be irrelevant.

The details of our model and results are as follows. Agent A produces a divisible good q for the Principal P, at a unit cost of θ that is privately observed by A. P's objective is a mixture of personal profit and the agent's payoff, with the latter occupying lower weight than the former.[3] Hence the model applies both to the organization of private firms as well as of regulatory institutions. S observes a signal η which is partially informative regarding θ. The realization of this signal is also observed by A so that asymmetric information within the (S, A) coalition is one-sided. P observes neither θ nor η, while their joint distribution is common knowledge. All parties are risk-neutral, and have zero outside options. In the absence of collusion, P designs a mechanism which stipulates production decisions and payments to S and A based on reports they send to P concerning realizations of their respective information as well as their willingness to participate in the mechanism. Collusion takes the form of a hidden side contract between S and A, which supplements any formal contracts. The side contract coordinates reports they respectively send to P, besides stipulating side payments exchanged between them. This side contract is negotiated at an ex ante stage, before S and A have received their respective signals. Liquidity constraints prevent lumpsum side payments at the ex ante stage; all side payments will be made *ex post* after S and A have received their respective payments from P. As is standard in the literature we assume these side contracts are costlessly enforceable, cannot be renegotiated and must respect interim participation constraints (i.e., allow S or A to exit from the

[3]If they are equally valued, the first-best can be achieved as in Baron and Myerson (1982).

collusive arrangement at the interim stage). Hence collusion is subject to frictions resulting from (one-sided) asymmetric information within the coalition. Owing to this, the Coase Theorem need not apply; collusive outcomes typically depend on the allocation of control rights and outside options available to the colluding parties.

Manipulating these provides P with an opportunity to control the outcomes of collusion. Consider an environment with a large number of ex ante identical applicants for the S position, as well as for the A position. If P appoints some S at the ex ante stage, and provides this S with the authority to appoint an A from the pool, it skews ex ante welfare weights in favor of S *vis-a-vis* A. While P cannot observe the precise contract offered by S to the appointed A, P thereby ensures that S can make a take-it-or-leave-it contract to the appointed A. Conversely, if P were to appoint an A and endow this agent with the authority to appoint an S from the pool, the allocation of bargaining power would be reversed to favor A. A more egalitarian arrangement would emerge if P were to appoint a specific S-A pair, and leaving them to negotiate a contract with one another under conditions of a bilateral monopoly.

When collusion is weak, a decision by either S or A to withdraw from colluding at the interim stage results in a noncooperative equilibrium play of P's mechanism: they exchange no side payments and submit reports that are a best response to the reports of the other. A previous paper of ours (Mookherjee et al. (2016)) provides a detailed analysis of optimal mechanism design by P under weak collusion, using the notion of a Perfect Bayesian Equilibrium that is Pareto-undominated for the coalition. It turns out that while weak collusion outcomes depend sensitively on interim outside options of S and A which can be manipulated by P (since the outside options are noncooperative equilibria of the underlying mechanism), they do not depend on the allocation of welfare weights between S and A at the ex ante stage. The intuitive reason is that once a given allocation is invulnerable to weak collusion, they do not allow any Pareto-improving deviations for the coalition. So if they are collusion proof for any set of welfare weights, they are also collusion proof for any other set of welfare weights. Hence it does not matter whether S appoints A or the other way around, or if they share power within a team.

This turns out to no longer be the case when collusion is strong. The consequences of strong collusion for auction design has been considered by some authors (Dequiedt (2007), Che and Kim (2009)), but we are not aware of any papers studying their consequences for design of supervisory mechanisms. Now the side contract allows each colluding party to stipulate a threat regarding how they would react to a unilateral exit by the other party at the interim stage, with respect to the report they will subsequently submit to P. Outside options will correspond to maximal punishments that can be inflicted by each party to exit by the other, rather than noncooperative equilibria of the non-collusive game. By lowering outside options, this enlarges the scope for collusion.

We provide two results for the strong collusion context. (1) If A has at least as much welfare weight as S, appointing a supervisor is worthless for P. In other words, P will attain the same payoff by contracting directly with A, without trying to elicit any information from any supervisor. (2) If S is assigned a higher welfare weight, and S's signal η is coarse (in the sense of having only two possible realizations while θ is continuously distributed), appointing a supervisor is valuable, i.e., P attains a strictly higher payoff compared to (1).

These results imply that in order to take advantage of the presence of a supervisor, it is necessary to bias ex ante control rights in favor of S. A 'bottom-up' organization where A appoints an S, or an egalitarian one in which S and A are appointed directly by P and share the same welfare weight, will not just perform worse than one where S is given the authority to appoint A — it will do no better than an organization which dispenses entirely with a supervisor. The intuitive reason is that when bargaining 'power' is not biased in favor of S, the coalition ends up maximizing rents of A while pushing S down to her minmax payoff (conditional on any signal realization). The former effectively becomes a residual claimant on aggregate coalitional rent, resulting in ex post efficient collusion. In other words, collusive outcomes are not subject to any frictions, and the coalition acts as a single agent (where S's signal is chosen to maximize A's rent). With S and A's interests perfectly aligned, appointing S is worthless.

Biasing the welfare weight in favor of S, on the other hand, preserves frictions in collusion. The coalition would ideally like to maximize S's payoff subject to A's willingness to participate, but is constrained by asymmetric information (owing to the coarseness of S's signal regarding A's cost). Not knowing A's cost realization, S is unable to punish A 'too severely'. The resulting conflict of interest can be utilized effectively by P, and thereby take advantage of S's presence. Hence it is advantageous to bias ex ante authority in favor of the relatively uninformed party.[4]

[4]This also helps explain the necessity of a 'coarse' information assumption in Result 2. If S were perfectly

This still leaves open the question whether it is optimal to skew welfare weights as much as possible in favor of S: i.e., is it optimal for P to give S the power to make a take-it-or-leave-it contract to A? Does such an organization dominate one where both parties have a positive welfare weight, thereby implying that delegating appointment rights to S dominates centralized appointment by P? We have not yet managed to answer this question in the context of a continuously distributed cost shock. But we have been able to show this in the context of the model of Celik (2009) where θ takes three possible values, η takes two possible values, and S's information has a partition structure. (To keep the paper short, the detailed analysis of this case is provided in the online Appendix.) In this setting, thus, our approach provides a novel argument for why delegation can outperform centralization.[5]

The preceding discussion pertains to allocation of *ex ante* control rights. With regard to the allocation of *interim* control rights, limits need to be imposed on the power that S exercises over A's appointment. So S is delegated the authority to hire, but not to fire A. The argument for not delegating the authority to fire is similar to a weak collusion context: the option exercised by A in deciding whether or not to accept the side contract offered by S is essential, otherwise 'double marginalization of rents' (DMR) sets in and agency costs get magnified by appointing a supervisor. The same continues to be true with strong collusion. The asymmetry between hiring and firing authority of managers is consistent with the design of many organizations, in which hierarchical authority of managers coexists with dispute settlement mechanisms that constrain their ex post authority to fire subordinates. The importance of internal dispute settlement mechanisms has been emphasized by Williamson (1975); e.g., as one of three positive attributes of hierarchies:

> First,...the incentives to behave opportunistically are accordingly attenuated. Second, and related, internal organization can be more effectively audited. Finally, when differences do arise, internal organization realizes an advantage over market mediated exchange in dispute setting respects. (Williamson (1975, pp 29-30))

The paper is organized as follows. Section 2 lays out the model and provides a charac-

informed regarding the realization of θ, there would again be no frictions within the coalition and P would not be able to take advantage of S's presence. However, this leaves open the question whether Result 2 extends to more than two (but a finite number) of possible realizations of S's signal.

[5]See Mookherjee (2006, 2013) for overviews of the theoretical literature comparing centralized and decentralized decision-making.

terization of allocations that can be supported with weak and strong collusion respectively. Section 3 then provides the two main results (1) and (2) and the detailed arguments underlying them. Section 4 concludes, while technical details of proofs are provided in the Appendix.

2 Model

2.1 Technology, Preferences and Information

We consider an organization composed of a principal (P), an agent (A), and a supervisor (S). P hires A who delivers an output q. P's return from q is $V(q)$, a twice continuously differentiable, increasing and strictly concave function satisfying the Inada condition ($\lim_{q\to 0} V^{'}(q) = +\infty$ and $\lim_{q\to +\infty} V^{'}(q) = 0$) and $V(0) = 0$. A's cost of supplying q is θq.

The realization of θ is privately observed by A. Θ denotes the support of θ. We assume that Θ is compact; apart from that we do not impose any structure on the support at this stage, as all but the last result do not depend on any specific features of Θ. It is common knowledge that everybody shares a common prior cdf $F(\theta)$ over Θ. S (as well as A) costlessly acquires an informative signal $\eta \in \Pi \equiv \{\eta_1, \eta_2, ..., \eta_m\}$ about A's cost θ with $m \geq 2$.[6] The cdf over θ conditional on η is denoted $F(\theta|\eta)$; since η is informative, there exists $\eta \in \Pi$ such that $F(.|\eta)$ differs from $F(.)$ over a set of positive (prior) measure. $a(\eta \mid \theta) \in [0, 1]$ denotes the likelihood function of η conditional on θ. $\Theta(\eta)$ denotes the support of θ conditional on η, which we assume is compact for every η. Let $K \equiv \{(\theta, \eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$ denote the set of possible states. We assume there exists η such that $\Theta(\eta)$ includes at least two elements; hence η does not provide complete information about θ.

All players are risk neutral. S's payoff is $u_S = X_S + t_S$ where t_S is a transfer received by S within the coalition. A's payoff is $u_A = X_A + t_A - \theta q$ where t_A is a transfer received by A within the coalition. A and S have outside options equal to 0. P's objective is a weighted average of profit ($\Pi \equiv V(q) - X_A - X_S$) and welfare of A and S ($u_A + u_S$), with a lower relative weight on the latter. With $k \in (\frac{1}{2}, 1]$ denotes the weight on profit, and $1 - k$ on welfare of A and S, P's payoff reduces to $k[V(q) - (X_S + X_A)] + (1 - k)[X_S + X_A - \theta q]$.[7]

[6] If S incurs a fixed cost c to acquire the signal, transfers received by S must be replaced by transfers net of this fixed cost while measuring S's payoff. Increases in c will of course lower the value of appointing the supervisor, but it is easy to see how the results will be modified.

[7] We exclude $k = \frac{1}{2}$ because in that case the first-best can be achieved (as in Baron and Myerson (1982)).

Hence the model applies both to the organization of private firms whose owners seek to maximize profit ($k = 1$), as well as regulation or taxation contexts where P is a social planner pursuing a welfare objective that includes payoffs of A and S as well as profit, but assigns a higher weight to the latter.[8]

In this economic environment, a (deterministic) allocation is denoted by

$$\{(u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta)) \mid (\theta,\eta) \in K\}.$$

2.2 Mechanism, Collusion Game and Equilibrium Concept

We focus attention on deterministic mechanisms or grand contract (GC) designed by P:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S); M_A, M_S)$$

where M_A (or M_S) denotes a message set for A (or S) which are compact subsets of finite dimensional Euclidean spaces.[9] Message spaces include exit options for A and S respectively ($e_A \in M_A, e_S \in M_S$), where $X_A = q = 0$ whenever $m_A = e_A$, and $X_S = 0$ whenever $m_S = e_S$). The set of grand contracts satisfying these restrictions is denoted by $\mathcal{GC}$.

It will frequently be necessary to allow for randomized message choices. Let $\Delta(M_A)$, $\Delta(M_S)$ and $\Delta(M)$ denote the set of probability measures on M_A, M_S and $M \equiv M_A \times M_S$ respectively. For $(\mu_A, \mu_S) \in \Delta(M_A) \times \Delta(M_S)$ and $\mu \in \Delta(M)$, we define the mixed strategy extensions of the grand contract, which are respectively described in the expected value of corresponding allocations, as follows:

$$\begin{aligned} \bar{GC} &\equiv (\bar{X}_A(\mu_A, \mu_S), \bar{X}_S(\mu_A, \mu_S), \bar{q}(\mu_A, \mu_S)) \\ &= \int_{M_A}\int_{M_S} (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S)) d\mu_A(m_A) d\mu_S(m_S) \end{aligned}$$

and

$$\tilde{GC} \equiv (\tilde{X}_A(\mu), \tilde{X}_S(\mu), \tilde{q}(\mu)) = \int_M (X_A(m), X_S(m), q(m)) d\mu(m).$$

Let NC denote the organization without any collusion. The timing of events in NC is as follows.

[8]The latter would be the case e.g., if P represents the interests of consumers, who need to be taxed to finance transfers to A and S, and these taxes involve deadweight losses.

[9]These assumptions on message spaces are imposed in order to apply the minimax theorem (Nikaido (1954)). We ignore stochastic mechanisms which randomize the allocation conditional on messages, since they lower P's welfare (owing to strict concavity of V) without affecting S or A's payoffs.

(NC1) P offers the grand contract $GC \in \mathcal{GC}$.

(NC2) A observes θ and η. S observes η.

(NC3) A and S play GC non-cooperatively.

If P offers a null contract to S (where M_S is the empty set and $X_S = 0$), this is an organization without S, which is referred to as NS (no supervision).

We now introduce collusion between S and A, which takes the form of a side contract SC between S and A that is unobserved by P. The side contract is designed at the *ex ante* stage. We do not formally model the 'appointment' game at the *ex ante* stage, where either one of the colluding parties may have the authority to make a take-it-or-leave-it offer to the other, or may have to share bargaining power more equally. We adopt a reduced form approach, summarizing the allocation of *ex ante* bargaining power by a parameter $\alpha \in [0, 1]$ that represents A's relative welfare weight. We treat α as a parameter than can be chosen by P: this is the key design question addressed in this paper.

The reduced form approach is represented by a fictional (uninformed) third party that acts as a mediator to design the side contract at the *ex ante* stage. The third party maximizes $\alpha u_A + (1 - \alpha)u_S$ ($\alpha \in [0, 1]$), where u_A and u_S respectively denote *ex ante* payoffs of A and S. The third party does not play any budget breaking role, hence transfers within the coalition must balance: $t_A + t_S \leq 0$. Owing to liquidity constraints, no side payments can be exchanged at the *ex ante* stage; they can only be exchanged at the *ex post* stage after they have received payments from P. The side contract cannot be renegotiated at the interim or *ex post* stage. It allows exchange of private messages between A and S, which determine a side payment and joint set of messages they respectively send to P. Since message spaces include exit as well as type reports, collusion takes the *ex ante* form studied in Mookherjee et al. (2016), rather than the interim form studied by Faure-Grimaud et al. (2003) or Celik (2009).

At the interim stage (where S observes η and A observes (θ, η)), S and A decide whether or not to accept the side contract. If both reject it (or if the third party offered a null side contract), S and A play GC noncooperatively. If both accept it, the SC is implemented. What happens when one of them accepts and the other rejects, depends on whether collusion is *weak* or *strong*. When collusion is weak, they play GC noncooperatively. When it is strong, SC specifies a reporting strategy of the party that accepted it, which can be interpreted as a threat that party commits to. The party that rejected it then plays a best response to this threat.

More formally, in strong collusion (referred to hereafter as S-Collusion), after stage NC1 where P offers GC, the third party offers a side-contract SC (or a null side-contract NSC) to A and S. If NSC is offered, A and S play GC non-cooperatively based on the prior belief. Otherwise, the game proceeds as follows. After A and S observing (θ, η) and η respectively, player $i = A, S$ selects a message $d_i \in D_i$ $(i = A, S)$ where D_i is i's message set specified in the side-contract. D_i includes i's exit option $\hat{e}_i$ from the side-contract. Then if $d_A \neq \hat{e}_A$ and $d_S \neq \hat{e}_S$, their reports to P are selected according to $\mu(d_A, d_S) \in \Delta(M)$, and side payments to A and S are determined according to functions $t_A(d_A, d_S)$ and $t_S(d_A, d_S)$ respectively. If $d_A = \hat{e}_A$ and $d_S = \hat{e}_S$, A and S play GC non-cooperatively. If $d_i \neq \hat{e}_i$ and $d_j = \hat{e}_j$ $(i, j = A, S)$, i's message to P is selected according to $\mu_i(d_i) \in \Delta(M_i)$, and the side payment to i is $t_i(d_i)$.[10] On the other hand, j plays GC without any constraint and any side transfer.

We focus on Weak Perfect Bayesian Equilibrium ($WPBE$) of this S-collusion game induced by the grand contract GC.[11] However, there may be multiple WPBE in a given game. We assume collusion permits parties to coordinate the choice of a WPBE, hence the third party can specify a selected WPBE to maximize the welfare-weighted sum of ex ante payoffs of S and A in the event of multiple WPBE. The resulting equilibrium concept is denoted by WPBE(sc). In case there are two WPBE(sc) where the third party receives the same payoff, we assume that P can select the more desirable one. With this convention, problems associated with multiple equilibria can be avoided.

Feasible allocations in S-collusion can now be defined:

Definition 1 *An allocation (u_A, u_S, q) is achievable in S-Collusion with α if it is realized in WPBE(sc) under α for some $GC \in \mathcal{GC}$.*

Let $\mathcal{A}^S(\alpha)$ denote the set of achievable allocation in S-Collusion with α.

2.3 Strong Collusion-Proof Allocations

We now define what it means for an allocation to be strong collusion proof, and show it is a necessary and sufficient condition for achievability in S-Collusion with α. This enables us

[10] Owing to the budget balance condition, $t_i(d_i) \leq 0$.

[11] For definition of WPBE, see Mas-Colell, Whinston and Green (1995, p.285).

to characterize achievable allocations by a set of incentive compatibility constraints.

First, it is evident that any achievable allocation must satisfy a set of individual incentive constraints, pertaining to truthful reporting of θ by A (IC_A), and participation constraints for A (PC_A) and S (PC_S) respectively:

$$u_A(\theta, \eta) \geq u_A(\theta^{'}, \eta) + (\theta^{'} - \theta)q(\theta^{'}, \eta)$$

for any $\theta, \theta^{'} \in \Theta(\eta)$ and any $\eta \in \Pi$,

$$u_A(\theta, \eta) \geq 0$$

for any $(\theta, \eta) \in K$ and

$$E[u_S(\theta, \eta) \mid \eta] \geq 0$$

for any $\eta \in \Pi$. These must be satisfied both when collusion does or does not take place. We say that (u_A, u_S, q) satisfies *individual incentive compatibility (IIC)* if and only if it satisfies IC_A, PC_A and PC_S. Incentive compatibility with respect to η reports do not have to be included since A and S observe the realization of η, so P can elicit this information by cross-checking their respective reports if they do not collude.

Now we turn to coalitional incentive constraints. Collusion proofness requires absence of any scope for the third party to offer a non-null side contract. In the context of S-collusion, threats not actually used on the equilibrium path play a role. To capture their role, we need to go beyond standard revelation mechanisms where each type report correspond to messages used on the equilibrium path, and augment them with some auxiliary non-type messages.

Formally, we augment the state space $K = \{(\theta, \eta) \mid \theta \in \Theta(\eta), \eta \in \Pi\}$ as follows. Define

$$\bar{K} \equiv \Theta \times \bar{\Pi}$$

where $\bar{\Pi} \equiv \Pi \cup \{\eta_0\}$. This includes two types of augmentation. One is the augmentation from $\Theta(\eta)$ to Θ, which allows inconsistent type reports to be submitted. Second is the augmentation from Π to $\bar{\Pi}$, which allows one auxiliary message η_0 regarding the signal realization to be submitted.

Recall that an allocation is defined over the type domain K. It can be extended to the augmented domain as follows.

Definition 2 *Given any allocation* (u_A, u_S, q) *which satisfies* IC_A *and* PC_A*,* (u_A^e, u_S^e, q^e) *is an incentive compatible augmentation of* (u_A, u_S, q) *on* $\bar{K}$*, if*

$$(u_A^e(\theta,\eta), u_S^e(\theta,\eta), q^e(\theta,\eta)) = (u_A(\theta,\eta), u_S(\theta,\eta), q(\theta,\eta))$$

for $(\theta,\eta) \in K$ *and* $(u_A^e(\theta,\eta), q^e(\theta,\eta))$ *satisfies* IC_A *and* PC_A *on* $\bar{K}$*.*

Note that the coalition can also collectively decide to exit from GC, which is represented by joint message $e \equiv (e_S, e_A)$. In the event that e is chosen, the autarkic allocation $(X_A = X_S = q = 0)$ results. Hence the augmented message space is $\bar{K} \cup \{e\}$.

For any allocation (defined over the type space K) we can define the corresponding *coalitional incentive scheme* by the aggregate transfers between P and the coalition: $(\hat{X}, \hat{q}) = (u_A + u_S + \theta q, q)$ which is also defined over K. We can also extend this to the augmented domain $\bar{K}$, while allowing A and S to randomize their messages according to the measure μ defined over $\bar{K} \cup \{e\}$. Let $\Delta(\bar{K} \cup \{e\})$ denote the corresponding set of measures. With a slight abuse of notation, we denote the corresponding expected values of the coalitional incentive scheme (defined over the augmented domain) as $(\hat{X}^e(\mu), \hat{q}^e(\mu))$ for any given $\mu \in \Delta(\bar{K} \cup \{e\})$, where $(\hat{X}^e(e), \hat{q}^e(e)) \equiv (0, 0)$.

The side contracting problem can be represented as follows. Given a coalitional incentive scheme, the coalition select a joint report $\mu \in \Delta(\bar{K} \cup \{e\})$ to send to P, and then redistribute the resulting rents $(\tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta))$ between A and S, i.e., such that $\tilde{u}_A(\theta,\eta) + \tilde{u}_S(\theta,\eta) = \hat{X}^e(\mu(\theta,\eta)) - \theta\hat{q}^e(\mu(\theta,\eta))$. Conditional on both A and S agreeing to participate, this joint decision is based on a θ report submitted by A to the third party. The side contract does not stipulate any coalitional decision in the event that both A and S reject it. If A rejects it while S does not, the side contract specifies a reporting strategy for S which acts as a threat. Let this strategy be denoted by $P(\cdot \mid \eta)$ for each $\eta \in \Pi$, which is a probability function defined on $\bar{\Pi}$ such that $\Sigma_{\eta' \in \bar{\Pi}} P(\eta' \mid \eta) = 1$ and $0 \leq P(\eta' \mid \eta) \leq 1$ for all $\eta' \in \bar{\Pi}$. Here $P(\eta' \mid \eta)$ denotes the probability that S reports η' in the state where η has been observed. The strategy of reporting truthfully in state η i.e., $(P(\eta \mid \eta) = 1$ and $P(\eta' \mid \eta) = 0$ for any $\eta' \neq \eta)$ is denoted by $I(\eta)$. Similarly, let $I(\theta,\eta)$ denote the strategy of reporting the state truthfully in state (θ,η).

Using these definitions and notations, we define strong collusion-proof (SCP) allocation as follows.

Definition 3 *Allocation* (u_A, u_S, q) *is strong collusion-proof (or SCP) for* $\alpha \in [0, 1]$*, if* (u_A, u_S, q) *is IIC, and there exists an outside option payoff* $\omega \geq 0$ *for S, and an incentive compatible augmentation* (u_A^e, u_S^e, q^e) *of* (u_A, u_S, q) *on* $\bar{K}$ *satisfying* $u_S^e(\theta, \eta_0) = \omega$ *for any* $\theta \in \Theta$*, such that for any* $\eta \in \Pi$*,*

$$(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), P(\cdot \mid \eta)) = (I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$$

solves problem $P^S(\alpha : \eta)$*:*

$$\max E[\alpha \tilde{u}_A(\theta, \eta) + (1 - \alpha)\tilde{u}_S(\theta, \eta) \mid \eta]$$

subject to $(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), P(\cdot \mid \eta))$ *satisfies for all* $\theta \in \Theta(\eta)$*:*

(i) $\mu(\theta, \eta) \in \Delta(\bar{K} \cup \{e\}), \tilde{u}_A(\theta, \eta) \in \Re, \tilde{u}_S(\theta, \eta) \in \Re, P(.|\eta) \in \Delta(\bar{\Pi})$

(ii) $\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}^e(\mu(\theta', \eta))$ *for any* $\theta' \in \Theta(\eta)$

(iii) $\tilde{u}_A(\theta, \eta) + \tilde{u}_S(\theta, \eta) = \hat{X}^e(\mu(\theta, \eta)) - \theta\hat{q}^e(\mu(\theta, \eta))$

(iv) $E[\tilde{u}_S(., \eta) \mid \eta] \geq \omega$

(v) $\tilde{u}_A(\theta, \eta) \geq \Sigma_{\eta' \in \bar{\Pi}} P(\eta' \mid \eta) u_A^e(\theta, \eta')$.

We provide an informal explanation of this notion. A non-null side contract is represented by the following components. Provided both A and S have agreed to participate, and following an internal type report θ by A and a common report η of the signal by A and S, the coalition submits a message report to P according to the strategy $\mu(\theta, \eta)$ (satisfying the first part of condition (i)), and then reallocates the resulting coalitional allocation via side payments to generate net payoffs $(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta))$ for A and S respectively (the budget balance condition (iii)). The side contract must provide A with an incentive to report θ truthfully within the coalition (condition (ii)).

Moreover, the side contract includes threats in the event of unilateral rejection by either party, that ensure their participation. In order to induce S to participate (conditional on A agreeing to participate) in the collusion, condition (iv) must be satisfied. S's participation is induced by a threat by A to subsequently report to P in some way if S refuses, which ensures that S cannot attain a payoff higher than ω. Since asymmetric information is one-sided, the standard minimax theorem ensures that S's minmax payoff is well-defined (given an associated grand contract GC), and A has a reporting strategy that guarantees S cannot earn more than ω. The minmax payoff ω of S must be non-negative since S can always exit from GC, and is effectively chosen by P while designing the mechanism. In particular, the mechanism can be augmented to ensure that ω is earned by S upon submitting the auxiliary message η_0, no matter what A reports. We show below that this way of designing the mechanism entails no loss of generality.

Finally, A's participation (conditional on S's participation) when both have observed the signal η, is ensured by the threat of S reporting according to the strategy $P(\cdot|\eta)$ if A refuses. A will then be a Stackelberg follower in noncooperative play of P's mechanism, and will select a best response to this threat. Since the augmented mechanism satisfies individual incentive constraints for A, it will be optimal for A to report truthfully, no matter what S reports.[12] This will generate A a payoff of $u_A^e(\theta, \eta')$ if S reports η'. Hence the right hand side of (v) represents the outside option payoff of A to participating in the collusion.

S-collusion proofness requires that the null side contract is an optimal choice for the third party. The null side-contract is represented by a choice of a side contract allocation which coincides with the allocation itself (i.e., there are no side-payments), and truthful reports submitted to P. Moreover, no threats need to be used by S to coerce A into accepting this contract, hence $P(\cdot \mid \eta) = I(\eta)$.

The set of strong collusion-proof (or SCP) allocations is denoted by $\mathcal{A}^S(\alpha)$. We now show that it coincides with the set of achievable allocations in S-collusion.

Lemma 1 *Allocation (u_A, u_S, q) is achievable in S-Collusion with α if and only if it is strongly collusion-proof for α.*

The proof is provided in the Appendix; it extends standard arguments to the context of strong collusion, which requires augmenting any given equilibrium allocation in a particular way that ensures the allocation satisfies the SCP property. Conversely, any SCP allocation can be achieved as a WPBE(sc) allocation in a GC which can be constructed on the basis of the incentive compatible augmentation of the allocation.

[12] Note that A reports after observing θ, so the realization of η does not affect A's preferences.

2.4 Comparison with Weak Collusion Proof Allocations

It is useful to compare the notion of strong collusion proofness to weak collusion-proofness, which has been analyzed extensively in our previous paper (Mookherjee et al. (2016)). In order to characterize weak collusion proof (WCP) allocations which are self-enforcing and not reliant on additional threats to induce S and A to agree to a collusive agreement, it suffices to consider revelation mechanisms without any augmentation. Hence the domain of the mechanism is $K \cup \{e\}$. Given a revelation mechanism (u_A, u_S, q) defined over $K \cup \{e\}$, and given a mixed reporting strategy $\mu \in \Delta(K \cup \{e\})$, the associated coalitional incentive scheme is denoted by $(\hat{X}(\mu), \hat{q}(\mu))$, where $(\hat{X}(\theta,\eta), \hat{q}(\theta,\eta)) \equiv (u_A(\theta,\eta)+u_S(\theta,\eta)+\theta q(\theta,\eta), q(\theta,\eta))$. and $(\hat{X}(e), \hat{q}(e)) \equiv (0,0)$. Using the terminology of this paper, a weak-collusion proof allocation can be defined as follows.[13]

Definition 4 *(u_A, u_S, q) is weak collusion-proof (or WCP) for α, if (u_A, u_S, q) is IIC, and for any $\eta \in \Pi$,*

$$(\mu(\theta,\eta), \tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta)) = (I(\theta,\eta), u_A(\theta,\eta), u_S(\theta,\eta))$$

solves problem $P^W(\alpha : \eta)$:

$$\max E[\alpha\tilde{u}_A(\theta,\eta) + (1-\alpha)\tilde{u}_S(\theta,\eta) \mid \eta]$$

subject to $(\mu(\theta,\eta), \tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta))$ satisfies for all $\theta \in \Theta(\eta)$:

(i) $\mu(\theta,\eta) \in \Delta(K \cup \{e\}), \tilde{u}_A(\theta,\eta) \in \Re, \tilde{u}_S(\theta,\eta) \in \Re$

(ii) $\tilde{u}_A(\theta,\eta) \geq \tilde{u}_A(\theta',\eta) + (\theta' - \theta)\hat{q}(\mu(\theta',\eta))$ *for any* $\theta' \in \Theta(\eta)$

(iii) $\tilde{u}_A(\theta,\eta) + \tilde{u}_S(\theta,\eta) = \hat{X}(\mu(\theta,\eta)) - \theta\hat{q}(\mu(\theta,\eta))$

(iv) $E[\tilde{u}_S(.,\eta) \mid \eta] \geq E[u_S(.,\eta) \mid \eta]$

(v) $\tilde{u}_A(\theta,\eta) \geq u_A(\theta,\eta)$.

Apart from the smaller range ($K \cup \{e\}$ instead of $\bar{K} \cup \{e\}$) of the reporting strategy $\mu(.)$, the main difference is in the participation constraints (iv) and (v). The outside options

[13] In Mookherjee et al. (2016), the definition of weak collusion-proof allocation did not include the individual participation constraints of A and S. Here we include them for purposes of comparability.

correspond to truthful reporting in GC, which forms a noncooperative equilibrium. Hence the outside options correspond exactly to interim payoffs associated with the allocation itself. Comparing A's participation constraint (v) between the two definitions, it is evident that A's outside option in strong collusion is lower, by an extent that can be controlled by the coalition by selecting an arbitrary reporting strategy $P(\cdot \mid \eta)$ by S in the event that A refuses to collude. Moreover S's outside option ω in strong collusion is also lower, as it is bounded above by S's equilibrium payoff.[14] Therefore strong collusion permits the third party to offer a wider range of allocations, implying that strong collusion proofness is a more restrictive property compared with weak collusion proofness.

The set of WCP allocations turns out to be independent of *ex ante* bargaining power α.[15] As we show in the next section, this is no longer true for SCP allocations. The WCP notion allows the (S,A) coalition to deviate only when they can find a Pareto improving allocation, while the SCP notion also allows deviations that reduce the welfare of one party if it increases the welfare of the other party sufficiently.

3 Results

One class of allocations that can always be attained by P irrespective of collusion corresponds to not utilizing reports regarding the supervisor's signal η at all. We refer to this as the *No Supervision (NS)* organization, in which the class of attainable allocations (denoted $\mathcal{A}^{NS}$) is defined as follows. There exists a nonnegative constant c and nonincreasing real-valued functions $(X(\theta), Q(\theta))$ defined on Θ such that for any (θ, η):

(a) $u_S(\theta, \eta) = c$

(b) $u_A(\theta, \eta) = X(\theta) - \theta Q(\theta) = \max_{\theta' \in \Pi}[X(\theta^{'}) - \theta Q(\theta^{'})]$.

It is evident that any feasible allocation in NS is also feasible with weak or strong collusion (irrespective of α), since it does not utilize any reports of η.

We now present our first main result.

[14]This follows from the requirement that the null side contract is feasible in the side contracting problem in strong collusion.

[15]If an allocation is not WCP for some $\alpha \in (0, 1)$, there must exist a feasible allocation that *ex ante* Pareto dominates it, so it will not be WCP for any other $\alpha^{'} \in (0, 1)$. As shown in Mookherjee et al. (2016), the argument can be extended to include corner values of α owing to the existence of side-transfers.

Proposition 1 *An allocation which is strong collusion proof for any $\alpha \geq \frac{1}{2}$ is also attainable in NS.*

Proof of Proposition 1: Consider any allocation $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$ which is strong collusion proof for $\alpha \geq \frac{1}{2}$. By Lemma 1, there exists $\omega \geq 0$ and an incentive compatible augmentation (u_A^e, u_S^e, q^e) of this allocation satisfying $u_S(\theta, \eta) = \omega$, such that for any η, $(I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$ solves $P^S(\alpha : \eta)$. Let the corresponding coalitional incentive scheme be $(\hat{X}^e(\mu), \hat{q}^e(\mu))$. Define

$$\mu^*(\theta) \in \arg \max_{\mu \in \Delta(\bar{K} \cup \{e\})} \hat{X}^e(\mu) - \theta \hat{q}^e(\mu).$$

i.e., a reporting strategy that maximizes the *ex post* joint payoff of A and S in every state.

We claim that

$$(\mu(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta)) = (\mu^*(\theta), \hat{X}(\mu^*(\theta)) - \theta \hat{q}(\mu^*(\theta), \omega)$$

is a solution of $P^S(\alpha : \eta)$ for any η. Upon setting $c = \omega$, $X(\theta) = \hat{X}(\mu^*(\theta))$ and $Q(\theta) = \hat{q}(\mu^*(\theta))$, it is evident this claim will imply that the allocation is attainable in NS.

To establish the claim, we first derive an upper bound for the objective function in the problem $P^S(\alpha : \eta)$. From the constraint $E[\tilde{u}_S(\theta, \eta) \mid \eta] \geq \omega$ and the assumption that $\alpha \geq 1/2$, for any reporting strategy $\mu(\theta, \eta)$ the following is true:

$$\begin{aligned} & E[\alpha \tilde{u}_A(\theta, \eta) + (1 - \alpha) \tilde{u}_S(\theta, \eta) \mid \eta] \\ = \quad & E[\alpha \{\hat{X}^e(\mu(\theta, \eta)) - \theta \hat{q}^e(\mu(\theta, \eta))\} + (1 - 2\alpha) \tilde{u}_S(\theta, \eta) \mid \eta] \\ \leq \quad & \alpha E[\hat{X}(\mu^*(\theta)) - \theta \hat{q}(\mu^*(\theta)) \mid \eta] + (1 - 2\alpha)\omega. \end{aligned}$$

This upper bound can be attained in $P^S(\alpha : \eta)$ by choosing $\mu(\theta, \eta) = \mu^*(\theta)$,

$$\tilde{u}_A(\theta, \eta) = \hat{X}^e(\mu^*(\theta)) - \theta \hat{q}^e(\mu^*(\theta)) - \omega,$$

$$\tilde{u}_S(\theta, \eta) = \omega$$

and $P(\eta_0 \mid \eta) = 1$ and $P(\eta' \mid \eta) = 0$ for any $\eta' \neq \eta_0$. This allocation satisfies A's participation constraint (v), since

$$\begin{aligned} & \tilde{u}_A(\theta, \eta) = \hat{X}^e(\mu^*(\theta)) - \theta \hat{q}^e(\mu^*(\theta)) - \omega \\ \geq \quad & \hat{X}^e(\theta, \eta_0) - \theta \hat{q}^e(\theta, \eta_0) - u_S^e(\theta, \eta_0) \\ = \quad & u_A^e(\theta, \eta_0). \end{aligned}$$

As the other constraints are obviously satisfied, the claim is established. ■

When A has at least as much bargaining power *ex ante* as S, it is optimal for the coalition to pin S down to her (constant) minmax payoff and provide all residual rents to A. Reports by the coalition are then chosen to maximize A's payoffs, implying that P cannot derive any benefit from appointing S. The one-sided asymmetric information within the coalition implies absence of any frictions in collusion when the informed party A has more bargaining power than S. For P to derive some value from appointing S, she has to exploit some frictions in coalitional bargaining.

Now consider the case where S has higher bargaining weight than A. To explore the value of appointing S, we need to impose some structure on A's type space and S's information. One necessary condition is that S should not be 'too well informed' about A's cost; for instance in the extreme case where S is perfectly informed about θ, there will again be no frictions in coalitional bargaining and appointing S will not yield any value to P. For the rest of this section, we focus on the context (denoted Context C) in which cost is continuously distributed, and two possible signal realizations η_1, η_2 satisfying a Monotone Likelihood Ratio Property (MLRP).

Specifically, Context C is described by the following properties: (a) Θ constitutes an interval $[\underline{\theta}, \bar{\theta}] \subset (0, \infty)$; (b) the prior distribution is represented by a density function $f(.)$ which is continuously differentiable and everywhere positive on Θ; (c) $H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$ is strictly increasing in θ; (d) the likelihood $a(\eta_i|\theta)$ of signal η_i conditional on θ is continuously differentiable and positive-valued on Θ; and (e) $a(\eta_1 \mid \theta)$ (respectively $a(\eta_2 \mid \theta)$) is decreasing (respectively increasing) in θ. Conditional on η_i, the density function and distribution function are respectively $f(\theta \mid \eta_i) \equiv f(\theta)a(\eta \mid \theta)/p(\eta)$ and $F(\theta \mid \eta) \equiv \int_{\underline{\theta}}^{\theta} f(\theta \mid \eta)d\theta$, where $p(\eta) \equiv \int_{\underline{\theta}}^{\bar{\theta}} f(\theta)a(\eta \mid \theta)d\theta$.

Our main result is that in this context, P can derive positive value from appointing S if S has greater bargaining weight than A, for a non-generic set of information structures. Given the previous result, this implies that (generically) P is better off when S has strictly higher bargaining weight than A, compared to when this is not true.

Proposition 2 *Consider Context C and assume $\alpha \in [0, 1/2)$. If there do not exist $(\rho, \nu, \gamma) \in \Re^3$ such that $a(\eta_1 \mid \theta) = \rho + \nu F(\theta)^\gamma$ for all $\theta \in \Theta$, P can attain a strictly higher expected payoff by appointing S, compared to not appointing S (i.e., the organization NS).*

As the proof is relegated to the Appendix, we outline the main steps in the argument here.

First we show that in the problem $P^S(\alpha : \eta)$, S's participation constraint is never binding for $\alpha \in [0, 1/2)$.

Lemma 2 *S's participation condition $E[\tilde{u}_S(\theta, \eta) \mid \eta] \geq \omega$ in $P^S(\alpha : \eta)$ is not binding for any $\alpha \in [0, 1/2)$.*

The reason for this is that if the lemma is false, the solution to the relaxed version of problem $P^S(\alpha : \eta)$ when S's participation constraint is dropped, must violate this constraint, implying that S ends up with an an expected payoff below his minmax payoff ω. The coalition has the option of switching to the 'A-residual-claimant' (ARC) side-contract (which is optimal for the coalition when A has more bargaining power, and has been used in the proof of Proposition 1) in which S receives a constant payoff of ω and A receives the rest of the aggregate coalitional rent. ARC induces ex post efficient reporting strategies, thereby (weakly) expanding the aggregate rent in every state. Given $\alpha < \frac{1}{2}$, the third party would not want to deviate to the ARC side-contract only if A appropriates a disproportionate share of the increase in coalitional rents. This implies that A must be better off in the ARC side-contract. But S is also better off in this side contract. It must therefore Pareto dominate the supposed solution, a contradiction.

We can therefore proceed to study problem $P^S(\alpha : \eta)$ in which S's participation constraint is dropped. P augments the mechanism in the manner described in Definition 3, where the auxiliary message η_0 is identified with the high-cost signal report η_2 (i.e., results in the same outcomes). Hence we can confine attention to two possible signal reports η_1, η_2 for A and S. If both report η_2, P selects the optimal allocation

$$(u_A^{NS}(\theta), u_S^{NS}(\theta), q^{NS}(\theta))$$

in NS satisfying

$$u_A^{NS}(\theta) = \int_\theta^{\bar{\theta}} \bar{q}(y) dy$$

$$u_S^{NS}(\theta) = 0$$

$$q^{NS}(\theta) = \bar{q}(\theta) \equiv \arg\max_q [V(q) - H_k(\theta) q],$$

where $H_k(\theta) \equiv \theta + \frac{2k-1}{k}\frac{F(\theta)}{f(\theta)}$ and k denotes the weight assigned to P's profit.[16] Then P's optimal payoff W^{NS} in NS is $E[V(\bar{q}(\theta)) - H_k(\theta)\bar{q}(\theta)]$.

When both S and A report the low-cost signal η_1, P selects the following variation on the optimal allocation in NS. Let $\beta \equiv \frac{1-2\alpha}{1-\alpha}$, which lies in the interval $(0,1)$. Let $\Lambda(\cdot) : \Theta \to \Re$ be such that (i) $\Lambda(\theta)$ is non-decreasing in θ with $\Lambda(\underline{\theta}) = 0$ and $\Lambda(\bar{\theta}) = 1$, and (ii) the function $z_\beta(\theta)$ defined by

$$z_\beta(\theta) = \theta + \beta\frac{F(\theta \mid \eta_1) - \Lambda(\theta)}{f(\theta \mid \eta_1)}$$

is nondecreasing. P can then select the output schedule $q(\theta, \eta_1) = \bar{q}(z_\beta(\theta))$. Below we shall describe how this Λ function can be chosen in more detail. $\Lambda(\cdot)$ can be thought of as a variation on the cdf $F(. \mid \eta_1)$. $z_\beta(\theta)$ exceeds or falls below θ according as $\Lambda(\theta)$ is smaller or larger than $F(\theta \mid \eta_1)$, implying in turn that $q(\theta, \eta_1)$ is smaller or larger than $q^{NS}(\theta)$. Given such a variation following reports of a low-cost signal, the payoffs are altered as follows:

$$u_A(\theta, \eta_1) = \int_\theta^{\bar{\theta}} \bar{q}(z_\beta(y))dy$$

$$u_S(\theta, \eta_1) = \bar{X}(z_\beta(\theta)) - \theta\bar{q}(z_\beta(\theta)) - \int_\theta^{\bar{\theta}} \bar{q}(z_\beta(y))dy$$

$$q(\theta, \eta_1) = \bar{q}(z_\beta(\theta))$$

where

$$\bar{X}(z) \equiv z\bar{q}(z) + \int_z^{\bar{\theta}} \bar{q}(y)dy.$$

If both S and A report η_2, the same allocation as in NS is selected:

$$(u_A(\theta, \eta_2), u_S(\theta, \eta_2), q(\theta, \eta_2)) = (u_A^{NS}(\theta), 0, q^{NS}(\theta)).$$

When S and A submit different reports $\eta^S \neq \eta^A$, A is offered the same allocation as in the case where the submitted η reports are η^S for both S and A, while S receives a payment equal to what he would have received if their η reports had been η^A for both S and A, minus a large positive number. This will ensure that the side contract will always involve submission of a common report by S and A, besides individual incentive compatibility for A to report θ truthfully.

[16]See Baron and Myerson (1982).

The aim is to construct $\Lambda(\cdot)$ with the properties stated above, such that the resulting allocation is SCP and improves P's payoff in state η_1:

$$\begin{aligned} & E[V(\bar{q}(z_\beta(\theta))) - \frac{2k-1}{k}\bar{X}(z_\beta(\theta)) - \frac{1-k}{k}\theta\bar{q}(z_\beta(\theta)) \mid \eta_1] \\ > \; & E[V(q^{NS}(\theta)) - \frac{2k-1}{k}\bar{X}(\theta) - \frac{1-k}{k}\theta q^{NS}(\theta) \mid \eta_1]. \end{aligned} \quad (1)$$

Since the allocation is unchanged in state η_2, P will achieve a higher payoff than in NS.

The proof shows that such a variation is indeed SCP provided the following two conditions are satisfied:

(a)$E[u_A(\theta,\eta_1) - u_A(\theta,\eta_2) \mid \eta_2] \geq 0$

(b) $\int_{\underline{\theta}}^{\bar{\theta}}[u_A(\theta,\eta_2) - u_A(\theta,\eta_1)]d\Lambda(\theta) \geq 0.$

Intuitively, these two conditions (in combination with the choice of allocation for A corresponding to conflicting η reports as specified above) ensure that a threat by S to report a different signal from the one actually observed in GC if collusion breaks down, does not lower A's expected payoff. S is unable to coerce A to accept a lower payoff in the collusive agreement, compared to a null side contract, thereby ensuring that the allocation is SCP for α.

Conditions (a) and (b) can be rewritten as follows:

$$E[\{\bar{q}(z_\beta(\theta)) - \bar{q}(\theta)\}\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)} \mid \eta_2] \geq 0 \quad (2)$$

and

$$E[(z_\beta(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z_\beta(\theta)) - \bar{q}(\theta)) \mid \eta_1] \geq 0 \quad (3)$$

where $h_\beta(\theta \mid \eta) = \theta + \beta\frac{F(\theta|\eta)}{f(\theta|\eta)}$.[17]

Consider a small variation of the $z_\beta(\theta)$ around θ. The corresponding pointwise variations in the left-hand-sides of (1)-(3) are as follows[18]

$$[V'(\bar{q}(z))\bar{q}'(z) - \frac{2k-1}{k}\bar{X}'(z) - \frac{1-k}{k}\theta\bar{q}'(z)]_{z=\theta}f(\theta \mid \eta_1) = \bar{q}'(\theta)\frac{2k-1}{k}\frac{F(\theta)}{f(\theta)}f(\theta \mid \eta_1), \quad (4)$$

$$[\bar{q}'(z)\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}]_{z=\theta}f(\theta \mid \eta_2) = \bar{q}'(\theta)F(\theta \mid \eta_2), \quad (5)$$

[17]We use $E[\int_{\theta}^{\bar{\theta}}\bar{q}(y)dy \mid \eta] = E[\frac{F(\theta|\eta)}{f(\theta|\eta)}\bar{q}(\theta) \mid \eta]$ to derive these equations.

[18]We use $V'(\bar{q}(z)) = H_k(z)$ to obtain (4).

and

$$[(\bar{q}(z) - \bar{q}(\theta)) + (z - h_\beta(\theta \mid \eta_1))\bar{q}^{'}(z)]_{z=\theta} f(\theta \mid \eta_1) = -\beta \bar{q}^{'}(\theta) F(\theta \mid \eta_1). \tag{6}$$

A necessary and sufficient condition for a variation which locally preserves the value of the left-hand-sides of (2) and (3), while increasing the value of the left-hand-side of (1), is that $\frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1)$ does not lie in the space spanned linearly by $F(\theta \mid \eta_2)$ and $-F(\theta \mid \eta_1)$, which turns out to be equivalent to the generic property stated in Proposition 2.

4 Conclusion

In summary, prospects of strong collusion between an informed agent and less-well-informed supervisor can rationalize asymmetric authority granted to the latter in designing contracts for the agent. If instead the agent has the upper hand, or at least the same welfare weight as the supervisor at the contract design stage, strong collusion allows the agent to push the supervisor down to her minmax payoff and extract all the residual rents. Collusion is then subject to no frictions, as reports are chosen to maximize *ex post* payoffs of the agent, completely undermining the role of the supervisor. Hence for the Principal to derive some benefit from appointing a supervisor, it is essential that the supervisor has the upper hand at the contract design stage. This ensures that collusion is subject to frictions resulting from asymmetric information, which induce trade-offs between the supervisor's rent and the agent's incentives. By contrast, when collusion is weak or absent, the allocation of control authority between supervisor and agent is irrelevant.

Some open questions remain. In the context of continuously distributed cost of the agent, we showed that the Principal can benefit from the presence of the supervisor if the latter's signal had only two possible realizations. We do not yet know if this result extends when the signal can take a finite number of realizations. We also do not know if the Principal's payoff is monotone with respect to the allocation of welfare weights over the range where the supervisor has a higher weight. However such a monotonicity result does obtain in a variant of the model with three possible cost types and a partition information structure akin to the model of Celik (2009). It would be interesting to know if this monotonicity property obtains more generally.

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Appendix: Proofs

Proof of Lemma 1

Proof of Necessity

Step 1: *Some definitions:*

Consider any WPBE(sc) allocation resulting from some grand contract GC. For this GC, define $\underline{w}_S(GC)$ as the minmax value of the S's payoff:

$$\underline{w}_S(GC) \equiv \min_{\mu_A \in \Delta(M_A)} \max_{\mu_S \in \Delta(M_S)} \bar{X}_S(\mu_A, \mu_S).$$

Since $\Delta(M_A), \Delta(M_S)$ (endowed with the weak convergence topology) are compact, we can apply the minimax theorem (Nikaido (1954)) to infer that there exists $(\underline{\mu}_A, \bar{\mu}_S)$ which satisfies

$$\begin{aligned} \underline{w}_S(GC) = \bar{X}_S(\underline{\mu}_A, \bar{\mu}_S) &= \min_{\mu_A \in \Delta(M_A)} \max_{\mu_S \in \Delta(M_S)} \bar{X}_S(\mu_A, \mu_S) \\ &= \max_{\mu_S \in \Delta(M_S)} \min_{\mu_A \in \Delta(M_A)} \bar{X}_S(\mu_A, \mu_S). \end{aligned}$$

where $\underline{\mu}_A$ is A's minmax strategy, and $\bar{\mu}_S$ is S's maxmin strategy. Since S always has the option to exit from the grand contract, $\underline{w}_S(GC) \geq 0$ for any GC.

Given grand contract GC, a reporting strategy for S in this GC: $\mu_S(\eta) \in \Delta(M_S)$ and a type of A: $\theta \in \Theta$, define

$$\hat{u}_A(\theta, \mu_S(\eta), GC) \equiv \max_{\mu_A \in \Delta(M_A)} \bar{X}_A(\mu_A, \mu_S(\eta)) - \theta \bar{q}(\mu_A, \mu_S(\eta)),$$

which is interpreted as the A's maximum payoff in the event that A is of type θ and exits from the side-contract (whence S chooses $\mu_S(\eta)$).

Step 2 *Characterization of allocations achievable for a given grand contract GC:*

Let (u_A, u_S, q) denote the allocation achieved as a WPBE(sc) outcome in GC, where the third party selects a side contract SC.

In this step, we show that for any η and for $\mu(\theta, \eta)$ which satisfies $q(\theta, \eta) = \tilde{q}(\mu(\theta, \eta))$ for all $\theta \in \Theta(\eta)$,

$$(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), \tilde{\mu}(\theta, \eta)) = (u_A(\theta, \eta), u_S(\theta, \eta), \mu(\theta, \eta)),$$

associated with the selection of some $\tilde{\mu}_S(\eta) = \mu_S(\eta)$, solves the following problem $P^S(\eta : \alpha, GC)$:

$$\max E[\alpha \tilde{u}_A(\theta, \eta) + (1 - \alpha)\tilde{u}_S(\theta, \eta) \mid \eta]$$

subject to the constraint that for some $\tilde{\mu}_S(\eta) \in \Delta(M_S)$ and for all $\theta \in \Theta(\eta)$:

(i) $\tilde{\mu}(\theta, \eta) \in \Delta(M_A \times M_S), \tilde{u}_A(\theta, \eta) \in \Re, \tilde{u}_S(\theta, \eta) \in \Re$

(ii) $\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\tilde{q}(\tilde{\mu}(\theta', \eta))$ for any $\theta' \in \Theta(\eta)$

(iii) $\tilde{u}_A(\theta, \eta) \geq \hat{u}_A(\theta, \tilde{\mu}_S(\eta), GC)$

(iv) $E[\tilde{u}_S(. \mid \eta)] \geq \underline{w}_S(GC)$ and

(v) $\tilde{X}_A(\tilde{\mu}(\theta, \eta)) + \tilde{X}_S(\tilde{\mu}(\theta, \eta)) - \theta\tilde{q}(\tilde{\mu}(\theta, \eta)) = \tilde{u}_A(\theta, \eta) + \tilde{u}_S(\theta, \eta)$.

Note that this problem $P^S(\eta : \alpha, GC)$ includes S's threat $\tilde{\mu}_S(\eta)$ in the event of A's non-participation as a choice variable. So the set of control variables can be written as $(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), \tilde{\mu}(\theta, \eta)), \tilde{\mu}_S(\eta))$.

Note also that the problem $P^S(\eta : \alpha, GC)$ refers to the given GC, and reporting strategies of the players are confined to mixed strategies available in GC. In later steps, the mechanism will be augmented so that the scope of collusion will be widened, as players will then be able to select mixed strategies in augmented message spaces.

Proof of Step 2:

Since (u_A, u_S, q) is an achievable allocation, it is straightforward to check that it is feasible in the above problem. If $(u_A(\theta, \eta), u_S(\theta, \eta), \mu(\theta, \eta), \mu_S(\eta))$ does not solve problem $P^S(\eta : \alpha, GC)$ for some η, we shall now show that there exists another side-contract and a continuation equilibrium in which the third party can achieve a higher payoff, which will contradict the hypothesis that the allocation resulted from a WPBE(sc) of GC. Suppose that for some η, the solution of $P(\eta : \alpha, GC)$ is instead some $(\tilde{u}^*_A(\theta, \eta), \tilde{u}^*_S(\theta, \eta), \tilde{\mu}^*(\theta, \eta), \tilde{\mu}^*_S(\eta)) \neq (u_A(\theta, \eta), u_S(\theta, \eta), \mu(\theta, \eta), \mu_S(\eta))$.

Construct a side-contract SC' as follows. If both S and A accept it, the third party requests a report from A of $(\theta_A, \eta_A) \in K$, and report from S of $\eta_S \in \Pi$. The report to P is subsequently selected according to $\tilde{\mu}^*(\theta_A, \eta_S)$, while side-transfers are selected as follows.

$$t_A(\theta_A, \eta_A, \eta_S) = \tilde{u}^*_A(\theta_A, \eta_S) - [\bar{X}_A(\tilde{\mu}^*(\theta_A, \eta_S)) - \theta_A\tilde{q}(\tilde{\mu}^*(\theta_A, \eta_S))] - L(\eta_A, \eta_S)$$

and

$$t_S(\theta_A, \eta_A, \eta_S) = \tilde{u}^*_S(\theta_A, \eta_A) - \bar{X}_S(\tilde{\mu}^*(\theta_A, \eta_S))$$

where $L(\eta_A, \eta_S)$ is zero for $\eta_A = \eta_S$ and a large positive number for $\eta_A \neq \eta_S$. If A were to accept and S were to reject SC', A would threaten to play $\underline{\mu}_A$. Conversely, if S accepts and reports η_S while A rejects SC', S threatens to play $\tilde{\mu}^*_S(\eta_S)$. It is easy to check that there exists a continuation equilibrium where nobody rejects SC' on the equilibrium path, and both A and S report truthfully to the third party, resulting in the allocation $(\tilde{u}^*_A(\theta, \eta), \tilde{u}^*_S(\theta, \eta))$. The third party attains a higher payoff, contradicting the hypothesis that we started with a WPBE(sc), completing the proof of Step 2. ■

The statement of Step 2 provides one characterization of allocations achievable for a given grand contract GC. Our aim is to find a more general characterization which does not depend on GC. This is the purpose of the following two steps.

Step 3: *Construction of augmented allocation:*

We continue with (u_A, u_S, q), an achievable allocation in GC in S-Collusion with α. Construct an incentive compatible augmentation (u^e_A, u^e_S, q^e) of (u_A, u_S, q) to the domain $\bar{K}$ satisfying the following conditions:

(i) $u^e_A(\theta, \eta) \geq \hat{u}_A(\theta, \mu_S(\eta), GC)$ on Θ, for any $\eta \in \Pi$.

(ii) $u^e_S(\theta, \eta) = -L$ on $\Theta \backslash \Theta(\eta)$ where $L > 0$ is sufficiently large (as explained later), for any $\eta \in \Pi$.

(iii) $(u^e_A(\theta, \eta_0), u^e_S(\theta, \eta_0), q^e(\theta, \eta_0)) = (\hat{u}_A(\theta, \bar{\mu}_S, GC), \omega, \bar{q}(\mu_A(\theta, \bar{\mu}_S), \bar{\mu}_S))$ where $\mu_A(\theta, \bar{\mu}_S)$ maximizes

$$\bar{X}_A(\mu_A, \bar{\mu}_S) - \theta \bar{q}(\mu_A, \bar{\mu}_S)$$

subject to $\mu_A \in \Delta(M_A)$, and $\omega \equiv \underline{w}_S(GC)$.

We sketch the argument for existence of an incentive compatible augmentation satisfying property (i), for the case where Θ is a closed interval $[\underline{\theta}, \bar{\theta}]$ and $\Theta(\eta)$ is the union of a set of closed intervals of Θ. The method can be extended in an obvious manner to other type spaces. Note that $\Theta(\eta)$ may not include $\underline{\theta}$ or $\bar{\theta}$. Define $\bar{\Theta}(\eta) \equiv \Theta(\eta) \cup \{\underline{\theta}, \bar{\theta}\}$. To simplify notation, we fix η and suppress it as an argument in the functions below. We are given functions $(u_A(\theta), q(\theta))$ and $\hat{u}_A(\theta) \equiv \hat{u}_A(\theta, \mu_S(\eta), GC)$ which satisfy

- $(u_A(\theta), q(\theta))$ is defined on $\Theta(\eta)$ and satisfies (IC)

$$u_A(\theta) \geq u_A(\theta^{'}) + (\theta^{'} - \theta)q(\theta^{'})$$

  for any $\theta, \theta^{'} \in \Theta(\eta)$.

- $\hat{u}_A(\theta)$ is convex and non-increasing on Θ by definition.

- $u_A(\theta) \geq \hat{u}_A(\theta)$ for $\theta \in \Theta(\eta)$.

Our purpose is to construct $(u_A^e(\theta), q^e(\theta))$ defined on Θ which satisfies

(a) $(u_A^e(\theta), q^e(\theta)) = (u_A(\theta), q(\theta))$ on $\Theta(\eta)$

(b) $u_A^e(\theta) \geq u_A^e(\theta^{'}) + (\theta^{'} - \theta)q^e(\theta^{'})$ for any $\theta, \theta^{'} \in \Theta$

(c) $u_A^e(\theta) \geq \hat{u}_A(\theta)$ for any $\theta \in \Theta$.

For any $\theta \in \Theta$, define $\underline{\theta}(\theta)$ and $\bar{\theta}(\theta)$ as follows:

$$\underline{\theta}(\theta) \equiv \max\{\theta^{'} \in \Theta(\eta) \cup \{\underline{\theta}\} \mid \theta^{'} \leq \theta\}$$

$$\bar{\theta}(\theta) \equiv \min\{\theta^{'} \in \Theta(\eta) \cup \{\bar{\theta}\} \mid \theta^{'} \geq \theta\}$$

Evidently if $\theta \in \Theta(\eta)$, $\underline{\theta}(\theta) = \bar{\theta}(\theta) = \theta$; otherwise $\underline{\theta}(\theta) < \theta < \bar{\theta}(\theta)$.

(IC) implies that $u_A(\theta)$ is non-increasing and convex on $\Theta(\eta)$ (i.e., it is convex on each interval contained in the latter).[19]

First in the case that $\underline{\theta} \notin \Theta(\eta)$ or $\bar{\theta} \notin \Theta(\eta)$, we augment $u_A(\theta)$ on $\bar{\Theta}(\eta)$ as follows. If $\underline{\theta} \notin \Theta(\eta)$,

$$u_A(\underline{\theta}) \equiv \max\{\hat{u}(\underline{\theta}(\eta)), u_A(\underline{\theta}(\eta)) + (\underline{\theta}(\eta) - \underline{\theta})q(\underline{\theta}(\eta))\}$$

while if $\bar{\theta} \notin \Theta(\eta)$,

$$u_A(\bar{\theta}) \equiv u_A(\bar{\theta}(\eta)).$$

This augmented function $u_A(\theta)$ on $\bar{\Theta}(\eta)$ is convex and non-increasing on $\bar{\Theta}(\eta)$ and $u_A(\theta) \geq \hat{u}_A(\theta)$ on $\bar{\Theta}(\eta)$.

Next select $u_A^e(\theta)$ as the maximum convex function defined on Θ such that $u_A(\theta) \geq u_A^e(\theta)$ for any $\theta \in \bar{\Theta}(\eta)$. This function has the following properties.

[19]The proof can be extended to a general type space including discrete type spaces by defining the convexity condition to be the following: $[u_A(\theta') - u_A(\theta)]/[\theta' - \theta] \leq [u_A(\theta'') - u_A(\theta')]/[\theta'' - \theta']$ for any $\theta < \theta' < \theta''$ in $\Theta(\eta)$, i.e., that the 'slope' is non-decreasing.

- $u_A(\theta) = u_A^e(\theta)$ on $\Theta(\eta)$.
- For any $\theta \notin \Theta(\eta)$, $u_A^e(\theta')$ is linear on $[\underline{\theta}(\theta), \bar{\theta}(\theta)]$ with constant slope $-\frac{u_A(\underline{\theta}(\theta))-u_A(\bar{\theta}(\theta))}{\bar{\theta}(\theta)-\underline{\theta}(\theta)}$.
- $u_A^e(\theta) \geq \hat{u}_A(\theta)$ on Θ, which implies property (c)). [Otherwise, consider $\max\{u_A^e(\theta), \hat{u}_A(\theta)\}$. This is convex on Θ and $u_A(\theta) \geq \max\{u_A^e(\theta), \hat{u}_A(\theta)\}$ on $\Theta(\eta)$, contradicting the definition of $u_A^e(\theta)$.]

The augmented function $q^e(\theta)$ is constructed as follows:

$$q^e(\theta) = q(\theta)$$

for $\theta \in \Theta(\eta)$ and

$$q^e(\theta) = \frac{u_A(\underline{\theta}(\theta)) - u_A(\bar{\theta}(\theta))}{\bar{\theta}(\theta) - \underline{\theta}(\theta)}.$$

for $\theta \notin \Theta(\eta)$. Then $q^e(\theta)$ has the following properties

(P1) $q^e(\theta)$ satisfies

$$u_A^e(\theta) = u_A^e(\bar{\theta}(\theta)) + (\theta - \bar{\theta}(\theta))q^e(\theta) = u_A^e(\underline{\theta}(\theta)) + (\underline{\theta}(\theta)) - \theta)q^e(\theta)$$

for any $\theta \in \Theta$.

(P2) $q^e(\theta)$ is non-increasing on Θ for the following reasons:

- $q^e(\theta)$ is non-increasing on $\Theta(\eta)$ from (IC)
- $q^e(\theta')$ is constant on $[\underline{\theta}(\theta), \bar{\theta}(\theta)]$
- (IC) implies that for any $\theta \notin \Theta(\eta)$,

$$q(\underline{\theta}(\theta)) \geq \frac{u_A(\underline{\theta}(\theta)) - u_A(\bar{\theta}(\theta))}{\bar{\theta}(\theta) - \underline{\theta}(\theta)} \geq q(\bar{\theta}(\theta))$$

or

$$q^e(\underline{\theta}(\theta)) \geq q^e(\theta) \geq q^e(\bar{\theta}(\theta)).$$

We now show that the augmented functions satisfies the IC property (b). For $\theta, \theta' \in \Theta$ $(\theta < \theta')$

- If $\underline{\theta}(\theta) \neq \underline{\theta}(\theta')$, $\theta \leq \bar{\theta}(\theta) < \underline{\theta}(\theta') \leq \theta'$ and

$$q^e(\theta) \geq q^e(\bar{\theta}(\theta)) \geq q^e(\underline{\theta}(\theta')) \geq q^e(\theta')$$

by (P2).

$$
\begin{aligned}
& u_A^e(\theta) = u_A^e(\bar{\theta}(\theta)) + (\bar{\theta}(\theta) - \theta)q^e(\theta) \\
\geq\ & u_A^e(\underline{\theta}(\theta^{'}) + (\underline{\theta}(\theta^{'}) - \bar{\theta}(\theta))q^e(\underline{\theta}(\theta^{'})) + (\bar{\theta}(\theta) - \theta)q^e(\theta) \\
=\ & u_A^e(\theta^{'}) + (\theta^{'} - \underline{\theta}(\theta^{'})))q^e(\theta^{'}) \\
+\ & (\underline{\theta}(\theta^{'}) - \bar{\theta}(\theta))q^e(\underline{\theta}(\theta^{'})) + (\bar{\theta}(\theta) - \theta)q^e(\theta) \\
\geq\ & u_A^e(\theta^{'}) + (\theta^{'} - \theta)q^e(\theta^{'})
\end{aligned}
$$

The first and third equalities use (P1). The second inequality uses (IC). The fourth inequality uses $\theta \leq \bar{\theta}(\theta) < \underline{\theta}(\theta^{'}) \leq \theta^{'}$ and

$$q^e(\theta) \geq q^e(\bar{\theta}(\theta)) \geq q^e(\underline{\theta}(\theta^{'})) \geq q^e(\theta^{'}).$$

Similarly.

$$
\begin{aligned}
& u_A^e(\theta^{'}) = u_A^e(\underline{\theta}(\theta^{'})) + (\underline{\theta}(\theta^{'})) - \theta^{'})q^e(\theta^{'}) \\
\geq\ & u_A^e(\bar{\theta}(\theta)) + (\bar{\theta}(\theta) - \underline{\theta}(\theta^{'}))q^e(\bar{\theta}(\theta)) + (\underline{\theta}(\theta^{'})) - \theta^{'})q^e(\theta^{'}) \\
=\ & u_A^e(\theta) + (\theta - \bar{\theta}(\theta))q^e(\theta) + (\bar{\theta}(\theta) - \underline{\theta}(\theta^{'}))q^e(\bar{\theta}(\theta)) + (\underline{\theta}(\theta^{'})) - \theta^{'})q^e(\theta^{'}) \\
\geq\ & u_A^e(\theta) + (\theta - \theta^{'})q^e(\theta)
\end{aligned}
$$

- If $\underline{\theta}(\theta) = \underline{\theta}(\theta^{'})$, θ and $\theta^{'}$ are not in $\Theta(\eta)$. By $q^e(\theta) = q^e(\theta^{'})$ and the linearity of u_A^e,

$$u_A^e(\theta) = u_A^e(\theta^{'}) + (\theta^{'} - \theta)q^e(\theta^{'}) = u_A^e(\theta^{'}) + (\theta^{'} - \theta)q^e(\theta).$$

Hence $(u_A^e(\theta), q^e(\theta))$ is IC on Θ.

Step 4

Now consider the problem $P^S(\alpha : \eta)$ defined by the augmented allocation (u_A^e, u_S^e, q^e) constructed in Step 3. Note that this problem differs from the one considered in Step 2 ($P^S(\eta : \alpha, GC)$), as it no longer refers to the original GC, and the coalition selects reports from the augmented message space $\bar{K}$ rather than $M_A \times M_S$.

We show in this step that

$$(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), P(\cdot \mid \eta)) = (I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$$

solves problem $P^S(\alpha : \eta)$.

It is straightforward to check that $(I(\theta,\eta), u_A(\theta,\eta), u_S(\theta,\eta), I(\eta))$ satisfies all constraints of $P^S(\alpha : \eta)$, and generates a payoff for the third party of

$$E[\alpha u_A(\theta,\eta) + (1-\alpha)u_S(\theta,\eta) \mid \eta].$$

Suppose that there exists some alternative choice of controls

$$(\mu^*(\theta,\eta), u_A^*(\theta,\eta), u_S^*(\theta,\eta), P^*(\cdot \mid \eta))$$

which is feasible in $P^S(\alpha : \eta)$, such that

$$E[\alpha u_A^*(\theta,\eta) + (1-\alpha)u_S^*(\theta,\eta) \mid \eta] > E[\alpha u_A(\theta,\eta) + (1-\alpha)u_S(\theta,\eta) \mid \eta].$$

We show that in such a case there would exist $\tilde{\mu}(\theta,\eta) : K \to \Delta(M_A \times M_S), \tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta), \tilde{\mu}_S(\eta)$ which would be feasible in $P^S(\eta : \alpha, GC)$ and generate a higher value in that problem compared to $(u_A(\theta,\eta), u_S(\theta,\eta), \mu(\theta,\eta))$, thereby contradicting the result established at Step 2.

Note to start with that with a sufficiently large L (selected in Step 3), we can confine attention to policies in which $\mu^*(\theta,\eta)$ does not assign positive probability to reports with $\eta \neq \eta_0$ and $\theta \in \Theta \backslash \Theta(\eta)$. Hence $\mu^*(\theta,\eta)$ divides all its weight between reports either in K or satisfying $\eta = \eta_0$. The former event corresponds to an outcome of GC that results when S and A's reports are chosen from M_S and M_A respectively. And the latter event corresponds (by (iii) in Step 3) to an outcome of GC resulting when S reports $\bar{\mu}_S \in \Delta(M_S)$ and A reports according to $\mu(\theta, \bar{\mu}_S) \in \Delta(M_A)$. In this case,

$$\bar{q}(\mu_A(\theta,\bar{\mu}_S), \bar{\mu}_S) = q(\theta,\eta_0)$$

while

$$\begin{aligned} \hat{X}(\theta,\eta_0) &= \omega + \bar{X}_A(\mu_A(\theta,\bar{\mu}_S), \bar{\mu}_S) \\ &\leq \bar{X}_S(\mu_A(\theta,\bar{\mu}_S), \bar{\mu}_S) + \bar{X}_A(\mu_A(\theta,\bar{\mu}_S), \bar{\mu}_S) \end{aligned}$$

since ω is S's minmax payoff in GC. Hence the outcome of $\mu^*(\theta,\eta)$ in $P^S(\alpha : \eta)$ can be attained by the coalition as an outcome of GC resulting from some reporting strategy $\tilde{\mu}(\theta,\eta) \in \Delta(M_A \times M_S)$ that satisfies

$$\bar{X}_A(\tilde{\mu}(\theta,\eta)) + \bar{X}_S(\tilde{\mu}(\theta,\eta)) \geq \hat{X}(\mu^*(\theta,\eta))$$

and

$$\bar{q}(\tilde{\mu}(\theta,\eta)) = \hat{q}(\mu^*(\theta,\eta)).$$

Let $\mu_S(\eta)$ denote the optimal threat chosen by S in the event that A does not participate in the side-contract, in the solution to problem $P^S(\eta : \alpha, GC)$. Define $\tilde{\mu}_S(\eta) \in \Delta(M_S)$ as the composite of the measures $\mu_S(\eta^{'})$ and $P^*(\eta^{'} \mid \eta)$. Then by (i) in Step 3 and the definition of $\hat{u}_A(\theta, \mu_S, GC)$,

$$\Sigma_{\eta^{'}\in\bar{\Pi}} P^*(\eta^{'} \mid \eta) u_A^e(\theta,\eta^{'}) \geq \Sigma_{\eta^{'}\in\bar{\Pi}} P^*(\eta^{'} \mid \eta)\hat{u}_A(\theta,\mu_S(\eta^{'}),GC) \geq \hat{u}_A(\theta,\tilde{\mu}_S(\eta),GC).$$

Since $u_A^*(\theta,\eta) \geq \Sigma_{\eta^{'}\in\bar{\Pi}} P^*(\eta^{'} \mid \eta) u_A^e(\theta,\eta^{'})$, it follows that

$$u_A^*(\theta,\eta) \geq \hat{u}_A(\theta,\tilde{\mu}_S(\eta),GC).$$

Defining

$$\tilde{u}_A(\theta,\eta) \equiv u_A^*(\theta,\eta)$$

and

$$\tilde{u}_S(\theta,\eta) \equiv \bar{X}_A(\tilde{\mu}(\theta,\eta)) + \bar{X}_S(\tilde{\mu}(\theta,\eta)) - \theta q(\tilde{\mu}(\theta,\eta)) - u_A^*(\theta,\eta),$$

we infer that $(\tilde{u}_A(\theta,\eta), \tilde{u}_S(\theta,\eta), \tilde{\mu}(\theta,\eta), \tilde{\mu}_S(\eta))$ is feasible in the problem $P^S(\eta : \alpha, GC)$, and $\tilde{u}_S(\theta,\eta) \geq u_S^*(\theta,\eta)$. Hence it generates a higher payoff for the third party than $E[\alpha u_A(\theta,\eta) + (1-\alpha)u_S(\theta,\eta) \mid \eta]$, and we obtain a contradiction to the result of Step 2. So

$$(I(\theta,\eta), u_A(\theta,\eta), u_S(\theta,\eta), I(\eta))$$

must be a solution of $P^S(\alpha : \eta)$, establishing the necessity of the SCP property.

Proof of Sufficiency

Let (u_A^e, u_S^e, q^e) be the incentive compatible augmentation of (u_A, u_S, q) for which the latter satisfies the SCP property. P can construct a grand contract GC as follows:

$$(X_A(m_A,m_S), X_S(m_A,m_S), q(m_A,m_S) : M_A, M_S)$$

where

$$M_A = K \cup \{e_A\}$$

$$M_S = \bar{\Pi} \cup \{e_S\}$$

- for any $(\theta,\eta) \in K$ and $\eta^{'} \in \Pi$, choose $(X_A((\theta,\eta),\eta^{'}), X_S((\theta,\eta),\eta^{'}), q((\theta,\eta),\eta^{'})) = (u_A^e(\theta,\eta^{'}) + \theta q^e(\theta,\eta^{'}), u_S^e(\theta,\eta) - L(\eta,\eta^{'}), q^e(\theta,\eta^{'}))$ where $L(\eta,\eta^{'}) = 0$ for $\eta = \eta^{'}$ and $L > 0$ (and sufficiently large) for $\eta \neq \eta^{'}$

- $(X_A((\theta,\eta),e_S), X_S((\theta,\eta),e_S), q((\theta,\eta),e_S)) = (u_A^e(\theta,\eta_0) + \theta q^e(\theta,\eta_0), 0, q^e(\theta,\eta_0))$.

- $(X_A((\theta,\eta),\eta_0), X_S((\theta,\eta),\eta_0), q((\theta,\eta),\eta_0)) = (u_A^e(\theta,\eta_0) + \theta q^e(\theta,\eta_0), \omega, q^e(\theta,\eta_0))$.

- $(X_A(e_A,m_S), X_S(e_A,m_S), q(e_A,m_S)) = (0,0,0)$ for any $m_S \neq \eta_0$

- $(X_A(e_A,\eta_0), X_S(e_A,\eta_0), q(e_A,\eta_0)) = (0,\omega,0)$

It is easy to check that $(\mu_A,\mu_S) = ((\theta,\eta),\eta)$ is a non-cooperative equilibrium of GC, and S's minmax payoff in GC is ω. The SCP property of (u_A,u_S,q) implies there is no room for the third party to improve its payoff by offering a deviating side-contract, so (u_A,u_S,q) is realized as the outcome of a WPBE(sc) under GC. ■

Proof of Proposition 2

We start with the proof of Lemma 2.

Proof of Lemma 2

Suppose that for some η, $(I(\theta,\eta), u_A(\theta,\eta), u_S(\theta,\eta), I(\eta))$ does not solve the relaxed version of $P^S(\alpha:\eta)$ where the constraint $E[\tilde{u}_S(\theta,\eta) \mid \eta] \geq \omega$ is dropped. It implies $E[\tilde{u}_S^r(\theta,\eta) \mid \eta] < \omega$ in the optimal solution of the relaxed problem represented by

$$(\mu^r(\theta,\eta), \tilde{u}_A^r(\theta,\eta), \tilde{u}_S^r(\theta,\eta), P^r(\cdot \mid \eta)).$$

As shown in the proof of Proposition 1, side contract $\tilde{SC}$ defined as follows is feasible in $P^S(\alpha:\eta)$, hence also in the relaxed problem:

- $\tilde{\mu}(\theta,\eta) = \mu^*(\theta)$ which maximizes $\hat{X}^e(\mu) - \theta\hat{q}^e(\mu)$ subject to $\mu \in \Delta(\bar{K} \cup \{e\})$

- $P(\eta_0 \mid \eta) = 1$ and $P(\eta^{'} \mid \eta) = 0$ for any $\eta^{'} \neq \eta_0$

- $\tilde{u}_A(\theta,\eta) = \hat{X}^e(\mu^*(\theta)) - \theta\hat{q}^e(\mu^*(\theta)) - \omega$ (denoted by $u_A^+(\theta,\eta)$ in later part)

- $\tilde{u}_S(\theta,\eta) = \omega$

Hence

$$\begin{aligned} & E[\alpha\tilde{u}_A^r(\theta,\eta) + (1-\alpha)\tilde{u}_S^r(\theta,\eta) \mid \eta] \\ = \;& E[(1-\alpha)\{\hat{X}^e(\mu^r(\theta,\eta)) - \theta\hat{q}^e(\mu^r(\theta,\eta))\} - (1-2\alpha)\tilde{u}_A^r(\theta,\eta) \mid \eta] \\ \geq \;& E[(1-\alpha)\{\hat{X}^e(\mu^*(\theta)) - \theta\hat{q}^e(\mu^*(\theta))\} - (1-2\alpha)u_A^+(\theta,\eta) \mid \eta]. \end{aligned}$$

But since

$$\begin{aligned} & E[\hat{X}^e(\tilde{\mu}^r(\theta,\eta)) - \theta\hat{q}^e(\tilde{\mu}^r(\theta,\eta)) \mid \eta] \\ \leq \;& E[\hat{X}^e(\mu^*(\theta)) - \theta\hat{q}^e(\mu^*(\theta)) \mid \eta] \end{aligned}$$

by the definition of $\mu^*(\theta)$, $\alpha < \frac{1}{2}$ implies that

$$E[u_A^+(\theta,\eta) \mid \eta] \geq E[\tilde{u}_A^r(\theta,\eta) \mid \eta].$$

This implies that the side contract $\tilde{SC}$ creates a Pareto improvement over the solution to the relaxed problem, yielding a strictly higher value of the third party's expected payoff, a contradiction. ■

The next step in the proof of Proposition 2 is to consider the specific mechanism described in the text; we establish this allocation is SCP provided conditions (a) and (b) are satisfied.

Owing to the previous lemma, we can drop S's participation constraint (iv) from problem $P^S(\alpha:\eta)$. So consider the relaxed problem denoted by $\bar{P}^S(\alpha:\eta)$, for this allocation defined on $\Theta \times \{\eta_1,\eta_2\}$, which selects $(\mu(\theta,\eta), \tilde{u}_A(\theta,\eta), p(\eta))$ to maximize

$$E[\hat{X}(\mu(\theta,\eta)) - \theta\hat{q}(\mu(\theta,\eta)) - \beta\tilde{u}_A(\theta,\eta) \mid \eta]$$

subject to $\mu(\theta,\eta) \in \Delta(\Theta \times \{\eta_1,\eta_2\} \cup \{e\})$ and $p(\eta) \in [0,1]$,

$$\tilde{u}_A(\theta,\eta) \geq p(\eta)u_A(\theta,\eta) + (1-p(\eta))u_A(\theta,\eta^{'})$$

and

$$\tilde{u}_A(\theta,\eta) \geq \tilde{u}_A(\theta^{'},\eta) + (\theta^{'}-\theta)\hat{q}(\mu(\theta^{'},\eta))$$

for any $\theta, \theta^{'} \in \Theta$.

Specifically, we aim to show that

$$(\mu(\theta,\eta), \tilde{u}_A(\theta,\eta), p(\eta)) = ((\theta,\eta), u_A(\theta,\eta), 1)$$

solves $\bar{P}^S(\alpha:\eta)$, if

(a)$E[u_A(\theta,\eta_1) - u_A(\theta,\eta_2) \mid \eta_2] \geq 0$

(b) $\int_{\underline{\theta}}^{\bar{\theta}}[u_A(\theta,\eta_2) - u_A(\theta,\eta_1)]d\Lambda(\theta) \geq 0.$

Upon choosing $\Lambda(.,\eta_1) \equiv \Lambda(.)$ and $\Lambda(.,\eta_2) \equiv F(.|\eta_2)$, we can unify conditions (a) and (b) into the following single condition

$$\int_{\underline{\theta}}^{\bar{\theta}}[u_A(\theta,\eta^{'}) - u_A(\theta,\eta)]d\Lambda(\theta,\eta) \geq 0$$

when $\eta,\eta^{'} \in \{\eta_1,\eta_2\}$ and $\eta \neq \eta^{'}$.

Since $\Lambda(\theta,\eta)$ is non-decreasing in θ, this condition implies that

$$\begin{aligned} 0 &\leq \int_{\underline{\theta}}^{\bar{\theta}}[\tilde{u}_A(\theta,\eta) - (1-p(\eta))u_A(\theta,\eta^{'}) - p(\eta)u_A(\theta,\eta)]d\Lambda(\theta,\eta) \\ &\leq \int_{\underline{\theta}}^{\bar{\theta}}[\tilde{u}_A(\theta,\eta) - u_A(\theta,\eta)]d\Lambda(\theta,\eta) \end{aligned}$$

for any $(\tilde{u}_A(\theta,\eta), p(\eta))$ satisfying constraints of $\bar{P}^S(\alpha:\eta)$. This result can be used to obtain an upper bound of the objective function in $\bar{P}^S(\alpha:\eta)$. First note that

$$\begin{aligned} & E[\hat{X}(\mu(\theta,\eta)) - \theta\hat{q}(\mu(\theta,\eta)) - \beta\tilde{u}_A(\theta,\eta) \mid \eta] \\ \leq\ & E[\hat{X}(\mu(\theta,\eta)) - \theta\hat{q}(\mu(\theta,\eta)) - \beta\tilde{u}_A(\theta,\eta) \mid \eta] \\ +\ & \beta\int_{\underline{\theta}}^{\bar{\theta}}[\tilde{u}_A(\theta,\eta) - u_A(\theta,\eta)]d\Lambda(\theta,\eta) \\ =\ & E[\hat{X}(\mu(\theta,\eta)) - z_\beta(\theta,\eta)\hat{q}(\mu(\theta,\eta)) \mid \eta] - \beta\int_{\underline{\theta}}^{\bar{\theta}} u_A(\theta,\eta)d\Lambda(\theta,\eta). \end{aligned}$$

The second equality uses the fact that

$$\tilde{u}_A(\theta,\eta) = \tilde{u}_A(\bar{\theta},\eta) + \int_{\theta}^{\bar{\theta}} \hat{q}(\tilde{\mu}(y,\eta))dy.$$

Next, note that $\tilde{\mu} = (\theta,\eta)$ maximizes $\hat{X}(\tilde{\mu}) - z_\beta(\theta,\eta)\hat{q}(\tilde{\mu})$. This implies that an upper bound to the value of the objective function is given by :[20]

$$E[\bar{X}(z_\beta(\theta,\eta)) - \theta\bar{q}(z_\beta(\theta,\eta)) - \beta u_A(\theta,\eta) \mid \eta].$$

[20]By definition of $(\hat{X}(\mu),\hat{q}(\mu))$,

$$\hat{X}(\theta,\eta) - z_\beta(\theta,\eta)\hat{q}(\theta,\eta) = \bar{X}(z_\beta(\theta,\eta)) - z_\beta(\theta,\eta)\bar{q}(z_\beta(\theta,\eta)).$$

But this is attainable with $(\mu(\theta,\eta),\tilde{u}_A(\theta,\eta),p(\eta)) = ((\theta,\eta),u_A(\theta,\eta),1)$ (which satisfies all constraints) in $\bar{P}^S(\alpha:\eta)$, implying that it is the optimal solution of this problem. This implies the allocation is SCP.

Let $Z(\eta_1)$ denote the set of non-decreasing functions $z:\Theta\to\Re$ such that $z(\theta) = \theta+\beta\frac{F(\theta|\eta_1)-\Lambda(\theta)}{f(\theta|\eta_1)}$ for some $\Lambda(\theta)$ which is non-decreasing in θ with $\Lambda(\underline{\theta},\eta)=0$ and $\Lambda(\bar{\theta},\eta)=1$. In order to prove Proposition 2, it suffices to construct $z_\beta(\cdot)\in Z(\eta_1)$ where (1), (2) and (3) are satisfied at the same time. The rest of the proof is devoted to this construction.

Step 1: There exists $(\lambda_1,\lambda_2)\neq 0$ *such that*

$$\frac{F(\theta)}{f(\theta)}f(\theta\mid\eta_1)+\lambda_1 F(\theta\mid\eta_2)-\lambda_2 F(\theta\mid\eta_1)=0$$

for all $\theta\in\Theta$, *if and only if if there exist* $(\rho,\nu,\gamma)\in\Re^3$ *such that* $a(\eta_1\mid\theta)=\rho+\nu F(\theta)^\gamma$ *for all* $\theta\in\Theta$.

Proof of Step 1

Proof of (If)

$a(\eta_1\mid\theta)=\rho+\nu F(\theta)^\gamma$ implies

$$\frac{F(\theta)}{f(\theta)}f(\theta\mid\eta_1)=\frac{\rho F(\theta)+\nu F(\theta)^{\gamma+1}}{\rho+\frac{\nu}{\gamma+1}}$$

$$F(\theta\mid\eta_1)=\frac{\rho F(\theta)+\frac{\nu}{\gamma+1}F(\theta)^{\gamma+1}}{\rho+\frac{\nu}{\gamma+1}}$$

$$F(\theta\mid\eta_2)=\frac{1}{1-\rho-\frac{\nu}{\gamma+1}}[(1-\rho)F(\theta)-\frac{\nu}{\gamma+1}F(\theta)^{\gamma+1}].$$

Then by choosing

$$\lambda_1=\rho\gamma\frac{1-\rho-\frac{\nu}{\gamma+1}}{\rho+\frac{\nu}{\gamma+1}}$$

$$\lambda_2=1+(1-\rho)\gamma,$$

we obtain

$$\frac{F(\theta)}{f(\theta)}f(\theta\mid\eta_1)+\lambda_1 F(\theta\mid\eta_2)-\lambda_2 F(\theta\mid\eta_1)=0$$

for any $\theta\in\Theta$.

Proof of (Only if)

Suppose that there exists $(\lambda_1, \lambda_2) \neq 0$ such that

$$\frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1) = 0 \tag{7}$$

for any $\theta \in \Theta$. Using $\frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1) = F(\theta) a(\eta_1 \mid \theta)/p(\eta_1)$, and taking the derivative of both sides of (7) with respect to θ, we obtain

$$\frac{F(\theta)}{p(\eta_1)} \frac{da(\eta_1 \mid \theta)}{d\theta} + \lambda_1 f(\theta \mid \eta_2) + (1 - \lambda_2) f(\theta \mid \eta_1) = 0.$$

for any θ. This can be rewritten as

$$\frac{\frac{da(\eta_1|\theta)}{d\theta}}{(\frac{\lambda_1 p(\eta_1)}{p(\eta_2)} - (1 - \lambda_2)) a(\eta_1 \mid \theta) - \frac{\lambda_1 p(\eta_1)}{p(\eta_2)}} = \frac{f(\theta)}{F(\theta)}. \tag{8}$$

Solving this differential equation, we obtain

$$a(\eta_1 \mid \theta) = \frac{1}{(\frac{\lambda_1 p(\eta_1)}{p(\eta_2)} - (1 - \lambda_2))} [F(\theta)^{\frac{\lambda_1 p(\eta_1)}{p(\eta_2)} - (1 - \lambda_2)} C + \frac{\lambda_1 p(\eta_1)}{p(\eta_2)}].$$

for some constant C. It implies that there exists $(\rho, \nu, \gamma) \in \Re^3$ such that $a(\eta_1 \mid \theta) = \rho + \nu F(\theta)^\gamma$.

Step 2: Under the hypothesis of Proposition 2, there exist (λ_1, λ_2) *and closed intervals on* Θ *(*$\Theta_1 = [\underline{\theta}_1, \bar{\theta}_1]$*,* $\Theta_2 = [\underline{\theta}_2, \bar{\theta}_2]$ *and* $\Theta_3 = [\underline{\theta}_3, \bar{\theta}_3]$*) such that* $\underline{\theta} < \underline{\theta}_i < \bar{\theta}_i < \underline{\theta}_{i+1} < \bar{\theta}_{i+1} < \bar{\theta}$ *(*$i = 1, 2$*), and the sign of*

$$\frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)$$

alternates among the interiors of Θ_1*,* Θ_2 *and* Θ_3*.*

Proof of Step 2

Under the conditions of Proposition 2, there exists $(\theta_1, \theta_2, \theta_3)$ with $\underline{\theta} < \theta_1 < \theta_2 < \theta_3 < \bar{\theta}$ such that

$$A(\theta_1, \theta_2, \theta_3) \equiv \begin{pmatrix} \frac{F(\theta_1)}{f(\theta_1)} f(\theta_1 \mid \eta_1) & F(\theta_1 \mid \eta_2) & -F(\theta_1 \mid \eta_1) \\ \frac{F(\theta_2)}{f(\theta_2)} f(\theta_2 \mid \eta_1) & F(\theta_2 \mid \eta_2) & -F(\theta_2 \mid \eta_1) \\ \frac{F(\theta_3)}{f(\theta_3)} f(\theta_3 \mid \eta_1) & F(\theta_3 \mid \eta_2) & -F(\theta_3 \mid \eta_1) \end{pmatrix}$$

is non-singular. To see this, for arbitrary $\theta^{'}$ and $\theta^{''}$ ($\theta^{'} \neq \theta^{''}$ and $\theta^{'}, \theta^{''} \in (\underline{\theta}, \bar{\theta})$), consider

$$\frac{|A(\theta, \theta^{'}, \theta^{''})|}{|B(\theta^{'}, \theta^{''})|} = \frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)$$

with

$$\lambda_1 \equiv -\frac{1}{|B(\theta',\theta'')|}\begin{vmatrix} \frac{F(\theta')}{f(\theta')}f(\theta' \mid \eta_1) & -F(\theta' \mid \eta_1) \\ \frac{F(\theta'')}{f(\theta'')}f(\theta'' \mid \eta_1) & -F(\theta'' \mid \eta_1) \end{vmatrix}$$

and

$$\lambda_2 \equiv -\frac{1}{|B(\theta',\theta'')|}\begin{vmatrix} \frac{F(\theta')}{f(\theta')}f(\theta' \mid \eta_1) & F(\theta' \mid \eta_2) \\ \frac{F(\theta'')}{f(\theta'')}f(\theta'' \mid \eta_1) & F(\theta'' \mid \eta_2) \end{vmatrix}$$

where

$$B(\theta',\theta'') \equiv \begin{pmatrix} F(\theta' \mid \eta_2) & -F(\theta' \mid \eta_1) \\ F(\theta'' \mid \eta_2) & -F(\theta'' \mid \eta_1) \end{pmatrix}.$$

Since $|B(\theta',\theta'')| \neq 0$ because of the monotone likelihood ratio property, the expressions above are well-defined. Our presumption and Step 1 imply that we can find $\theta \neq \theta',\theta''$ ($\theta \in (\underline{\theta},\bar{\theta})$) such that the above equation is not zero, i.e., $A(\theta,\theta',\theta'')$ is non-singular.

Next for $\bar{\theta} < \theta_1 < \theta_2 < \theta_3 < \bar{\theta}$ such that $|A(\theta_1,\theta_2,\theta_3)| \neq 0$ and for arbitrary $(b_1,b_2,b_3) \neq 0$ such that Sign $b_1 =$ Sign $b_3 \neq$ Sign b_2, consider the set of equations

$$A(\theta_1,\theta_2,\theta_3)\begin{pmatrix} \tilde{\lambda}_0 \\ \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Since $|A(\theta_1,\theta_2,\theta_3)| \neq 0$, these equations have a unique solution for $(\tilde{\lambda}_0,\tilde{\lambda}_1,\tilde{\lambda}_2)$. Moreover we can show $\tilde{\lambda}_0 \neq 0$. Otherwise, suppose that $\tilde{\lambda}_0 = 0$. Then there must exist $(\tilde{\lambda}_1,\tilde{\lambda}_2)$ such that the sign of $\tilde{\lambda}_1 F(\theta \mid \eta_2) - \tilde{\lambda}_2 F(\theta \mid \eta_1)$ alternates between $\theta_1,\theta_2,\theta_3$. However this contradicts the monotone likelihood ratio property which states that $\frac{F(\theta|\eta_1)}{F(\theta|\eta_2)}$ is monotone in θ. So we can define $\lambda_1 \equiv \tilde{\lambda}_1/\tilde{\lambda}_0$ and $\lambda_2 \equiv \tilde{\lambda}_2/\tilde{\lambda}_0$, and the sign of

$$\frac{F(\theta_i)}{f(\theta_i)}f(\theta_i \mid \eta_1) + \lambda_1 F(\theta_i \mid \eta_2) - \lambda_2 F(\theta_i \mid \eta_1) = b_i/\tilde{\lambda}_0$$

alternates among $i = 1,2,3$.

By the continuity of $\frac{F(\theta)}{f(\theta)}f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)$ for θ, we can choose closed intervals Θ_1, Θ_2 and Θ_3 ($\Theta_i \cap \Theta_{i+1} = \phi$ and $\underline{\theta} < \underline{\theta}_1 < \bar{\theta}_3 < \bar{\theta}$) such that

$$\frac{F(\theta)}{f(\theta)}f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)$$

has the same sign as at θ_i on the interior of Θ_i ($i = 1,2,3$).

In later analysis, our focus is restricted to the case that there exists (λ_1, λ_2) such that

$$\frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)$$

is negative on the interior of Θ_1 and Θ_3, and positive on the interior of Θ_2. We can adopt the same analysis for the opposite case. ■

Step 3: For any closed interval $[\theta', \theta''] \subset \Theta$ such that $\underline{\theta} < \theta' < \theta'' < \bar{\theta}$, there exists $\delta > 0$ so that $z(\cdot) \in Z(\eta_1)$ for any function $z(\cdot)$ satisfying the following properties:

(i) $z(\cdot)$ is increasing and differentiable with $|z(\theta) - \theta| < \delta\beta$ and $|z'(\theta) - 1| < \delta\beta$ for any $\theta \in \Theta$

(ii) $z(\theta) = \theta$ for any $\theta \notin [\theta', \theta'']$.

Proof of Step 3

(i) and (ii) means that a function $z(\cdot)$ is sufficiently close to identity function $\hat{\theta}(\cdot)$ (with $\hat{\theta}(\theta) = \theta$) in both distance and the slope. For arbitrary closed interval $[\theta', \theta''] \subset \Theta$ such that $\underline{\theta} < \theta' < \theta'' < \bar{\theta}$, we choose ϵ_1 and ϵ_2 such that

$$\epsilon_1 \equiv \min_{\theta \in [\theta', \theta'']} f(\theta \mid \eta)$$

and

$$\epsilon_2 \equiv \max_{\theta \in [\theta', \theta'']} |f'(\theta \mid \eta)|.$$

From our assumptions that $f(\theta \mid \eta)$ is continuously differentiable and positive on Θ, $\epsilon_1 > 0$, and ϵ_2 is non-negative and bounded above. We choose $\delta > 0$ such that

$$\delta \in (0, \frac{\epsilon_1}{\epsilon_1 + \epsilon_2}).$$

For this δ, consider a function $z(\cdot)$ which satisfies the condition (i) and (ii) of the statement. Define

$$\Lambda(\theta) \equiv \frac{(\theta - z(\theta))}{\beta} f(\theta \mid \eta_1) + F(\theta \mid \eta_1).$$

Since $z(\theta)$ is differentiable on Θ, $\Lambda(\theta)$ is also so. It is equal to $\Lambda(\theta) = F(\theta \mid \eta_1)$ on $\theta \notin [\theta', \theta'']$. For $\theta \in [\theta', \theta'']$,

$$\begin{aligned}
\frac{\partial \Lambda(\theta)}{\partial \theta} &= (\frac{1 - z'(\theta)}{\beta} + 1) f(\theta \mid \eta_1) + \frac{(\theta - z(\theta))}{\beta} f'(\theta \mid \eta_1) \\
&> (1 - \delta) f(\theta \mid \eta_1) - \delta |f'(\theta \mid \eta_1)| \\
&\geq (1 - \delta)\epsilon_1 - \delta\epsilon_2.
\end{aligned}$$

This is positive by the definition of $(\epsilon_1, \epsilon_2, \delta)$. Then $\Lambda(\theta)$ is increasing in θ on Θ with $\Lambda(\underline{\theta}) = 0$ and $\Lambda(\bar{\theta}) = 1$. Since $z(\theta)$ is increasing in θ by the definition, $z(\cdot) \in Z(\eta_1)$ by the definition of $Z(\eta_1)$. ■

Step 4: Construction of $z_\beta(\cdot)$

Here we construct $z_\beta(\cdot) \in Z(\eta_1)$ where (1) – (3) are satisfied at the same time. To simplify the notation, we use $z(\cdot)$ instead of $z_\beta(\cdot)$ in later argument. The construction of $z(\theta)$ has the following four steps.

(i) Construction of $\bar{z}(\cdot)$

First let us define $\Phi(z, \theta)$ by

$$\Phi(z,\theta) \equiv [H_k(z) - \frac{2k-1}{k}z - \frac{(1-k)}{k}\theta + \frac{(2k-1)\lambda_2}{k\beta}(z - h_\beta(\theta \mid \eta_1)) + \frac{2k-1}{k}\lambda_1 \frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_1)}]\bar{q}^{'}(z)$$
$$+ \quad \frac{(2k-1)\lambda_2}{k\beta}[\bar{q}(z) - \bar{q}(\theta)]$$

where $h_\beta(\theta \mid \eta) \equiv \theta + \beta \frac{F(\theta|\eta)}{f(\theta|\eta)}$. With $z = \theta$,

$$\Phi(\theta, \theta) \equiv \frac{2k-1}{kf(\theta \mid \eta_1)}[\frac{F(\theta)}{f(\theta)}f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)]\bar{q}^{'}(\theta).$$

Since $\Phi(z, \theta)$ is differentiable in z and θ, the statement in Step 2 guarantees the existence of $\bar{z}(\theta)$ such that (i) $\bar{z}(\theta)$ is differentiable on Θ, (ii) $\bar{z}(\theta) > \theta$ on $(\underline{\theta}_1, \bar{\theta}_1)$ and $\Phi(z, \theta) > 0$ for any $z \in [\theta, \bar{z}(\theta)]$ and any $\theta \in (\underline{\theta}_1, \bar{\theta}_1)$, (iii) $\bar{z}(\theta) < \theta$ on $(\underline{\theta}_2, \bar{\theta}_2)$ and $\Phi(z, \theta) < 0$ for any $z \in [\bar{z}(\theta), \theta]$ and any $\theta \in (\underline{\theta}_2, \bar{\theta}_2)$, (iv) $\bar{z}(\theta) > \theta$ on $(\underline{\theta}_3, \bar{\theta}_3)$ and $\Phi(z, \theta) > 0$ for any $z \in [\theta, \bar{z}(\theta)]$ and any $\theta \in (\underline{\theta}_3, \bar{\theta}_3)$, and (v) $\bar{z}(\theta) = \theta$ elsewhere.

(ii) Construction of $z_1(\cdot)$

For $\hat{\theta}_1 \in (\bar{\theta}_1, \underline{\theta}_2)$ and $\hat{\theta}_2 \in (\bar{\theta}_2, \underline{\theta}_3)$ (chosen arbitrary), ρ_1 and ρ_2 are defined by

$$\rho_1 \equiv \frac{F(\hat{\theta}_1 \mid \eta_2)}{F(\hat{\theta}_1 \mid \eta_1)}$$

and

$$\rho_2 \equiv \frac{F(\hat{\theta}_2 \mid \eta_2)}{F(\hat{\theta}_2 \mid \eta_1)}.$$

Then define

$$\Psi_i(z, \theta) \equiv [\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_1)} + \frac{\rho_i}{\beta}(z - h_\beta(\theta \mid \eta_1))]\bar{q}^{'}(z) + \frac{\rho_i}{\beta}(\bar{q}(z) - \bar{q}(\theta)).$$

$z_1(\theta)$ is defined such that $\Psi_1(z_1(\theta),\theta)=0$ is satisfied. There always exists such a $z_1(\theta)$, since for each θ, $\Psi_i(z,\theta)$ is continuous for z and is negative for $z>\max\{\theta, h_\beta(\theta\mid\eta_1)-\frac{\beta F(\theta|\eta_2)}{\rho_1 f(\theta|\eta_1)}\}$ and is positive for $z<\min\{\theta, h_\beta(\theta\mid\eta_1)-\frac{\beta F(\theta|\eta_2)}{\rho_1 f(\theta|\eta_1)}\}$. It also implies that $z_1(\theta)<h_\beta(\theta\mid\eta)$ for any θ. If there are multiple z which satisfies $\Psi_1(z,\theta)=0$, we choose one which is the closest to θ. Then rewriting $\Psi_1(z_1(\theta),\theta)=0$, we obtain

$$z_1(\theta)-\theta+\frac{\bar{q}(z_1(\theta))-\bar{q}(\theta)}{\bar{q}^{'}(z_1(\theta))}=\frac{\beta F(\theta\mid\eta_1)}{\rho_1 f(\theta\mid\eta_1)}[\rho_1-\frac{F(\theta\mid\eta_2)}{F(\theta\mid\eta_1)}].$$

Since $\frac{F(\theta|\eta_2)}{F(\theta|\eta_1)}$ is increasing in θ by the monotone likelihood ratio assumption, $z_1(\theta)>\theta$ for $\theta<\hat{\theta}_1$ and $z_1(\theta)<\theta$ for $\theta>\hat{\theta}_1$. Since $\Psi_1(\theta,\theta)>0$ (or <0) for $\theta<\hat{\theta}_1$ (or $\theta>\hat{\theta}_1$), $\Psi_1(z,\theta)>0$ for any $z\in(\theta,z_1(\theta))$ and for any $\theta<\hat{\theta}_1$ and $\Psi_1(z,\theta)<0$ for any $z\in(z_1(\theta),\theta)$ and for any $\theta>\hat{\theta}_1$. On the other hand, $\Psi_2(z,\theta)$ is positive for (θ,z) such that $z<\theta<\hat{\theta}_2$ and negative for (θ,z) such that $\hat{\theta}_2<\theta<z$ from the definition of $\Psi_2(z,\theta)$ and θ_2. Then the argument is summarized as

- For $z\in(\theta,z_1(\theta))$, $\Psi_1(z,\theta)>0$ for any $\theta\in\Theta_1$.
- For $z\in(z_1(\theta),\theta)$, $\Psi_1(z,\theta)<0$ and $\Psi_2(z,\theta)>0$ for any $\theta\in\Theta_2$
- For $z>\theta$, $\Psi_2(z,\theta)<0$ for any $\theta\in\Theta_3$.

(iii) Construction of $z_2(\cdot)$

Next let us define

$$\Gamma(z,\theta)\equiv\frac{d[(z-h_\beta(\theta\mid\eta_1))(\bar{q}(z)-\bar{q}(\theta))]}{dz}=\bar{q}(z)-\bar{q}(\theta)+(z-h_\beta(\theta\mid\eta_1))\bar{q}^{'}(z).$$

$\Gamma(z,\theta)>0$ for $z\le\theta$ and $\Gamma(z,\theta)<0$ at $z=h_\beta(\theta\mid\eta_1)$. Then we can choose $z_2(\theta)(>\theta)$ which is the minimum z such that $\Gamma(z,\theta)=0$. Therefore $(z-h_\beta(\theta\mid\eta_1))(\bar{q}(z)-\bar{q}(\theta))$ is increasing in z on $z<z_2(\theta)$.

(iv) Construction of $z(\cdot)$

Finally let us construct $z(\cdot)$, based on $\bar{z}(\cdot), z_1(\cdot)$ and $z_2(\cdot)$. According to the procedure in Step 3, for $[\theta^{'},\theta^{''}]=[\underline{\theta}_1,\bar{\theta}_3]$, choose $\delta>0$. We construct $z(\theta)$ as follows:

(i) $z(\theta)$ is differentiable and increasing in θ on Θ with $|z(\theta)-\theta|<\delta\beta$ and $|z^{'}(\theta)-1|<\delta\beta$

(ii) $z(\theta)\in(\theta,\min\{\bar{z}(\theta),z_1(\theta),z_2(\theta)\})$ on $(\underline{\theta}_1,\bar{\theta}_1)$

(iii) $z(\theta) \in (\max\{\bar{z}(\theta), z_1(\theta)\}, \theta)$ on $(\underline{\theta}_2, \bar{\theta}_2)$

(iv) $z(\theta) \in (\theta, \min\{\bar{z}(\theta), z_2(\theta)\})$ on $(\underline{\theta}_3, \bar{\theta}_3)$

(v) $z(\theta) = \theta$ elsewhere

(vi) $E[(z(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_1] = 0$

(vii) $E[(\bar{q}(\theta) - \bar{q}(z(\theta)))\frac{F(\theta|\eta_2)}{f(\theta|\eta_2)} \mid \eta_2] = 0.$

(i) implies $z(\theta) \in Z(\eta)$. We argue that there exists $z(\theta)$ which satisfies (i)-(vii). It is evident that there exists $z(\cdot)$ which satisfies (i)-(v). In addition, since $(z - h_\beta(\theta \mid \eta_1))(\bar{q}(z) - \bar{q}(\theta))$ is increasing in z for $z < z_2(\theta)$, $z(\theta) > \theta$ on Θ_1 and Θ_3 (or $z(\theta) < \theta$ on Θ_2) has the effect on raising (or reducing) $E[(z(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_1]$ away from zero. By making a balance between two effects, $z(\cdot)$ can also satisfy (vi).

Suppose $z(\cdot)$ which satisfies (i)-(vi), but does not satisfy (vii). It is shown that we can construct a new function which satisfies all of (i)-(vii) with small adjustment of $z(\cdot)$. First we define $\tilde{z}(\cdot, \epsilon)$ $(\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3))$ as $\tilde{z}(\theta, \epsilon) \equiv \theta + \epsilon_i(z(\theta) - \theta)$ on Θ_i $(i = 1, 2, 3)$ and $\tilde{z}(\theta, \epsilon) = \theta$ elsewhere. It is evident that for any $\epsilon_i \in (0, 1]$ $(i = 1, 2, 3)$, $\tilde{z}(\cdot, \epsilon)$ satisfies (i)-(v), since $\tilde{z}(\cdot, \epsilon)$ is closer to $\hat{\theta}(\cdot)$ than $z(\cdot)$ in both the distance and the slope. For the convenience of the exposition, define $\Pi(\epsilon_1, \epsilon_2, \epsilon_3)$ as

$$\Pi(\epsilon_1, \epsilon_2, \epsilon_3) \equiv E[(\tilde{z}(\theta, \epsilon) - h_\beta(\theta \mid \eta_1))(\bar{q}(\tilde{z}(\theta, \epsilon)) - \bar{q}(\theta)) \mid \eta_1].$$

It is evident that $\Pi(1, 1, 1) = 0$, since $z(\cdot)$ satisfies (vi), and $\Pi(0, 0, 0) = 0$. $\Pi(\epsilon_1, \epsilon_2, \epsilon_3)$ is continuous for each ϵ_i $(i = 1, 2, 3)$, increasing in ϵ_1 and ϵ_3 and decreasing in ϵ_2. Then since $\Pi(1, 0, 0) > 0$ and $\Pi(1, 1, 0) < 0$, there exists $\epsilon_2^{'} \in (0, 1)$ such that $\Pi(1, \epsilon_2^{'}, 0) = 0$. Similarly since $\Pi(0, 0, 1) > 0$ and $\Pi(0, 1, 1) < 0$, there exists $\epsilon_2^{''} \in (0, 1)$ such that $\Pi(0, \epsilon_2^{''}, 1) = 0$. Define $\epsilon^{'} \equiv (1, \epsilon_2^{'}, 0)$ and $\epsilon^{''} \equiv (0, \epsilon_2^{''}, 1)$. It is shown that there exists a function $\epsilon(t)$ on $t \in [0, 1]$ such that $\epsilon(t)$ is continuous and monotonic function with $\epsilon(0) = \epsilon^{'}$ and $\epsilon(1) = \epsilon^{''}$, and $\Pi(\epsilon(t)) = 0$ for any $t \in [0, 1]$. Evidently $\epsilon(t) \neq 0$ for any $t \in [0, 1]$. Suppose the case that $\epsilon_2^{'} < \epsilon_2^{''}$. (The same argument is applied for the case of $\epsilon_2^{'} \geq \epsilon_2^{''}$, and so we omit the argument for the latter case.) We choose arbitrary continuous and monotonic functions $(\epsilon_1(t), \epsilon_2(t))$ with $(\epsilon_1(0), \epsilon_2(0)) = (1, \epsilon_2^{'})$ and $(\epsilon_1(0), \epsilon_2(0)) = (0, \epsilon_2^{''})$. $\epsilon_2(t)$ is increasing in t. Then for $t \in (0, 1)$,

$$\Pi(\epsilon_1(t), \epsilon_2(t), 0) < \Pi(1, \epsilon_2^{'}, 0) = 0 = \Pi(0, \epsilon_2^{''}, 1) < \Pi(\epsilon_1(t), \epsilon_2(t), 1).$$

It implies that there exists $\epsilon_3(t) \in (0,1)$ such that $\Pi(\epsilon_1(t), \epsilon_2(t), \epsilon_3(t)) = 0$. The continuity of $\Pi(\epsilon), \epsilon_1(t), \epsilon_2(t)$ implies that $\epsilon_3(t)$ is continuous. For $t, t' \in [0,1]$ such that $t < t'$, and for any $\epsilon_3 \in (0,1)$,

$$\Pi(\epsilon_1(t), \epsilon_2(t), \epsilon_3) > \Pi(\epsilon_1(t'), \epsilon_2(t'), \epsilon_3),$$

implying that $\epsilon_3(t)$ is increasing in t.

For $\epsilon' \equiv (1, \epsilon_2', 0)$,

$$\begin{aligned}
& E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon')) - \bar{q}(\theta)) \mid \eta_2] \\
= \ & E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon')) - \bar{q}(\theta)) \mid \eta_2] \\
+ \ & \frac{\rho_1}{\beta} E[(\tilde{z}(\theta, \epsilon') - h_\beta(\theta \mid \eta_1))(\bar{q}(\tilde{z}(\theta, \epsilon')) - \bar{q}(\theta)) \mid \eta_1] \\
= \ & E[\int_\theta^{\tilde{z}(\theta, \epsilon')} [\{\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_1)} + \frac{\rho_1}{\beta}(z - h_\beta(\theta \mid \eta_1))\}\bar{q}'(z) \\
+ \ & \frac{\rho_1}{\beta}(\bar{q}(z) - \bar{q}(\theta))]dz \mid \eta_1] \\
= \ & E[\int_\theta^{\tilde{z}(\theta, \epsilon')} \Psi_1(z, \theta) dz \mid \eta_1] > 0,
\end{aligned}$$

since $\Psi_1(z, \theta) > 0$ for any $z \in (\theta, z(\theta))$ and any $\theta \in \Theta_1$ and $\Psi_1(z, \theta) < 0$ for any $z \in (\theta + \epsilon_2'(z(\theta) - \theta), \theta)$ and any $\theta \in \Theta_2$. Similarly for $\epsilon'' \equiv (0, \epsilon_2'', 1)$.

$$\begin{aligned}
& E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon'')) - \bar{q}(\theta)) \mid \eta_2] \\
= \ & E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon'')) - \bar{q}(\theta)) \mid \eta_2] \\
+ \ & \frac{\rho_2}{\beta} E[(\tilde{z}(\theta, \epsilon'') - h_\beta(\theta \mid \eta_1))(\bar{q}(\tilde{z}(\theta, \epsilon'')) - \bar{q}(\theta)) \mid \eta_1] \\
= \ & E[\int_\theta^{\tilde{z}(\theta, \epsilon'')} \Psi_2(z, \theta) dz \mid \eta_1] < 0,
\end{aligned}$$

since $\Psi_2(z, \theta) > 0$ for any $z \in (\theta + \epsilon_2''(z(\theta) - \theta), \theta)$ and any $\theta \in \Theta_2$ and $\Psi_2(z, \theta) < 0$ for any $z \in (\theta, z(\theta))$ and any $\theta \in \Theta_3$. Moreover $E[\frac{F(\theta|\eta_2)}{f(\theta|\eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon)) - \bar{q}(\theta)) \mid \eta_2]$ is continuous for ϵ. Therefore there exists $t \in (0,1)$ such that

$$E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon(t))) - \bar{q}(\theta)) \mid \eta_2] = 0.$$

This argument implies that there exists $\epsilon \neq 0$ such that both (vi) and (vii) are satisfied under $\tilde{z}(\cdot, \epsilon)$. For this $\tilde{z}(\cdot, \epsilon)$, all conditions (i)-(vii) are satisfied.

Step 5: Improvement of P's payoff

Finally we check that under $z(\theta)$ which is constructed in Step 4,

$$\begin{aligned} & E[V(\bar{q}(z(\theta))) - \frac{2k-1}{k}\bar{X}(z(\theta)) - \frac{1-k}{k}\theta\bar{q}(z(\theta)) \mid \eta] \\ > \quad & E[V(\bar{q}(\theta)) - \frac{2k-1}{k}\bar{X}(\theta) - \frac{1-k}{k}\theta\bar{q}(\theta) \mid \eta] \end{aligned}$$

For (λ_1, λ_2) specified in Step 2,

$$\begin{aligned} & E[V(\bar{q}(z(\theta))) - \frac{2k-1}{k}\bar{X}(z(\theta)) - \frac{1-k}{k}\theta\bar{q}(z(\theta)) \mid \eta_1] \\ - \quad & E[V(\bar{q}(\theta)) - \frac{2k-1}{k}\bar{X}(\theta) - \frac{1-k}{k}\theta\bar{q}(\theta) \mid \eta_1] \\ = \quad & E[V(\bar{q}(z(\theta))) - \frac{2k-1}{k}\bar{X}(z(\theta)) - \frac{1-k}{k}\theta\bar{q}(z(\theta)) \mid \eta_1] \\ + \quad & \frac{(2k-1)\lambda_2}{k\beta_1}E[(z(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_1] \\ + \quad & \frac{2k-1}{k}\lambda_1[E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_2] \\ - \quad & E[V(\bar{q}(\theta)) - \frac{2k-1}{k}\bar{X}(\theta) - \frac{1-k}{k}\theta\bar{q}(\theta) \mid \eta_1] \\ = \quad & E[\int_\theta^{z(\theta)} \Phi(z,\theta)dz \mid \eta_1] > 0. \end{aligned}$$

The first equality comes from (vi) and (vii) in Step 4. Therefore P's payoff is improved over the optimal NS. It completes the proof. ■