Hierarchical Control Rights and Strong Collusion[1]

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Abstract

In a setting with complete contracts, we study strong collusion between an informed agent and less well-informed supervisor, where each can commit to threats to punish the other for refusing to enter into a hidden side-contract. We show it is beneficial for the Principal to skew bargaining power in favor of the supervisor, e.g. by delegating the right to appoint the agent, in order to increase bargaining frictions within the coalition. By contrast, in contexts of weak collusion where such commitments are not possible, the allocation of bargaining power is irrelevant.

KEYWORDS: mechanism design, supervision, collusion, bargaining power

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1 Introduction

It is well known (e.g., Myerson (1982)) that delegation of authority to privately informed agents is not worthwhile for a Principal that can commit perfectly to 'complete contracts' — i.e., without any restrictions on message spaces of agents and absence of collusion between agents. The effect of any mechanism which delegates authority to some agents can be replicated by an incentive compatible revelation mechanism where the Principal retains full authority over all decisions as a function of reports of private information made by agents. Owing to this result, theories of decentralized authority have been based on limited commitment power of the Principal[4], contexts with unforeseen contingencies, and communication or information processing costs that restrict message spaces.[5] The consequences of collusion among agents have not been hitherto explored, with few exceptions.[6]

In this paper we explore the value of delegating authority as a means of controlling the consequences of collusive behavior among agents. Collusion refers to hidden side-contracts, wherein the Principal cannot prevent agents from side-contracts allowing them to communicate with one another, coordinate their responses to the Principal and exchange side-payments. We impose no restrictions on the size of message spaces, and abstract entirely from 'complexity' costs or lack of commitment. We consider a context with a supervisor (S) and agent (A), in which the latter is privately informed about the unit cost of delivering a good to the Principal, and S obtains a noisy signal

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[4]See e.g., Poitevin (2000), Aghion and Tirole (1997), Dessein (2002).

[5]Marschak and Radner (1972), Green and Laffont (1986, 1987), Melumad, Mookherjee and Reichelstein (1992), Radner (1993), Van Zandt (1999), Battigali and Maggi (2002), Fadel and Segal (2009), Mookherjee and Tsumagari (2014).

[6]A number of papers on decentralization and collusion present models where delegation of authority is equivalent to centralization in the presence of collusion: e.g., Baliga and Sjostrom (1998) or Faure-Grimaud, Laffont and Martimort (2003). The only paper we are aware of that shows superiority of decentralization in the presence of collusion is Laffont and Martimort (1998), but their theory requires presence of communication costs along with collusion. This paper focuses on collusion *per se*, without any restrictions on message spaces.

of this cost.[7] A also observes the realization of S's signal; hence collusion takes the form of a side contract with one-sided asymmetric information between S and A. The relevant notion of 'authority' refers to allocation of bargaining power over the side contract. For instance, consider a setting where P needs to select one A and one S from a large pool of *ex ante* identical agents and supervisors respectively. P could select an S from the supervisor pool, and delegate to it the authority to select A from the agent pool. This amounts to giving all the bargaining power in side-contracting to S, since the option to select between different agents enables S to push A down to the latter's outside option. At the other extreme, P could skew bargaining power to the agent by appointing an A and authorizing A to select an S. An intermediate option would be for P to personally select an S and an A and then allow them to bargain over the side-contract, which would allocate bargaining power more symmetrically.

The value of delegating 'appointment authority' thus translates into the value of skewing bargaining power in favor of either S or A. We explore the idea that favoring the relatively less informed party (S, in this case) generates greater bargaining frictions, which is valuable to P by reducing the impact of collusion. We show this is the case when (a) side contracting takes place at the ex ante stage, allowing S and A to collude on both participation and reporting decisions at the interim stage, and (b) collusion is 'strong', in the sense that the side contract allows each party to 'threaten' the other in the event of the latter refusing to accept the offered side contract at the interim stage (by committing to specific messages to be sent thereafter to P). Such threats enlarge the scope of collusion relative to 'weak' collusion where refusal of the side contract by one party is followed by noncooperative behavior thereafter. While papers on optimal auction design have explored the implications of strong and weak collusion[8], the existing literature on mechanism design with supervision and

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[7]In contrast to Tirole (1986) or Laffont and Tirole (1993), and in line with subsequent literature on supervision, we consider contexts of 'soft' information where supervisors and agents can send any messages to the Principal.

[8]See Dequiedt (2007), Pavlov (2008) and Che and Kim (2009).

collusion has been confined to 'weak' collusion. Under weak collusion the allocation of bargaining power between colluding parties does not matter.[9]

Hence the main contributions of this paper are the exploration of strong ex ante collusion in the context of supervisor-agent relationships, and the result that skewing bargaining power in favor of the supervisor can enhance the Principal's welfare by reducing the scope for collusion. Moreover, we show that if the agent has greater bargaining power, the presence of the supervisor is worthless to the Principal, as the chosen side contract will not be subject to any frictions. Intuitively, this is because the side contract will maximize the residual rents of the agent, while pushing the supervisor down to a constant outside option payoff which is common knowledge within the coalition. Skewing authority in favor of the supervisor, by contrast, ensures that coalitional bargaining is subject to frictions owing to asymmetric information, since the agent is better informed than the supervisor.

While delegation of authority to the supervisor over agent selection is valuable, it turns out to not be optimal for the Principal to also delegate the choice of the contract for the agent. In order for P to obtain value from the presence of the supervisor, it is essential that P contracts personally with both agent and supervisor. This enables P to manipulate the outside options of colluding parties and thereby restrict the scope of collusion. In this respect there is no difference between weak and strong collusion.[10]

Relevant applications of our model and results include internal organization of firms (where P is a firm owner, S a manager or supervisor and A a worker), government procurement or regulation (P is the government, S a bureaucrat or regulator, and A a private utility or contractor), or financial intermediaries (P is an investor, S an auditor/financial advisor and A a borrower seeking to finance a project).[11] The model sug-

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[9]Faure-Grimaud, Laffont and Martimort (2003) and Mookherjee, Motta and Tsumagari (2018) show this in settings of interim and ex ante collusion respectively.

[10]With weak collusion, the necessity of P contracting with both A and S was shown in Mookherjee, Motta and Tsumagari (2018, Proposition 1 and 4) when collusion arises at the *ex ante* stage, and by Celik (2009) when it arises at the *interim* stage.

[11]Evidence for such collusion is available in many real world contexts, e.g., between outside Di-

gests that biasing the allocation of authority in favor of managers/regulators/intermediaries over selection of workers/contractors/borrowers is beneficial in limiting the harmful consequences of strong collusion. At the same time, firm owners/governments need to retain control over the design of formal contracts.

The paper is organized as follows. Section 2 lays out the model, followed by Section 3 which verifies the Collusion Proofness Principle holds for both strong and weak collusion: that any feasible allocation in either setting can be achieved by a collusion proof mechanism for some grand contract. This provides a characterization of allocations that can be supported with strong and weak collusion respectively. Using this characterization, Section 4 provides our two main results for strong collusion proof allocations for alternative ranges of bargaining power allocation. First, if A has at least as much welfare weight as S, appointing a supervisor is worthless for P. In other words, P can attain the same payoff by contracting directly with A, without trying to elicit any information from any supervisor. Second, if S is assigned a higher welfare weight, and S's signal is coarse (in the sense of having only two possible realizations while A's cost is continuously distributed), appointing a supervisor is valuable, i.e., P attains a strictly higher payoff compared to the option of not hiring S at all. Section 5 then shows that in order to derive any value from the supervisor it is essential for P to contract with both parties. Section 6 concludes, while technical details of proofs and relevant extensions of the model are provided in the online Appendix.[12]

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rectors and CEOs (Hallock (1997), Hwang and Kim (2009), Fracassi and Tate (2012), Kramarz and Thesmar (2013), Schmidt (2015)), between management and workers (Bertrand and Mullainathan (1999, 2003), Atanassov and Kim (2009), Cronqvist et al. (2009)), 'revolving doors' between credit-rating agencies and firms (de Haan et al. (2015), Cornaggia et al. (2016)) and between auditors and their clients (Lennox (2005), Lennox and Park (2007), Firth et al. (2012)).

[12]The online Appendix is located at http://people.bu.edu/dilipm/wkpap/index.html.

2   Model

2.1   Technology, Preferences and Information

An appointed agent A delivers an output q to the Principal P at a personal cost of θq. P's return from q is $V(q)$, a twice continuously differentiable, increasing and strictly concave function satisfying the Inada condition ($\lim_{q \to 0} V'(q) = +\infty$ and $\lim_{q \to +\infty} V'(q) = 0$) and $V(0) = 0$. The realization of θ is privately observed by A. Θ, which denotes the support of θ, constitutes an interval $[\underline{\theta}, \bar{\theta}] \subset (0, \infty)$.[13] It is common knowledge that everybody shares a common distribution function $F(\theta)$ over Θ. It has a density function $f(\theta)$ which is continuously differentiable and everywhere positive on Θ. It is assumed that $H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$ is strictly increasing in θ.

An appointed supervisor S costlessly acquires an informative signal $\eta \in \Pi \equiv \{\eta_1, \eta_2, ..., \eta_m\}$ about A's cost θ with $m \geq 2$.[14] The realization of S's signal is observed by A. $a(\eta \mid \theta) \in [0, 1]$, which denotes the likelihood function of η conditional on θ, is continuously differentiable and positive-valued on Θ.[15] We assume that for any $\eta \in \Pi$, $a(\eta \mid \theta)$ is not a constant function on Θ, and there are some subsets of θ with positive measure satisfying $a(\eta \mid \theta) \neq a(\eta' \mid \theta)$ for every $\eta, \eta' \in \Pi$. In this sense each possible signal realization conveys information about the agent's cost. The information conveyed is partial, since Π is finite. The cdf over θ conditional on η is denoted $F(\theta|\eta)$. Conditional on η, the density function and distribution function are respectively $f(\theta \mid \eta) \equiv f(\theta)a(\eta \mid \theta)/p(\eta)$ and $F(\theta \mid \eta) \equiv \int_{\underline{\theta}}^{\theta} f(\theta \mid \eta)d\theta$, where $p(\eta) \equiv \int_{\underline{\theta}}^{\bar{\theta}} f(\theta)a(\eta \mid \theta)d\theta$. Let $K \equiv \Theta \times \Pi$ denote the set of possible states.

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[13]All but the last result do not depend on any specific features of Θ. More general argument is provided in the online Appendix.

[14]If S incurs a fixed cost c to acquire the signal, transfers received by S must be replaced by transfers net of this fixed cost while measuring S's payoff. Increases in c will of course lower the value of appointing the supervisor, but it is easy to see how the results will be modified.

[15]This assumes that the support of θ given η is Θ for all η, in which sense θ has full support. We adopt this assumption purely to simplify the exposition; our results extend to the case of non-full support, as shown in the online Appendix.

All players are risk neutral. S's payoff is $u_S = X_S + t_S$ where t_S is a transfer received by S within the coalition. A's payoff is $u_A = X_A + t_A - \theta q$ where t_A is a transfer received by A within the coalition. P's objective is a weighted average of profit ($\Pi \equiv V(q) - X_A - X_S$) and welfare of A and S ($u_A + u_S$), with a lower relative weight on the latter. With $k \in (\frac{1}{2}, 1]$ which denotes the weight on profit, and $1 - k$ on welfare of A and S, P's payoff reduces to $k[V(q) - (X_S + X_A)] + (1-k)[X_S + X_A - \theta q]$.[16] Hence the model applies both to the organization of private firms whose owners seek to maximize profit ($k = 1$), as well as regulation or taxation contexts where P is a social planner pursuing a welfare objective that includes payoffs of A and S as well as profit, but assigns a higher weight to the latter.[17]

In this economic environment, a (deterministic) allocation is denoted by

$$\{(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta)) \mid (\theta, \eta) \in K\}.$$

2.2 Mechanism, Collusion Game and Equilibrium Concept

P designs a grand contract (GC) played by an appointed pair of A and S, describing production decisions and transfers made by P in response to message sent by S and A. We focus on deterministic mechanisms:

$$GC = (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S); M_A, M_S)$$

where M_A (resp. M_S) denotes a message set for A (resp. S).[18] Message spaces include exit options for A and S respectively ($e_A \in M_A, e_S \in M_S$), where $X_A = q = 0$ whenever $m_A = e_A$, and $X_S = 0$ whenever $m_S = e_S$. The set of grand contracts satisfying these restrictions is denoted by $\mathcal{GC}$. As a special case, P has the option to

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[16]We exclude $k = \frac{1}{2}$ because in that case the first-best can be achieved (as in Baron and Myerson (1982)).

[17]The latter would be the case e.g., if P represents the interests of consumers, who need to be taxed to finance transfers to A and S, and these taxes involve deadweight losses.

[18]We ignore stochastic mechanisms which randomize the allocation conditional on messages, since they lower P's welfare (owing to strict concavity of V) without affecting S or A's payoffs.

not hire S, which we denote by *No Supervision (NS)*, where M_S is null and $X_S \equiv 0$. Another special case considered in Section 5 is of full delegation, where P contracts only with either S or A, not communicating or transacting with the other party. For instance, if P contracts only with S, A contracts only with S and becomes a pure subcontractor — effectively the entire authority to contract with A is delegated to S. Centralization corresponds to P contracting with both S and A, whence M_A and M_S are both non-null and X_A, X_S assume non-zero values for some message pairs.

It will be convenient to allow for randomized message choices. Let $\Delta(M_A)$, $\Delta(M_S)$ and $\Delta(M)$ denote the set of probability measures on M_A, M_S and $M \equiv M_A \times M_S$ respectively. For $(\mu_A, \mu_S) \in \Delta(M_A) \times \Delta(M_S)$ and $\mu \in \Delta(M)$, we define the mixed strategy extensions of the grand contract, which are respectively described in the expected value of corresponding allocations, as follows:[19]

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\begin{aligned}
\bar{GC} &\equiv (\bar{X}_A(\mu_A, \mu_S), \bar{X}_S(\mu_A, \mu_S), \bar{q}(\mu_A, \mu_S)) \\
&= \int_{M_A} \int_{M_S} (X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S)) d\mu_A(m_A) d\mu_S(m_S)
\end{aligned}
$$

and

$$
\tilde{GC} \equiv (\tilde{X}_A(\mu), \tilde{X}_S(\mu), \tilde{q}(\mu)) = \int_M (X_A(m), X_S(m), q(m)) d\mu(m).
$$

Collusion between S and A takes the form of a side contract (SC) which is unobserved by P. The allocation of bargaining power between S and A can be chosen (or influenced) by P, by controlling the process by which S and A are appointed. For now, we treat the allocation of bargaining power as a parameter, and represent it by relative welfare weights on the ex ante payoffs of S and A at the time that the side contract is chosen by the coalition in response to the contract GC offered by P. Formally, the side contract is selected at the ex ante stage by a fictional (uninformed) third party acting as a mediator, who maximizes $\alpha u_A + (1-\alpha)u_S$ ($\alpha \in [0, 1]$), where

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[19]In order to avoid technical complications, we assume that M_A and M_S are compact subsets of finite dimensional Euclidean spaces, and $(\bar{X}_A, \bar{X}_S, \bar{q})$ are continuous for each of μ_A and μ_S and $(\tilde{X}_A, \tilde{X}_S, \tilde{q})$ for each of μ. These assumptions enable us to apply the minimax theorem (Nikaido (1954)) and guarantee the existence of an optimal side contract.

u_A and u_S respectively denote ex ante payoffs of A and S, in response to choices of α and GC made by P. The third party does not play any budget breaking role, hence transfers within the coalition must balance: $t_A + t_S \leq 0$. No side payments can be exchanged at the *ex ante* stage; they can only be exchanged at the *ex post* stage after payments from P have been received. The side contract cannot be renegotiated at the interim or *ex post* stage. It allows exchange of private messages between A and S, which determine a side payment and joint set of messages they respectively send to P. Since message spaces include exit as well as type reports, collusion takes the *ex ante* form studied in Mookherjee et al. (2018), rather than the interim form studied by Faure-Grimaud et al. (2003) or Celik (2009).

The stages of the game are as follows. Following the choice of GC and α by P, at stage 1 (the ex ante stage), the third party offers a side contract to S and A. A null side contract (NSC) could also be offered.

Next at stage 2 (the interim stage) S observes η and A observes (θ, η). If a NSC was offered, they play the GC noncooperatively based on their prior beliefs, just as in a game without any collusion. If a non-null side contract was offered, S and A independently decide whether to accept it. Specifically, the game proceeds as follows. $i = A, S$ selects a message $d_i \in D_i$ ($i = A, S$) where D_i is i's message set specified in the side-contract. D_i includes i's exit option $\hat{e}_i$ from the side-contract. If $d_A \neq \hat{e}_A$ and $d_S \neq \hat{e}_S$, their reports to P are selected according to $\mu(d_A, d_S) \in \Delta(M)$, and side payments to A and S are determined according to functions $t_A(d_A, d_S)$ and $t_S(d_A, d_S)$ respectively. If $d_A = \hat{e}_A$ and $d_S = \hat{e}_S$, A and S play GC non-cooperatively.

What happens when one accepts and the other does not, depends on whether collusion is strong or weak. If it is strong, SC specifies a reporting strategy of the party that accepted it, which can be interpreted as a threat that party commits to. The party that rejected it then plays a best response to this threat. Hence with strong collusion, if $d_i \neq \hat{e}_i$ and $d_j = \hat{e}_j$ ($i, j = A, S$), i's message to P is selected according to

$\mu_i(d_i) \in \Delta(M_i)$, and the side payment to i is $t_i(d_i)$.[20] On the other hand, j plays GC without any constraint imposed by the side contract, and without any side transfer.

When collusion is weak instead, the side contract ceases to apply for the subsequent messages for either player when one of them exits — S and A play GC noncooperatively.

We focus on Perfect Bayesian Equilibrium (PBE) of this strong collusion game induced by the grand contract GC and bargaining weight parameter α.[21] However, there may be multiple PBE in a given game. We assume collusion permits parties to coordinate the choice of a PBE, hence the third party can specify a selected PBE to maximize the welfare-weighted sum of ex ante payoffs of S and A in the event of multiple PBE. The resulting equilibrium concept is denoted by PBE(sc). In case there are two PBE(sc) where the third party receives the same payoff, we assume that P can select the more desirable one.

Feasible allocations in strong collusion can now be defined:

Definition 1 *An allocation (u_A, u_S, q) is achievable in strong collusion with α if it is realized in PBE(sc) under α for some $GC \in \mathcal{GC}$.*

$\mathcal{A}^S(\alpha)$ will denote the set of achievable allocations in strong collusion with α.

The definition of feasible allocations in weak collusion is analogous.

2.3   Interpreting Choice of Bargaining Parameter α

Here is one possible setting which describes processes by which S and A can be appointed, specific versions of which correspond to alternative values of the parameter α representing allocation of bargaining power between S and A. Suppose there is a pool C_i $(i = A, S)$ of a countably infinite number of ex-ante identical candidates for i. Each of them has an ex-ante outside option equal to zero. P can select one of the following three options:

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[20]Owing to the budget balance condition, $t_i(d_i) \le 0$.

[21]For definition of PBE, see Fudenberg and Tirole (1991).

- DSS (Delegation of Selection to S): P selects S randomly from C_S and delegates the selection of A from C_A to S.

- DSA (Delegation of Selection to D): P selects A randomly from C_A and delegates the selection of S from C_S to A.

- CS (Centralized Selection): P selects A and S randomly from C_A and C_S respectively.

In DSS the S appointed by P has the authority to select any A in C_A, and offer it a side contract at the ex ante stage. The chosen A can accept or reject the offer; in the latter case, S can select any other agent and make it an offer, and so on. The ex ante stage allows an infinite number of rounds of such side contract offers to be made. It is easy to see that S will end up with all the bargaining power over the choice of a side contract: the outcome must maximize S's ex ante payoff over the set of achievable allocations (corresponding to the particular collusion concept), which corresponds to $\alpha = 0$.

The DSA game is the same as DSS, except that the roles of S and A are interchanged: this corresponds to $\alpha = 1$.

In CS, the S and A appointed by P negotiate a side contract. We do not formally model the bargaining game between S and A in CS, but invoke Binmore et al. (1986) who show specific bargaining protocols whose outcomes reduce to the maximization of a (weighted) sum of payoffs between S and A over the set of achievable allocations. This corresponds to an interior value of α. As Binmore et al. (1986) show, the bargaining weights correspond to the exact sequence of moves and underlying frictions in the bargaining process (such as delays) and how they affect the participants (e.g., their relative impatience). Alternative assumptions about these details of the bargaining process cause different bargaining weights to be generated. Hence there will exist bargaining protocols which will end up assigning a welfare weight to the agent which is not lower than the weight assigned to the supervisor. As we shall see, the

outcome of CS will then coincide with the outcome of DSA under strong collusion, while DSS will generate higher expected profits to P.

3 Collusion-Proofness Principle and Collusion-Proof Allocations

We now show that the Collusion-Proofness Principle (see Tirole (1992)) holds for strong collusion — i.e., that the outcome of any achievable allocation (for any given value of α) can be replicated by a PBE of the strong collusion game following choice of some grand contract in which it is optimal for the third-party to offer a null side contract. This is useful in characterizing the set of achievable allocations with strong collusion by a set of individual and coalitional incentive and participation constraints.

3.1 Strong Collusion-Proof Allocations

We start by defining a class of allocations which are not vulnerable to either collusive or individual deviations from contract acceptance and truthful reporting, and later show that these characterize the class of all achievable allocations.

First, any achievable allocation must satisfy a set of individual incentive constraints, pertaining to truthful reporting of θ by A (IC_A), and interim participation constraints for A (PC_A) and S (PC_S) respectively:

$$u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$$

for any $\theta, \theta' \in \Theta$ and any $\eta \in \Pi$,

$$u_A(\theta, \eta) \geq 0$$

for any $(\theta, \eta) \in K$ and

$$E[u_S(\theta, \eta) \mid \eta] \geq 0$$

for any $\eta \in \Pi$. We say that (u_A, u_S, q) satisfies *individual incentive compatibility (IIC)* if and only if it satisfies IC_A, PC_A and PC_S. Incentive compatibility with respect to η reports do not have to be included since A and S observe the realization of η, so P can elicit this information by cross-checking their respective reports if they do not collude.

Now turn to coalitional incentive constraints. Collusion-proofness requires absence of any scope for the third party to benefit by offering a non-null side contract. In the context of strong collusion, threats not actually used on the equilibrium path play a role. To capture their role, we need to go beyond standard revelation mechanisms where each type report correspond to messages used on the equilibrium path, and augment them with an auxiliary non-type message. Define $\bar{K} \equiv \Theta \times \bar{\Pi}$ where $\bar{\Pi} \equiv \Pi \cup \{\eta_0\}$.[22] The augmentation from Π to $\bar{\Pi}$ allows one auxiliary message η_0 regarding the signal realization to be submitted. An augmentation of allocation (u_A, u_S, q) on $\bar{K}$ is represented by (u_A^e, u_S^e, q^e) with the selection of $(u_A(\theta, \eta_0), u_S(\theta, \eta_0), q(\theta, \eta_0))$.

Note that the coalition can also collectively decide to exit from GC, which is represented by joint message $e \equiv (e_S, e_A)$. In the event that e is chosen, the autarkic allocation $(X_A = X_S = q = 0)$ results. Hence the augmented message space is $\bar{K} \cup \{e\}$.

For any allocation (u_A, u_S, q) (defined over the type space K) and its augmented allocation (u_A^e, u_S^e, q^e) (defined over the type space $\bar{K}$), we can define the corresponding *coalitional incentive scheme* by the aggregate transfers between P and the coalition: $(\hat{X}^e, \hat{q}^e) = (u_A^e + u_S^e + \theta q^e, q^e)$ which is also defined over $\bar{K}$. We can also select $(\hat{X}^e(e), \hat{q}^e(e)) \equiv (0, 0)$. We also allow A and S to randomize their messages according to the measure μ defined over $\bar{K} \cup \{e\}$. Let $\Delta(\bar{K} \cup \{e\})$ denote the corresponding set of measures. With a slight abuse of notation, we denote the corresponding expected values of the coalitional incentive scheme (defined over the augmented domain) as

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[22] This argument is based on our assumption of full support that the support of θ given η is always Θ. The case of non-full support is provided in the online Appendix.

$(\hat{X}^e(\mu), \hat{q}^e(\mu))$ for any given $\mu \in \Delta(\bar{K} \cup \{e\})$.

The side contracting problem can be represented as follows. Given a coalitional incentive scheme, the coalition select a joint report $\mu \in \Delta(\bar{K} \cup \{e\})$ to send to P, and then redistribute the resulting rents $(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta))$ between A and S, i.e., such that $\tilde{u}_A(\theta, \eta) + \tilde{u}_S(\theta, \eta) = \hat{X}^e(\mu(\theta, \eta)) - \theta \hat{q}^e(\mu(\theta, \eta))$. Conditional on both A and S agreeing to participate, this joint decision is based on a θ report submitted by A to the third party. The side contract does not stipulate any coalitional decision in the event that both A and S reject it. If A rejects it while S does not, the side contract specifies a reporting strategy for S which acts as a threat. Let this strategy be denoted by $P(\cdot \mid \eta)$ for each $\eta \in \Pi$, which is a probability function defined on $\bar{\Pi}$ such that $\Sigma_{\eta' \in \bar{\Pi}} P(\eta' \mid \eta) = 1$ and $0 \leq P(\eta' \mid \eta) \leq 1$ for all $\eta' \in \bar{\Pi}$. Here $P(\eta' \mid \eta)$ denotes the probability that S reports η' in the state where η has been observed. The strategy of reporting truthfully in state η (i.e. $P(\eta \mid \eta) = 1$ and $P(\eta' \mid \eta) = 0$ for any $\eta' \neq \eta$) is denoted by $I(\eta)$. Similarly, let $I(\theta, \eta)$ denote the strategy of reporting the state truthfully in state (θ, η).

Using these definitions and notations, we define strong collusion-proof (SCP) allocation as follows.

Definition 2 *Allocation* (u_A, u_S, q) *is strong collusion-proof (or SCP) for* $\alpha \in [0, 1]$, *if* (u_A, u_S, q) *is IIC, and there exists an outside option payoff* $\omega \geq 0$ *for S, and an augmentation* (u_A^e, u_S^e, q^e) *of* (u_A, u_S, q) *on* $\bar{K}$ *with* $u_S^e(\theta, \eta_0) = \omega$ *for any* $\theta \in \Theta$ *and* $(u_A^e(\theta, \eta_0), q^e(\theta, \eta_0))$ *satisfying (IC$_A$) and (PC$_A$) such that for any* $\eta \in \Pi$,

$$(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), P(\cdot \mid \eta)) = (I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$$

solves problem $P^S(\alpha : \eta)$:

$$\max E[\alpha \tilde{u}_A(\theta, \eta) + (1 - \alpha) \tilde{u}_S(\theta, \eta) \mid \eta]$$

subject to $(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), P(\cdot \mid \eta))$ *satisfies for all* $\theta \in \Theta$:

(i) $\mu(\theta, \eta) \in \Delta(\bar{K} \cup \{e\}), \tilde{u}_A(\theta, \eta) \in \Re, \tilde{u}_S(\theta, \eta) \in \Re, P(.|\eta) \in \Delta(\bar{\Pi})$

(ii) $\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}^e(\mu(\theta', \eta))$ *for any* $\theta' \in \Theta$

(iii) $\tilde{u}_A(\theta, \eta) + \tilde{u}_S(\theta, \eta) = \hat{X}^e(\mu(\theta, \eta)) - \theta\hat{q}^e(\mu(\theta, \eta))$

(iv) $E[\tilde{u}_S(., \eta) \mid \eta] \geq \omega$

(v) $\tilde{u}_A(\theta, \eta) \geq \Sigma_{\eta' \in \bar{\Pi}} P(\eta' \mid \eta) u_A^e(\theta, \eta')$.

We provide an informal explanation of this notion. A non-null side contract is represented by the following components. Provided both A and S have agreed to participate at the interim stage, and following an internal type report θ by A and a common report η of the signal by A and S, the coalition submits a message report to P according to the strategy $\mu(\theta, \eta)$ (satisfying the first part of condition (i)), and then reallocates the resulting coalitional allocation via side payments to generate net payoffs $(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta))$ for A and S respectively (the budget balance condition (iii)). The side contract must provide A with an incentive to report θ truthfully within the coalition (condition (ii)).

Moreover, the side contract includes threats in the event of unilateral rejection by either party, that ensure their participation. In order to induce S to participate (conditional on A agreeing to participate) in the collusion, condition (iv) must be satisfied. S's participation is induced by a threat by A to subsequently report to P in some way if S refuses, which ensures that S cannot attain a payoff higher than ω. Since asymmetric information is one-sided, the standard minimax theorem ensures that S's minmax payoff is well-defined (given an associated grand contract GC), and A has a reporting strategy that guarantees S cannot earn more than ω. The minmax payoff ω of S must be non-negative since S can always exit from GC, and is effectively chosen by P while designing the mechanism. In particular, the mechanism can be augmented to ensure that ω is earned by S upon submitting the auxiliary message η_0, no matter what A reports. We show below that this way of designing the mechanism entails no loss of generality. Hence ω is the outside option payoff for S.

Finally, A's participation (conditional on S's participation) when both have observed the signal η, is ensured by the threat of S reporting according to the strategy $P(\cdot|\eta)$ if A refuses. A will then be a Stackelberg follower in noncooperative play of P's mechanism, and will select a best response to this threat. Since the augmented mechanism satisfies individual incentive constraints for A, it will be optimal for A to report truthfully, no matter what S reports.[23] This will generate A a payoff of $u_A^e(\theta, \eta')$ if S reports η'. Hence the right hand side of (v) represents the outside option payoff of A to participating in the collusion.

Strong collusion proofness requires that the null side contract is an optimal choice for the third party. The null side-contract is represented by a choice of a side contract allocation which coincides with the allocation itself (i.e., there are no side-payments), and truthful reports submitted to P. Moreover, no threats need to be used by S to coerce A into accepting this contract, hence $P(\cdot \mid \eta) = I(\eta)$.

We now present the main result of this section.

Lemma 1 *An allocation (u_A, u_S, q) is achievable in strong collusion with α if and only if it is strongly collusion-proof for α.*

The proof is provided in the Appendix; it extends standard arguments to the context of strong collusion, which requires augmenting any given equilibrium allocation in a particular way that ensures the allocation satisfies the SCP property. Conversely, any SCP allocation can be achieved as a PBE(sc) allocation in a GC which can be constructed on the basis of the incentive compatible augmentation of the allocation.

3.2   Comparison with Weak Collusion Proof Allocations

It is useful to compare the notion of strong collusion proofness to weak collusion-proofness, which has been analyzed extensively in Mookherjee et al. (2018). In order to characterize weak collusion proof (WCP) allocations, it suffices to consider reve-

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[23]Note that A reports after observing θ, so the realization of η does not affect A's preferences.

lation mechanisms without any augmentation. Hence the domain of the mechanism is $K \cup \{e\}$. Given a revelation mechanism (u_A, u_S, q) defined over $K \cup \{e\}$, and given a mixed reporting strategy $\mu \in \Delta(K \cup \{e\})$, the associated coalitional incentive scheme is denoted by $(\hat{X}(\mu), \hat{q}(\mu))$, where $(\hat{X}(\theta, \eta), \hat{q}(\theta, \eta)) \equiv (u_A(\theta, \eta) + u_S(\theta, \eta) + \theta q(\theta, \eta), q(\theta, \eta))$ and $(\hat{X}(e), \hat{q}(e)) \equiv (0, 0)$. Using the terminology of this paper, a weak-collusion proof allocation can be defined as follows.[24]

Definition 3 (u_A, u_S, q) *is weak collusion-proof (or WCP) for* α, *if* (u_A, u_S, q) *is IIC, and for any* $\eta \in \Pi$,

$$(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta)) = (I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta))$$

solves problem $P^W(\alpha : \eta)$:

$$\max E[\alpha \tilde{u}_A(\theta, \eta) + (1 - \alpha)\tilde{u}_S(\theta, \eta) \mid \eta]$$

subject to $(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta))$ *satisfies for all* $\theta \in \Theta(\eta)$:

(i) $\mu(\theta, \eta) \in \Delta(K \cup \{e\}), \tilde{u}_A(\theta, \eta) \in \Re, \tilde{u}_S(\theta, \eta) \in \Re$

(ii) $\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}(\mu(\theta', \eta))$ *for any* $\theta' \in \Theta$

(iii) $\tilde{u}_A(\theta, \eta) + \tilde{u}_S(\theta, \eta) = \hat{X}(\mu(\theta, \eta)) - \theta\hat{q}(\mu(\theta, \eta))$

(iv) $E[\tilde{u}_S(., \eta) \mid \eta] \geq E[u_S(., \eta) \mid \eta]$

(v) $\tilde{u}_A(\theta, \eta) \geq u_A(\theta, \eta)$.

Apart from the smaller range ($K \cup \{e\}$ instead of $\bar{K} \cup \{e\}$) of the reporting strategy $\mu(.)$, the main difference is in the participation constraints (iv) and (v). The outside options correspond to truthful reporting in GC, which forms a noncooperative equilibrium. Hence the outside options correspond exactly to interim payoffs associated

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[24]In Mookherjee et al. (2018), the definition of weak collusion-proof allocation did not include the individual participation constraints of A and S. Here we include them for purposes of comparability.

with the allocation itself. Comparing A's participation constraint (v) between the two definitions, it is evident that A's outside option in strong collusion is lower, by an extent that can be controlled by the coalition by selecting an arbitrary reporting strategy $P(\cdot \mid \eta)$ by S in the event that A refuses to collude. Moreover S's outside option ω in strong collusion is also lower, as it is bounded above by S's equilibrium payoff.[25] Therefore strong collusion permits the third party to offer a wider range of allocations, implying that strong collusion proofness is a more restrictive property compared with weak collusion proofness.

The set of WCP allocations turns out to be independent of *ex ante* bargaining power α.[26] As we show in the next section, this is no longer true for SCP allocations. The WCP notion allows the (S,A) coalition to deviate only when they can find a Pareto improving allocation, while the SCP notion also allows deviations that reduce the welfare of one party if it increases the welfare of the other party sufficiently.

4   The Main Results

One class of allocations that can always be attained by P irrespective of collusion corresponds to not utilizing reports regarding the supervisor's signal η at all. This is the *No Supervision (NS)* organization, in which the class of attainable allocations (denoted $\mathcal{A}^{NS}$) is as follows. There exists a nonnegative constant c and nonincreasing real-valued functions $(X(\theta), Q(\theta))$ defined on Θ such that for any (θ, η):

(a) $u_S(\theta, \eta) = c$

(b) $u_A(\theta, \eta) = X(\theta) - \theta Q(\theta) = \max_{\theta' \in \Theta}[X(\theta') - \theta Q(\theta')]$.

---

[25]This follows from the requirement that the null side contract is feasible in the side contracting problem in strong collusion.

[26]If an allocation is not WCP for some $\alpha \in (0,1)$, there must exist a feasible allocation that *ex ante* Pareto dominates it, so it will not be WCP for any other $\alpha' \in (0,1)$. As shown in Mookherjee et al. (2018), the argument can be extended to include corner values of α owing to the existence of side-transfers.

It is evident that any feasible allocation in NS is also feasible with weak or strong collusion (irrespective of α), since it does not utilize any reports of η.

We now present our first main result.

Proposition 1 *An allocation which is strong collusion proof for any $\alpha \geq \frac{1}{2}$ is also attainable in NS.*

Proof of Proposition 1: Consider any allocation $(u_A(\theta, \eta), u_S(\theta, \eta), q(\theta, \eta))$ which is strong collusion proof for $\alpha \geq \frac{1}{2}$. By Lemma 1, there exists $\omega \geq 0$ and an incentive compatible augmentation (u_A^e, u_S^e, q^e) of this allocation satisfying $u_S(\theta, \eta_0) = \omega$, such that for any η, $(I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$ solves $P^S(\alpha : \eta)$. Let the corresponding coalitional incentive scheme be $(\hat{X}^e(\mu), \hat{q}^e(\mu))$. Define

$$\mu^*(\theta) \in \arg \max_{\mu \in \Delta(\bar{K} \cup \{e\})} [\hat{X}^e(\mu) - \theta \hat{q}^e(\mu)]$$

i.e., a reporting strategy that maximizes the *ex post* joint payoff of A and S in every state.

We claim that

$$(\mu(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta)) = (\mu^*(\theta), \hat{X}(\mu^*(\theta)) - \theta \hat{q}(\mu^*(\theta)) - \omega, \omega)$$

is a solution of $P^S(\alpha : \eta)$ for any η. Upon setting $c = \omega$, $X(\theta) = \hat{X}(\mu^*(\theta)) - \omega$ and $Q(\theta) = \hat{q}(\mu^*(\theta))$, it is evident this claim will imply that the allocation is attainable in NS.

To establish the claim, we first derive an upper bound for the objective function in the problem $P^S(\alpha : \eta)$. From the constraint $E[\tilde{u}_S(\theta, \eta) \mid \eta] \geq \omega$ and the assumption that $\alpha \geq 1/2$, for any reporting strategy $\mu(\theta, \eta)$ the following is true:

$$
\begin{aligned}
& E[\alpha \tilde{u}_A(\theta, \eta) + (1 - \alpha) \tilde{u}_S(\theta, \eta) \mid \eta] \\
= \ & E[\alpha \{\hat{X}^e(\mu(\theta, \eta)) - \theta \hat{q}^e(\mu(\theta, \eta))\} + (1 - 2\alpha) \tilde{u}_S(\theta, \eta) \mid \eta] \\
\leq \ & \alpha E[\hat{X}(\mu^*(\theta)) - \theta \hat{q}(\mu^*(\theta)) \mid \eta] + (1 - 2\alpha)\omega.
\end{aligned}
$$

This upper bound can be attained in $P^S(\alpha : \eta)$ by choosing $\mu(\theta, \eta) = \mu^*(\theta)$,

$$\tilde{u}_A(\theta, \eta) = \hat{X}^e(\mu^*(\theta)) - \theta\hat{q}^e(\mu^*(\theta)) - \omega$$

and $\tilde{u}_S(\theta, \eta) = \omega$ for any $\theta \in \Theta$, and $P(\eta_0 \mid \eta) = 1$ and $P(\eta' \mid \eta) = 0$ for any $\eta' \neq \eta_0$. This allocation satisfies A's participation constraint (v), since

$$\tilde{u}_A(\theta, \eta) = \hat{X}^e(\mu^*(\theta)) - \theta\hat{q}^e(\mu^*(\theta)) - \omega$$
$$\geq \hat{X}^e(\theta, \eta_0) - \theta\hat{q}^e(\theta, \eta_0) - u_S^e(\theta, \eta_0) = u_A^e(\theta, \eta_0).$$

As the other constraints are obviously satisfied, the claim is established. ∎

When A has at least as much bargaining power *ex ante* as S, it is optimal for the coalition to pin S down to her (constant) minmax payoff and provide all residual rents to A. Reports by the coalition are then chosen to maximize A's payoffs, implying that P cannot derive any benefit from appointing S. The one-sided asymmetric information within the coalition implies absence of any frictions in collusion when the informed party A has more bargaining power than S. For P to derive some value from appointing S, she has to exploit some frictions in coalitional bargaining.

Now consider the case where S has higher bargaining weight than A. We impose some structure on A's type space and S's information. One condition is that S should not be 'too well informed' about A's cost; for instance in the extreme case where S is perfectly informed about θ, there will again be no frictions in coalitional bargaining and appointing S will not yield any value to P. For the rest of this section, we focus on the context (denoted *Context C* hereafter) with two possible signal realizations η_1, η_2 satisfying a Monotone Likelihood Ratio Property (MLRP) such that $a(\eta_1 \mid \theta)$ is decreasing (while $a(\eta_2 \mid \theta)$ is increasing) in θ.

Our main result is that in this context, P can derive positive value from appointing S if S has greater bargaining weight than A, for a generic set of information structures. Given the previous result, this implies that (generically) P is better off when S has strictly higher bargaining weight than A, compared to when this is not true.

Proposition 2 *Consider Context C and assume $\alpha \in [0, 1/2)$. If there do not exist $(\rho, \nu, \gamma) \in \Re^3$ such that $a(\eta_1 \mid \theta) = \rho + \nu F(\theta)^\gamma$ for all $\theta \in \Theta$, P can attain a strictly higher expected payoff by appointing S, compared to not appointing S.*

As the proof is relegated to the Appendix, we outline the main steps in the argument here.

First we show that in the problem $P^S(\alpha : \eta)$, S's participation constraint is never binding for $\alpha \in [0, 1/2)$.

Lemma 2 *S's participation condition $E[\tilde{u}_S(\theta, \eta) \mid \eta] \geq \omega$ in $P^S(\alpha : \eta)$ is not binding for any $\alpha \in [0, 1/2)$.*

The reason for this is as follows. If the lemma is false, the solution to the relaxed version of problem $P^S(\alpha : \eta)$ when S's participation constraint is dropped, must violate this constraint, implying that S ends up with an expected payoff below his minmax payoff ω. The coalition has the option of switching to the 'A-residual-claimant' (ARC) side-contract (which is optimal for the coalition when A has more bargaining power, and has been used in the proof of Proposition 1) in which S receives a constant payoff of ω and A receives the rest of the aggregate coalitional rent. ARC induces ex post efficient reporting strategies, thereby (weakly) expanding the aggregate rent in every state. Given $\alpha < \frac{1}{2}$, the third party would not want to deviate to the ARC side-contract only if A appropriates a disproportionate share of the increase in coalitional rents. This implies that A must be better off in the ARC side-contract. But S is also better off in this side contract. It must therefore Pareto dominate the supposed solution, a contradiction.

We can therefore proceed to study problem $P^S(\alpha : \eta)$ in which S's participation constraint is dropped. P augments the mechanism in the manner described in Definition 2, where the auxiliary message η_0 is identified with the high-cost signal report η_2 (i.e., results in the same outcomes). Hence we can confine attention to two possible

signal reports η_1, η_2 for A and S. If both report η_2, P selects the optimal allocation

$$(u_A^{NS}(\theta), u_S^{NS}(\theta), q^{NS}(\theta))$$

in NS satisfying

$$u_A^{NS}(\theta) = \int_\theta^{\bar{\theta}} \bar{q}(y)dy$$

$$u_S^{NS}(\theta) = 0$$

$$q^{NS}(\theta) = \bar{q}(\theta) \equiv \arg\max_q [V(q) - H(\theta)q],$$

where $H(\theta) \equiv \theta + \frac{2k-1}{k}\frac{F(\theta)}{f(\theta)}$ and k denotes the weight assigned to P's profit.[27] Then P's optimal payoff W^{NS} in NS is $E[V(\bar{q}(\theta)) - H(\theta)\bar{q}(\theta)]$.

When both S and A report the low-cost signal η_1, P selects the following variation on the optimal allocation in NS. Let $\beta \equiv \frac{1-2\alpha}{1-\alpha}$, which lies in the interval $(0, 1)$. Let $\Lambda(\cdot) : \Theta \to \Re$ be such that (i) $\Lambda(\theta)$ is non-decreasing in θ with $\Lambda(\underline{\theta}) = 0$ and $\Lambda(\bar{\theta}) = 1$, and (ii) the function $z_\beta(\theta)$ defined by

$$z_\beta(\theta) = \theta + \beta\frac{F(\theta \mid \eta_1) - \Lambda(\theta)}{f(\theta \mid \eta_1)}$$

is nondecreasing. P can then select the output schedule $q(\theta, \eta_1) = \bar{q}(z_\beta(\theta))$. Below we shall describe how this Λ function can be chosen in more detail. $\Lambda(\cdot)$ can be thought of as a variation on the cdf $F(. \mid \eta_1)$. $z_\beta(\theta)$ exceeds or falls below θ according as $\Lambda(\theta)$ is smaller or larger than $F(\theta \mid \eta_1)$, implying in turn that $q(\theta, \eta_1)$ is smaller or larger than $q^{NS}(\theta)$. Given such a variation following reports of a low-cost signal, the payoffs are altered as follows:

$$u_A(\theta, \eta_1) = \int_\theta^{\bar{\theta}} \bar{q}(z_\beta(y))dy$$

$$u_S(\theta, \eta_1) = \bar{X}(z_\beta(\theta)) - \theta\bar{q}(z_\beta(\theta)) - \int_\theta^{\bar{\theta}} \bar{q}(z_\beta(y))dy$$

$$q(\theta, \eta_1) = \bar{q}(z_\beta(\theta))$$

where

$$\bar{X}(z) \equiv z\bar{q}(z) + \int_z^{\bar{\theta}} \bar{q}(y)dy.$$

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[27]See Baron and Myerson (1982).

If both S and A report η_2, the same allocation as in NS is selected:

$$(u_A(\theta, \eta_2), u_S(\theta, \eta_2), q(\theta, \eta_2)) = (u_A^{NS}(\theta), 0, q^{NS}(\theta)).$$

When S and A submit different reports $\eta^S \neq \eta^A$, A is offered the same allocation as in the case where the submitted η reports are η^S for both S and A, while S receives a payment equal to what he would have received if their η reports had been η^A for both S and A, minus a large positive number. This will ensure that the side contract will always involve submission of a common report by S and A, besides individual incentive compatibility for A to report θ truthfully.

The aim is to construct $\Lambda(\cdot)$ with the properties stated above, such that the resulting allocation is SCP and improves P's payoff in state η_1:

$$
\begin{aligned}
& E[V(\bar{q}(z_\beta(\theta))) - \frac{2k-1}{k}\bar{X}(z_\beta(\theta)) - \frac{1-k}{k}\theta\bar{q}(z_\beta(\theta)) \mid \eta_1] \\
> \ & E[V(q^{NS}(\theta)) - \frac{2k-1}{k}\bar{X}(\theta) - \frac{1-k}{k}\theta q^{NS}(\theta) \mid \eta_1].
\end{aligned}
\tag{1}
$$

Since the allocation is unchanged in state η_2, P will achieve a higher payoff than in NS.

The proof shows that such a variation is indeed SCP provided the following two conditions are satisfied:

(a) $E[u_A(\theta, \eta_1) - u_A(\theta, \eta_2) \mid \eta_2] \geq 0$

(b) $\int_{\underline{\theta}}^{\bar{\theta}} [u_A(\theta, \eta_2) - u_A(\theta, \eta_1)] d\Lambda(\theta) \geq 0.$

Intuitively, these two conditions (in combination with the choice of allocation for A corresponding to conflicting η reports as specified above) ensure that a threat by S to report a different signal from the one actually observed in GC if collusion breaks down, does not lower A's expected payoff. S is unable to coerce A to accept a lower payoff in the collusive agreement, compared to a null side contract, thereby ensuring that the allocation is SCP for α.

Conditions (a) and (b) can be rewritten as follows:

$$E[\{\bar{q}(z_\beta(\theta)) - \bar{q}(\theta)\}\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)} \mid \eta_2] \geq 0 \tag{2}$$

and

$$E[(z_\beta(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z_\beta(\theta)) - \bar{q}(\theta)) \mid \eta_1] \geq 0 \tag{3}$$

where $h_\beta(\theta \mid \eta) = \theta + \beta \frac{F(\theta|\eta)}{f(\theta|\eta)}$.[28]

Consider a small variation of the $z_\beta(\theta)$ around θ. The corresponding point-wise variations in the left-hand-sides of (1)-(3) are as follows[29]

$$[V'(\bar{q}(z))\bar{q}'(z) - \frac{2k-1}{k}\bar{X}'(z) - \frac{1-k}{k}\theta\bar{q}'(z)]_{z=\theta}f(\theta \mid \eta_1) = \bar{q}'(\theta)\frac{2k-1}{k}\frac{F(\theta)}{f(\theta)}f(\theta \mid \eta_1), \tag{4}$$

$$[\bar{q}'(z)\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}]_{z=\theta}f(\theta \mid \eta_2) = \bar{q}'(\theta)F(\theta \mid \eta_2), \tag{5}$$

and

$$[(\bar{q}(z) - \bar{q}(\theta)) + (z - h_\beta(\theta \mid \eta_1))\bar{q}'(z)]_{z=\theta}f(\theta \mid \eta_1) = -\beta\bar{q}'(\theta)F(\theta \mid \eta_1). \tag{6}$$

A necessary and sufficient condition for a variation which locally preserves the value of the left-hand-sides of (2) and (3), while increasing the value of the left-hand-side of (1), is that $\frac{F(\theta)}{f(\theta)}f(\theta \mid \eta_1)$ does not lie in the space spanned linearly by $F(\theta \mid \eta_2)$ and $-F(\theta \mid \eta_1)$, which turns out to be equivalent to the generic property stated in Proposition 2.

5 Consequences of Full Delegation of Contracting Authority to One Party

We now show that to realize some benefit from engaging a supervisor, it is essential for P to personally contract with both S and A. Otherwise she would contract with

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[28]We use $E[\int_\theta^{\bar{\theta}} \bar{q}(y)dy \mid \eta] = E[\frac{F(\theta|\eta)}{f(\theta|\eta)}\bar{q}(\theta) \mid \eta]$ to derive these equations.

[29]We use $V'(\bar{q}(z)) = H(z)$ to obtain (4).

only one of them, and effectively fully delegate authority to subcontract with the other. Call an organization where P contracts only with S (resp. A) *delegation of contract to S* (DCS) (resp. *delegation of contract to A* (DCA)). DCS is a special case of GC with $M_A = \phi$ and $X_A = 0$, while DCA is a special case with $M_S = \phi$ and $X_S = 0$. P could continue to retain control over the appointment process, thereby determining α, which determines the objective of the third party that designs the subcontract between A and S.

Consider first DCS, where P contracts only with S. If $\alpha \geq 1/2$, the same argument employed in Proposition 1 shows that there would be no value from employing S. So consider the case where $\alpha < 1/2$, If S exits from the side contract, there is no scope for A to deliver the good to P owing to the lack of a direct contract between P and A. Hence in that event, there will be no production or payments made by P to either S or A, implying both will attain a payoff of zero. For the same reason there will be no production either if A exits from the side contract. Hence outside options for both S and A in the side contract are zero. Moreover, with $X_A \equiv 0$, we have $u_S = X_S - u_A - \theta q$, so the weighted average of S and A's payoff reduces to $\alpha u_A + (1 - \alpha)(X_S - u_A - \theta q) = (1 - \alpha)[X_S - \frac{1-2\alpha}{1-\alpha} u_A - \theta q]$. Hence the optimal subcontract in DCS solves the following problem:

$$\max E[\tilde{X}_S(\mu_S(\theta, \eta)) - \theta \tilde{q}(\mu_S(\theta, \eta)) - \frac{1 - 2\alpha}{1 - \alpha} u_A(\theta, \eta)]$$

subject to $\mu_S(\theta, \eta) \in \Delta(M_S)$ for all (θ, η),

$$u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)\tilde{q}(\mu_S(\theta', \eta))$$

for any $\theta, \theta' \in \Theta$ and any η,

$$u_A(\theta, \eta) \geq 0$$

for all (θ, η) and

$$E[\tilde{X}_S(\mu_S(\theta, \eta)) - \theta \tilde{q}(\mu_S(\theta, \eta)) - u_A(\theta, \eta) \mid \eta] \geq 0$$

25

for all η. As we saw earlier, no incentive constraint for η is needed, since P can elicit this information by cross-checking S and A's reports. The third and the fourth inequalities are the participation constraints of A and S respectively based on zero outside option payoffs.

P's problem of designing a contract for S then reduces to the problem of contracting with a single supplier whose unit supply cost is $\theta + \frac{1-2\alpha}{1-\alpha} \frac{F(\theta|\eta)}{f(\theta|\eta)}$. Since $\alpha < \frac{1}{2}$, this exceeds θ for almost all values of θ. P would then earn higher profit by dispensing with S and contracting directly with A, whose unit cost is almost surely lower.[30] For the same reason as in weak collusion (Mookherjee et al. (2018, Proposition 1)), full delegation of authority to S to contract with A results in double marginalization of rents, which is dominated by the organization NS which does not employ any S.

Next, consider DCA where P does not contract directly with S. With $M_S = \phi$ and $X_S = 0$, S always receives zero payoff by rejecting a subcontract. Moreover, S does not have any ability to penalize A in the event that A exits from the side contract in strong collusion. Hence in the event of either exiting from the subcontract, GC will involve A contracting unilaterally with P according to GC in the absence of any S: A will select a report m_A to maximize $X_A(m_A) - \theta q(m_A)$, while S will receive a zero payoff. These payoffs are independent of η. There is then no scope for collusion, and S's information does not help create any value in the organization, regardless of α.

These arguments are summarized in the following proposition.

Proposition 3 *DCA and DCS are weakly dominated by the organization NS where S is not employed, regardless of α.*

---

[30]Formally, applying standard methods (Myerson (1981), Baron and Myerson (1982)), the optimal subcontract selects a message μ_S which maximizes $\tilde{X}_S(\mu_S) - \hat{h}_\alpha(\theta \mid \eta)\tilde{q}(\mu_S)$ where $\hat{h}_\alpha(\theta \mid \eta)$ is a transformation of $\theta + \frac{1-2\alpha}{1-\alpha} \frac{F(\theta|\eta)}{f(\theta|\eta)}$ by the ironing rule based on $F(\theta \mid \eta)$. Then $\hat{h}_\alpha(\theta \mid \eta) > \theta$ for every $\theta > \underline{\theta}$, and we can apply the argument in Mookherjee et al. (2018, Appendix).

6 Conclusion

In summary, prospects of strong collusion between an informed agent and less-well-informed supervisor can rationalize asymmetric authority granted to the latter over the appointment process. Otherwise if the agent had at least as much bargaining power as the supervisor, the optimal side contract pushes the supervisor down to her minmax payoff and allows the agent to extract all the residual rents. Collusion would then be subject to no frictions, with reports chosen to maximize *ex post* payoffs of the agent, thereby rendering the supervisor redundant. Hence for the Principal to derive some benefit from the supervisor, it is essential that the supervisor has the upper hand at the side-contract design stage. This ensures collusion will be subject to frictions resulting from asymmetric information, inducing trade-offs between the supervisor's rent and the agent's incentives. By contrast, when collusion is weak or absent, the allocation of control authority between supervisor and agent is irrelevant. We also showed the importance of P contracting directly with both parties, instead of delegating choice of the agent's contract to the supervisor. Hence the optimal organizational design blends centralized control over contracts with some delegation of authority over the appointment process.

Some open questions remain. In the context of continuously distributed cost of the agent, we showed that the Principal can benefit from the presence of the supervisor if the latter's signal had only two possible realizations. We do not yet know if this result extends when the signal can take a finite number of realizations. We also do not know if the Principal's payoff is monotone with respect to the allocation of welfare weights over the range where the supervisor has a higher weight. However (as shown in the online Appendix) such a monotonicity result does obtain in a variant of the model with three possible cost types and a partition information structure akin to the model of Celik (2009). It would be interesting to know if this monotonicity property obtains more generally.

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Appendix: Proofs

Proof of Lemma 1

Proof of Necessity

Step 1: Here we define some notations. Consider any PBE(sc) allocation resulting from some grand contract GC. For this GC, define $\underline{w}_S(GC)$ as the minmax value of the S's payoff:

$$\underline{w}_S(GC) \equiv \min_{\mu_A \in \Delta(M_A)} \max_{\mu_S \in \Delta(M_S)} \bar{X}_S(\mu_A, \mu_S).$$

Since $\Delta(M_A), \Delta(M_S)$ (endowed with the weak convergence topology) are compact and $\bar{X}_S$ is continuous for μ_A and μ_S, we can apply the minimax theorem (Nikaido (1954)) to infer that there exists $(\underline{\mu}_A, \bar{\mu}_S)$ which satisfies

$$
\begin{aligned}
\underline{w}_S(GC) = \bar{X}_S(\underline{\mu}_A, \bar{\mu}_S) &= \min_{\mu_A \in \Delta(M_A)} \max_{\mu_S \in \Delta(M_S)} \bar{X}_S(\mu_A, \mu_S) \\
&= \max_{\mu_S \in \Delta(M_S)} \min_{\mu_A \in \Delta(M_A)} \bar{X}_S(\mu_A, \mu_S).
\end{aligned}
$$

where $\underline{\mu}_A$ is A's minmax strategy, and $\bar{\mu}_S$ is S's maxmin strategy. Since S always has the option to exit from the grand contract, $\underline{w}_S(GC) \geq 0$ for any GC.

Given grand contract GC, a reporting strategy for S in this GC: $\mu_S(\eta) \in \Delta(M_S)$ and a type of A: $\theta \in \Theta$, define

$$\hat{u}_A(\theta, \mu_S(\eta), GC) \equiv \max_{\mu_A \in \Delta(M_A)} \bar{X}_A(\mu_A, \mu_S(\eta)) - \theta \bar{q}(\mu_A, \mu_S(\eta)),$$

which is interpreted as the A's maximum payoff in the event that A is of type θ and exits from the side-contract (whence S chooses $\mu_S(\eta)$).

Step 2: We characterize allocations achievable for a given grand contract GC. Let (u_A, u_S, q) denote the allocation achieved as a PBE(sc) outcome in GC, where the third party selects a side contract SC. In this step, we show that for any η and for $\mu(\theta, \eta)$ which satisfies $q(\theta, \eta) = \tilde{q}(\mu(\theta, \eta))$ for all $\theta \in \Theta$,

$$(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), \tilde{\mu}(\theta, \eta)) = (u_A(\theta, \eta), u_S(\theta, \eta), \mu(\theta, \eta)),$$

associated with the selection of some $\tilde{\mu}_S(\eta) = \mu_S(\eta)$, solves the following problem $P^S(\alpha : \eta, GC)$:

$$\max E[\alpha \tilde{u}_A(\theta, \eta) + (1 - \alpha)\tilde{u}_S(\theta, \eta) \mid \eta]$$

subject to the constraint that for some $\tilde{\mu}_S(\eta) \in \Delta(M_S)$ and for all $\theta \in \Theta(\eta)$:

(i) $\tilde{\mu}(\theta, \eta) \in \Delta(M_A \times M_S), \tilde{u}_A(\theta, \eta) \in \Re, \tilde{u}_S(\theta, \eta) \in \Re$

(ii) $\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\tilde{q}(\tilde{\mu}(\theta', \eta))$ for any $\theta' \in \Theta$

(iii) $\tilde{u}_A(\theta, \eta) \geq \hat{u}_A(\theta, \tilde{\mu}_S(\eta), GC)$

(iv) $E[\tilde{u}_S(\theta, \eta) \mid \eta] \geq \underline{w}_S(GC)$ and

(v) $\tilde{X}_A(\tilde{\mu}(\theta, \eta)) + \tilde{X}_S(\tilde{\mu}(\theta, \eta)) - \theta\tilde{q}(\tilde{\mu}(\theta, \eta)) = \tilde{u}_A(\theta, \eta) + \tilde{u}_S(\theta, \eta).$

Note that this problem $P^S(\alpha : \eta, GC)$ includes S's threat $\tilde{\mu}_S(\eta)$ in the event of A's non-participation as a choice variable. So the set of control variables can be written as $(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), \tilde{\mu}(\theta, \eta), \tilde{\mu}_S(\eta))$.

Note also that the problem $P^S(\alpha : \eta, GC)$ refers to the given GC, and reporting strategies of the players are confined to mixed strategies available in GC. In later steps, the mechanism will be augmented so that the scope of collusion will be widened, as players will then be able to select mixed strategies in augmented message spaces.

Proof of Step 2: Since (u_A, u_S, q) is an achievable allocation, it is straightforward to check that it is feasible in the above problem. If $(u_A(\theta, \eta), u_S(\theta, \eta), \mu(\theta, \eta), \mu_S(\eta))$

does not solve problem $P^S(\eta : \alpha, GC)$ for some η, we shall now show that there exists another side-contract and a continuation equilibrium in which the third party can achieve a higher payoff, which will contradict the hypothesis that the allocation resulted from a PBE(sc) of GC. Suppose that for some η, the solution of $P(\eta : \alpha, GC)$ is instead some $(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta), \tilde{\mu}^*(\theta, \eta), \tilde{\mu}_S^*(\eta)) \neq (u_A(\theta, \eta), u_S(\theta, \eta), \mu(\theta, \eta), \mu_S(\eta))$.

Construct a side-contract SC' as follows. If both S and A accept it, the third party requests a report from A of $(\theta_A, \eta_A) \in K$, and report from S of $\eta_S \in \Pi$. The report to P is subsequently selected according to $\tilde{\mu}^*(\theta_A, \eta_S)$, while side-transfers are selected as follows.

$$t_A(\theta_A, \eta_A, \eta_S) = \tilde{u}_A^*(\theta_A, \eta_S) - [\tilde{X}_A(\tilde{\mu}^*(\theta_A, \eta_S)) - \theta_A \tilde{q}(\tilde{\mu}^*(\theta_A, \eta_S))] - L(\eta_A, \eta_S)$$

and

$$t_S(\theta_A, \eta_A, \eta_S) = \tilde{u}_S^*(\theta_A, \eta_A) - \tilde{X}_S(\tilde{\mu}^*(\theta_A, \eta_S))$$

where $L(\eta_A, \eta_S)$ is zero for $\eta_A = \eta_S$ and a large positive number for $\eta_A \neq \eta_S$. If A were to accept and S were to reject SC', A would threaten to play $\underline{\mu}_A$. Conversely, if S accepts and reports η_S while A rejects SC', S threatens to play $\tilde{\mu}_S^*(\eta_S)$. It is easy to check that there exists a continuation equilibrium where nobody rejects SC' on the equilibrium path, and both A and S report truthfully to the third party, resulting in the allocation $(\tilde{u}_A^*(\theta, \eta), \tilde{u}_S^*(\theta, \eta))$. The third party attains a higher payoff, contradicting the hypothesis that we started with a PBE(sc), completing the proof of Step 2. ∎

The statement of Step 2 provides one characterization of allocations achievable for a given grand contract GC. Our aim is to find a more general characterization which does not depend on GC. This is the purpose of the following step.

Step 3: We continue with (u_A, u_S, q), an achievable allocation in GC in strong collusion with α. Evidently (u_A, u_S, q) is IIC. Consider an augmentation (u_A^e, u_S^e, q^e) of (u_A, u_S, q) to the domain $\bar{K}$ with the selection of

$$(u_A^e(\theta, \eta_0), u_S^e(\theta, \eta_0), q^e(\theta, \eta_0)) \equiv (\hat{u}_A(\theta, \bar{\mu}_S, GC), \omega, \bar{q}(\mu_A(\theta, \bar{\mu}_S), \bar{\mu}_S))$$

where $\mu_A(\theta, \bar{\mu}_S)$ maximizes $\bar{X}_A(\mu_A, \bar{\mu}_S) - \theta \bar{q}(\mu_A, \bar{\mu}_S)$ subject to $\mu_A \in \Delta(M_A)$, and $\omega \equiv \underline{w}_S(GC)$. By the definition, $\omega \geq 0$ and $(u_A^e(\theta, \eta_0), q^e(\theta, \eta_0))$ satisfies (IC_A) and (PC_A).

Now consider the problem $P^S(\alpha : \eta)$ defined by the augmented allocation (u_A^e, u_S^e, q^e). Note that this problem differs from the one considered in Step 2 $(P^S(\alpha : \eta, GC))$, as it no longer refers to the original GC, and the coalition selects reports from the augmented message space $\bar{K} \cup \{e\}$ rather than $M_A \times M_S$.

We show in this step that

$$(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), P(\cdot \mid \eta)) = (I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$$

solves problem $P^S(\alpha : \eta)$.

It is straightforward to check that $(I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$ satisfies all constraints of $P^S(\alpha : \eta)$, and generates a payoff for the third party of

$$E[\alpha u_A(\theta, \eta) + (1 - \alpha)u_S(\theta, \eta) \mid \eta].$$

Suppose that there exists some alternative choice of controls

$$(\mu^*(\theta, \eta), u_A^*(\theta, \eta), u_S^*(\theta, \eta), P^*(\cdot \mid \eta))$$

which is feasible in $P^S(\alpha : \eta)$, such that

$$E[\alpha u_A^*(\theta, \eta) + (1 - \alpha)u_S^*(\theta, \eta) \mid \eta] > E[\alpha u_A(\theta, \eta) + (1 - \alpha)u_S(\theta, \eta) \mid \eta].$$

We show that in such a case there would exist $\tilde{\mu}(\theta, \eta) : K \to \Delta(M_A \times M_S), \tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), \tilde{\mu}_S(\eta)$ which would be feasible in $P^S(\alpha : \eta, GC)$ and generate a higher value in that problem compared to $(\mu(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), \mu_S(\eta))$, thereby contradicting the result established at Step 2.

$\mu^*(\theta, \eta)$, which is a probability measure on $\bar{K} \cup \{e\}$, divides its weight between reports either in $K \cup \{e\}$ or satisfying $\eta = \eta_0$. The former event corresponds to an outcome of GC that results when S and A's reports are chosen from M_S and M_A respectively. And the latter event corresponds (by specification of $(u_A^e(\theta, \eta_0), u_S^e(\theta, \eta_0), q^e(\theta, \eta_0))$)

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to an outcome of GC resulting when S reports $\bar{\mu}_S \in \Delta(M_S)$ and A reports according to $\mu_A(\theta, \bar{\mu}_S) \in \Delta(M_A)$. In this case,

$$\bar{q}(\mu_A(\theta, \bar{\mu}_S), \bar{\mu}_S) = q^e(\theta, \eta_0)$$

while

$$\hat{X}^e(\theta, \eta_0) = \omega + \bar{X}_A(\mu_A(\theta, \bar{\mu}_S), \bar{\mu}_S) \leq \bar{X}_S(\mu_A(\theta, \bar{\mu}_S), \bar{\mu}_S) + \bar{X}_A(\mu_A(\theta, \bar{\mu}_S), \bar{\mu}_S)$$

since ω is S's minmax payoff in GC. Hence the outcome of $\mu^*(\theta, \eta)$ in $P^S(\alpha : \eta)$ can be attained by the coalition as an outcome of GC resulting from some reporting strategy $\tilde{\mu}(\theta, \eta) \in \Delta(M_A \times M_S)$ that satisfies

$$\tilde{X}_A(\tilde{\mu}(\theta, \eta)) + \tilde{X}_S(\tilde{\mu}(\theta, \eta)) \geq \hat{X}^e(\mu^*(\theta, \eta))$$

and

$$\tilde{q}(\tilde{\mu}(\theta, \eta)) = \hat{q}^e(\mu^*(\theta, \eta)).$$

Let $\mu_S(\eta)$ denote the optimal threat chosen by S in the event that A does not participate in the side-contract, in the solution to problem $P^S(\alpha : \eta, GC)$. Let us select $\mu_S(\eta_0) \equiv \bar{\mu}_S$. Then $u_A^e(\theta, \eta) \geq \hat{u}_A(\theta, \mu_S(\eta), GC)$ for any $\eta \in \bar{\Pi}$. Define $\tilde{\mu}_S(\eta) \in \Delta(M_S)$ as the composite of the measures $\mu_S(\eta')$ and $P^*(\eta' \mid \eta)$. Then by the definition of $\hat{u}_A(\theta, \mu_S, GC)$,

$$\Sigma_{\eta' \in \bar{\Pi}} P^*(\eta' \mid \eta) u_A^e(\theta, \eta') \geq \Sigma_{\eta' \in \bar{\Pi}} P^*(\eta' \mid \eta) \hat{u}_A(\theta, \mu_S(\eta'), GC) \geq \hat{u}_A(\theta, \tilde{\mu}_S(\eta), GC).$$

Since $u_A^*(\theta, \eta) \geq \Sigma_{\eta' \in \bar{\Pi}} P^*(\eta' \mid \eta) u_A^e(\theta, \eta')$, it follows that $u_A^*(\theta, \eta) \geq \hat{u}_A(\theta, \tilde{\mu}_S(\eta), GC)$. Defining $\tilde{u}_A(\theta, \eta) \equiv u_A^*(\theta, \eta)$ and

$$\tilde{u}_S(\theta, \eta) \equiv \tilde{X}_A(\tilde{\mu}(\theta, \eta)) + \tilde{X}_S(\tilde{\mu}(\theta, \eta)) - \theta q(\tilde{\mu}(\theta, \eta)) - u_A^*(\theta, \eta),$$

we infer that $(\tilde{u}_A(\theta, \eta), \tilde{u}_S(\theta, \eta), \tilde{\mu}(\theta, \eta), \tilde{\mu}_S(\eta))$ is feasible in the problem $P^S(\alpha : \eta, GC)$, and $\tilde{u}_S(\theta, \eta) \geq u_S^*(\theta, \eta)$. Hence it generates a higher payoff for the third party than $E[\alpha u_A(\theta, \eta) + (1 - \alpha) u_S(\theta, \eta) \mid \eta]$, and we obtain a contradiction to the result of Step

2. So $(I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$ must be a solution of $P^S(\alpha : \eta)$, establishing the necessity of the SCP property.

Proof of Sufficiency: Let (u_A^e, u_S^e, q^e) be an augmentation of (u_A, u_S, q) for which the latter satisfies the SCP property. By Definition 3, $u_S^e(\theta, \eta_0) = \omega$ for any $\theta \in \Theta$ and $(u_A^e(\theta, \eta_0), q^e(\theta, \eta_0))$ satisfies (IC_A) and (PC_A). P can construct a grand contract GC as follows:

$$(X_A(m_A, m_S), X_S(m_A, m_S), q(m_A, m_S) : M_A, M_S)$$

where $M_A = K \cup \{e_A\}$, $M_S = \bar{\Pi} \cup \{e_S\}$ and

- for any $(\theta, \eta) \in K$ and $\eta' \in \Pi$, choose $(X_A((\theta, \eta), \eta'), X_S((\theta, \eta), \eta'), q((\theta, \eta), \eta')) = (u_A^e(\theta, \eta') + \theta q^e(\theta, \eta'), u_S^e(\theta, \eta) - L(\eta, \eta'), q^e(\theta, \eta'))$ where $L(\eta, \eta') = 0$ for $\eta = \eta'$ and $L > 0$ (and sufficiently large) for $\eta \neq \eta'$

- $(X_A((\theta, \eta), e_S), X_S((\theta, \eta), e_S), q((\theta, \eta), e_S)) = (u_A^e(\theta, \eta_0) + \theta q^e(\theta, \eta_0), 0, q^e(\theta, \eta_0))$.

- $(X_A((\theta, \eta), \eta_0), X_S((\theta, \eta), \eta_0), q((\theta, \eta), \eta_0)) = (u_A^e(\theta, \eta_0) + \theta q^e(\theta, \eta_0), \omega, q^e(\theta, \eta_0))$.

- $(X_A(e_A, m_S), X_S(e_A, m_S), q(e_A, m_S)) = (0, 0, 0)$ for any $m_S \neq \eta_0$

- $(X_A(e_A, \eta_0), X_S(e_A, \eta_0), q(e_A, \eta_0)) = (0, \omega, 0)$.

It is easy to check that $(\mu_A, \mu_S) = ((\theta, \eta), \eta)$ is a non-cooperative equilibrium of GC, and S's minmax payoff in GC is ω. The SCP property of (u_A, u_S, q) implies there is no room for the third party to improve its payoff by offering a deviating side-contract, so (u_A, u_S, q) is realized as the outcome of a PBE(sc) under GC. ∎

Proof of Proposition 2

We start with the proof of Lemma 2.

Proof of Lemma 2: Suppose that for some η, $(I(\theta, \eta), u_A(\theta, \eta), u_S(\theta, \eta), I(\eta))$ does not solve the relaxed version of $P^S(\alpha : \eta)$ where the constraint $E[\tilde{u}_S(\theta, \eta) \mid \eta] \geq \omega$ is

dropped. It implies $E[\tilde{u}_S^r(\theta, \eta) \mid \eta] < \omega$ in the optimal solution of the relaxed problem represented by

$$(\mu^r(\theta, \eta), \tilde{u}_A^r(\theta, \eta), \tilde{u}_S^r(\theta, \eta), P^r(\cdot \mid \eta)).$$

As shown in the proof of Proposition 1, side contract $\tilde{SC}$ defined as follows is feasible in $P^S(\alpha : \eta)$, hence also in the relaxed problem:

- $\tilde{\mu}(\theta, \eta) = \mu^*(\theta)$ which maximizes $\hat{X}^e(\mu) - \theta \hat{q}^e(\mu)$ subject to $\mu \in \Delta(\bar{K} \cup \{e\})$

- $P(\eta_0 \mid \eta) = 1$ and $P(\eta' \mid \eta) = 0$ for any $\eta' \neq \eta_0$

- $\tilde{u}_A(\theta, \eta) = \hat{X}^e(\mu^*(\theta)) - \theta \hat{q}^e(\mu^*(\theta)) - \omega$ (denoted by $u_A^+(\theta, \eta)$ in later part)

- $\tilde{u}_S(\theta, \eta) = \omega$

Hence

$$E[\alpha \tilde{u}_A^r(\theta, \eta) + (1 - \alpha)\tilde{u}_S^r(\theta, \eta) \mid \eta]$$
$$= E[(1 - \alpha)\{\hat{X}^e(\mu^r(\theta, \eta)) - \theta \hat{q}^e(\mu^r(\theta, \eta))\} - (1 - 2\alpha)\tilde{u}_A^r(\theta, \eta) \mid \eta]$$
$$\geq E[(1 - \alpha)\{\hat{X}^e(\mu^*(\theta)) - \theta \hat{q}^e(\mu^*(\theta))\} - (1 - 2\alpha)u_A^+(\theta, \eta) \mid \eta].$$

But since $E[\hat{X}^e(\tilde{\mu}^r(\theta, \eta)) - \theta \hat{q}^e(\tilde{\mu}^r(\theta, \eta)) \mid \eta] \leq E[\hat{X}^e(\mu^*(\theta)) - \theta \hat{q}^e(\mu^*(\theta)) \mid \eta]$ by the definition of $\mu^*(\theta)$, $\alpha < \frac{1}{2}$ implies that $E[u_A^+(\theta, \eta) \mid \eta] \geq E[\tilde{u}_A^r(\theta, \eta) \mid \eta]$. This implies that the side contract $\tilde{SC}$ creates a Pareto improvement over the solution to the relaxed problem, yielding a strictly higher value of the third party's expected payoff, a contradiction. ∎

The next step in the proof of Proposition 2 is to consider the specific mechanism described in the text; we establish this allocation is SCP provided conditions (a) and (b) are satisfied.

Owing to the previous lemma, we can drop S's participation constraint (iv) from problem $P^S(\alpha : \eta)$. So consider the relaxed problem denoted by $\bar{P}^S(\alpha : \eta)$, for this allocation defined on $\Theta \times \{\eta_1, \eta_2\}$, which selects $(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), p(\eta))$ to maximize

$$E[\hat{X}(\mu(\theta, \eta)) - \theta \hat{q}(\mu(\theta, \eta)) - \beta \tilde{u}_A(\theta, \eta) \mid \eta]$$

subject to $\mu(\theta, \eta) \in \Delta(\Theta \times \{\eta_1, \eta_2\} \cup \{e\})$ and $p(\eta) \in [0, 1]$,

$$\tilde{u}_A(\theta, \eta) \geq p(\eta)u_A(\theta, \eta) + (1 - p(\eta))u_A(\theta, \eta')$$

and

$$\tilde{u}_A(\theta, \eta) \geq \tilde{u}_A(\theta', \eta) + (\theta' - \theta)\hat{q}(\mu(\theta', \eta))$$

for any $\theta, \theta' \in \Theta$.

Specifically, we aim to show that $(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), p(\eta)) = ((\theta, \eta), u_A(\theta, \eta), 1)$ solves $\bar{P}^S(\alpha : \eta)$, if

(a) $E[u_A(\theta, \eta_1) - u_A(\theta, \eta_2) \mid \eta_2] \geq 0$

(b) $\int_{\underline{\theta}}^{\bar{\theta}}[u_A(\theta, \eta_2) - u_A(\theta, \eta_1)]d\Lambda(\theta) \geq 0$.

Upon choosing $\Lambda(., \eta_1) \equiv \Lambda(.)$ and $\Lambda(., \eta_2) \equiv F(.|\eta_2)$, we can unify conditions (a) and (b) into the following single condition

$$\int_{\underline{\theta}}^{\bar{\theta}} [u_A(\theta, \eta') - u_A(\theta, \eta)]d\Lambda(\theta, \eta) \geq 0$$

when $\eta, \eta' \in \{\eta_1, \eta_2\}$ and $\eta \neq \eta'$.

Since $\Lambda(\theta, \eta)$ is non-decreasing in θ, this condition implies that

$$
\begin{aligned}
0 &\leq \int_{\underline{\theta}}^{\bar{\theta}} [\tilde{u}_A(\theta, \eta) - p(\eta)u_A(\theta, \eta) - (1 - p(\eta))u_A(\theta, \eta')]d\Lambda(\theta, \eta) \\
&\leq \int_{\underline{\theta}}^{\bar{\theta}} [\tilde{u}_A(\theta, \eta) - u_A(\theta, \eta)]d\Lambda(\theta, \eta)
\end{aligned}
$$

for any $(\tilde{u}_A(\theta, \eta), p(\eta))$ satisfying constraints of $\bar{P}^S(\alpha : \eta)$. This result can be used to obtain an upper bound of the objective function in $\bar{P}^S(\alpha : \eta)$. First note that

$$
\begin{aligned}
&E[\hat{X}(\mu(\theta, \eta)) - \theta\hat{q}(\mu(\theta, \eta)) - \beta\tilde{u}_A(\theta, \eta) \mid \eta] \\
&\leq E[\hat{X}(\mu(\theta, \eta)) - \theta\hat{q}(\mu(\theta, \eta)) - \beta\tilde{u}_A(\theta, \eta) \mid \eta] + \beta \int_{\underline{\theta}}^{\bar{\theta}} [\tilde{u}_A(\theta, \eta) - u_A(\theta, \eta)]d\Lambda(\theta, \eta) \\
&= E[\hat{X}(\mu(\theta, \eta)) - z_\beta(\theta, \eta)\hat{q}(\mu(\theta, \eta)) \mid \eta] - \beta \int_{\underline{\theta}}^{\bar{\theta}} u_A(\theta, \eta)d\Lambda(\theta, \eta).
\end{aligned}
$$

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The second equality uses the fact that

$$\tilde{u}_A(\theta, \eta) = \tilde{u}_A(\bar{\theta}, \eta) + \int_\theta^{\bar{\theta}} \hat{q}(\tilde{\mu}(y, \eta)) dy.$$

Next, note that $\tilde{\mu} = (\theta, \eta)$ maximizes $\hat{X}(\tilde{\mu}) - z_\beta(\theta, \eta)\hat{q}(\tilde{\mu})$. This implies that an upper bound to the value of the objective function is given by :[31]

$$E[\bar{X}(z_\beta(\theta, \eta)) - \theta\bar{q}(z_\beta(\theta, \eta)) - \beta u_A(\theta, \eta) \mid \eta].$$

But this is attainable with $(\mu(\theta, \eta), \tilde{u}_A(\theta, \eta), p(\eta)) = ((\theta, \eta), u_A(\theta, \eta), 1)$ (which satisfies all constraints) in $\bar{P}^S(\alpha : \eta)$, implying that it is the optimal solution of this problem. This implies the allocation is SCP.

Let $Z(\eta_1)$ denote the set of non-decreasing functions $z : \Theta \to \Re$ such that $z(\theta) = \theta + \beta\frac{F(\theta|\eta_1) - \Lambda(\theta)}{f(\theta|\eta_1)}$ for some $\Lambda(\theta)$ which is non-decreasing in θ with $\Lambda(\underline{\theta}) = 0$ and $\Lambda(\bar{\theta}) = 1$. In order to prove Proposition 2, it suffices to construct $z_\beta(\cdot) \in Z(\eta_1)$ where (1), (2) and (3) are satisfied at the same time. The rest of the proof is devoted to this construction.

Step 1: Under the hypothesis of Proposition 2, there exist (λ_1, λ_2) and closed intervals on Θ ($\Theta_1 = [\underline{\theta}_1, \bar{\theta}_1]$, $\Theta_2 = [\underline{\theta}_2, \bar{\theta}_2]$ and $\Theta_3 = [\underline{\theta}_3, \bar{\theta}_3]$) such that $\underline{\theta} < \underline{\theta}_i < \bar{\theta}_i < \underline{\theta}_{i+1} < \bar{\theta}_{i+1} < \bar{\theta}$ ($i = 1, 2$), and the sign of

$$\frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)$$

alternates among the interiors of Θ_1, Θ_2 and Θ_3. The proof is provided in the online Appendix. In later analysis, our focus is restricted to the case that there exists (λ_1, λ_2) such that

$$\frac{F(\theta)}{f(\theta)} f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)$$

---

[31]By definition of $(\hat{X}(\mu), \hat{q}(\mu))$,

$$\hat{X}(\theta, \eta) - z_\beta(\theta, \eta)\hat{q}(\theta, \eta) = \bar{X}(z_\beta(\theta, \eta)) - z_\beta(\theta, \eta)\bar{q}(z_\beta(\theta, \eta)).$$

is negative on the interior of Θ_1 and Θ_3, and positive on the interior of Θ_2. We can adopt the same analysis for the opposite case.

Step 2: For any closed interval $[\theta', \theta''] \subset \Theta$ such that $\underline{\theta} < \theta' < \theta'' < \bar{\theta}$, there exists $\delta > 0$ so that $z(\cdot) \in Z(\eta_1)$ for any function $z(\cdot)$ satisfying the following properties:

(i) $z(\cdot)$ is increasing and differentiable with $|z(\theta) - \theta| < \delta\beta$ and $|z'(\theta) - 1| < \delta\beta$ for any $\theta \in \Theta$

(ii) $z(\theta) = \theta$ for any $\theta \notin [\theta', \theta'']$.

The proof is provided in the online Appendix.

Step 3: Here we construct $z_\beta(\cdot) \in Z(\eta_1)$ where $(1) - (3)$ are satisfied at the same time. To simplify the notation, we use $z(\cdot)$ instead of $z_\beta(\cdot)$ in later argument. The construction of $z(\theta)$ has the following four steps.

(i) Construction of $\bar{z}(\cdot)$

First let us define $\Phi(z, \theta)$ by

$$\Phi(z, \theta) \equiv [H(z) - \frac{2k-1}{k}z - \frac{(1-k)}{k}\theta + \frac{(2k-1)\lambda_2}{k\beta}(z - h_\beta(\theta \mid \eta_1)) + \frac{2k-1}{k}\lambda_1\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_1)}]\bar{q}'(z)$$
$$+ \frac{(2k-1)\lambda_2}{k\beta}[\bar{q}(z) - \bar{q}(\theta)]$$

where $h_\beta(\theta \mid \eta) \equiv \theta + \beta\frac{F(\theta \mid \eta)}{f(\theta \mid \eta)}$. With $z = \theta$,

$$\Phi(\theta, \theta) \equiv \frac{2k-1}{kf(\theta \mid \eta_1)}[\frac{F(\theta)}{f(\theta)}f(\theta \mid \eta_1) + \lambda_1 F(\theta \mid \eta_2) - \lambda_2 F(\theta \mid \eta_1)]\bar{q}'(\theta).$$

Since $\Phi(z, \theta)$ is differentiable in z and θ, the statement in Step 1 guarantees the existence of $\bar{z}(\theta)$ such that (i) $\bar{z}(\theta)$ is differentiable on Θ, (ii) $\bar{z}(\theta) > \theta$ on $(\underline{\theta}_1, \bar{\theta}_1)$ and $\Phi(z, \theta) > 0$ for any $z \in [\theta, \bar{z}(\theta)]$ and any $\theta \in (\underline{\theta}_1, \bar{\theta}_1)$, (iii) $\bar{z}(\theta) < \theta$ on $(\underline{\theta}_2, \bar{\theta}_2)$ and $\Phi(z, \theta) < 0$ for any $z \in [\bar{z}(\theta), \theta]$ and any $\theta \in (\underline{\theta}_2, \bar{\theta}_2)$, (iv) $\bar{z}(\theta) > \theta$ on $(\underline{\theta}_3, \bar{\theta}_3)$ and $\Phi(z, \theta) > 0$ for any $z \in [\theta, \bar{z}(\theta)]$ and any $\theta \in (\underline{\theta}_3, \bar{\theta}_3)$, and (v) $\bar{z}(\theta) = \theta$ elsewhere.

(ii) Construction of $z_1(\cdot)$

For $\hat{\theta}_1 \in (\bar{\theta}_1, \underline{\theta}_2)$ and $\hat{\theta}_2 \in (\bar{\theta}_2, \underline{\theta}_3)$ (chosen arbitrary), ρ_1 and ρ_2 are defined by

$$\rho_1 \equiv \frac{F(\hat{\theta}_1 \mid \eta_2)}{F(\hat{\theta}_1 \mid \eta_1)}$$

and

$$\rho_2 \equiv \frac{F(\hat{\theta}_2 \mid \eta_2)}{F(\hat{\theta}_2 \mid \eta_1)}.$$

Then define

$$\Psi_i(z, \theta) \equiv [\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_1)} + \frac{\rho_i}{\beta}(z - h_\beta(\theta \mid \eta_1))]\bar{q}'(z) + \frac{\rho_i}{\beta}(\bar{q}(z) - \bar{q}(\theta)).$$

$z_1(\theta)$ is defined such that $\Psi_1(z_1(\theta), \theta) = 0$ is satisfied. There always exists such a $z_1(\theta)$, since for each θ, $\Psi_i(z, \theta)$ is continuous for z and is negative for $z > \max\{\theta, h_\beta(\theta \mid \eta_1) - \frac{\beta F(\theta|\eta_2)}{\rho_1 f(\theta|\eta_1)}\}$ and is positive for $z < \min\{\theta, h_\beta(\theta \mid \eta_1) - \frac{\beta F(\theta|\eta_2)}{\rho_1 f(\theta|\eta_1)}\}$. It also implies that $z_1(\theta) < h_\beta(\theta \mid \eta)$ for any θ. If there are multiple z which satisfies $\Psi_1(z, \theta) = 0$, we choose one which is the closest to θ. Then rewriting $\Psi_1(z_1(\theta), \theta) = 0$, we obtain

$$z_1(\theta) - \theta + \frac{\bar{q}(z_1(\theta)) - \bar{q}(\theta)}{\bar{q}'(z_1(\theta))} = \frac{\beta F(\theta \mid \eta_1)}{\rho_1 f(\theta \mid \eta_1)}[\rho_1 - \frac{F(\theta \mid \eta_2)}{F(\theta \mid \eta_1)}].$$

Since $\frac{F(\theta|\eta_2)}{F(\theta|\eta_1)}$ is increasing in θ by the monotone likelihood ratio assumption, $z_1(\theta) > \theta$ for $\theta < \hat{\theta}_1$ and $z_1(\theta) < \theta$ for $\theta > \hat{\theta}_1$. Since $\Psi_1(\theta, \theta) > 0$ (or < 0) for $\theta < \hat{\theta}_1$ (or $\theta > \hat{\theta}_1$), $\Psi_1(z, \theta) > 0$ for any $z \in (\theta, z_1(\theta))$ and for any $\theta < \hat{\theta}_1$ and $\Psi_1(z, \theta) < 0$ for any $z \in (z_1(\theta), \theta)$ and for any $\theta > \hat{\theta}_1$. On the other hand, $\Psi_2(z, \theta)$ is positive for (θ, z) such that $z < \theta < \hat{\theta}_2$ and negative for (θ, z) such that $\hat{\theta}_2 < \theta < z$ from the definition of $\Psi_2(z, \theta)$ and θ_2. Then the argument is summarized as

- For $z \in (\theta, z_1(\theta))$, $\Psi_1(z, \theta) > 0$ for any $\theta \in \Theta_1$

- For $z \in (z_1(\theta), \theta)$, $\Psi_1(z, \theta) < 0$ and $\Psi_2(z, \theta) > 0$ for any $\theta \in \Theta_2$

- For $z > \theta$, $\Psi_2(z, \theta) < 0$ for any $\theta \in \Theta_3$.

(iii) Construction of $z_2(\cdot)$

Next let us define

$$\Gamma(z,\theta) \equiv \frac{d[(z - h_\beta(\theta \mid \eta_1))(\bar{q}(z) - \bar{q}(\theta))]}{dz} = \bar{q}(z) - \bar{q}(\theta) + (z - h_\beta(\theta \mid \eta_1))\bar{q}'(z).$$

$\Gamma(z,\theta) > 0$ for $z \leq \theta$ and $\Gamma(z,\theta) < 0$ at $z = h_\beta(\theta \mid \eta_1)$. Then we can choose $z_2(\theta)(> \theta)$ which is the minimum z such that $\Gamma(z,\theta) = 0$. Therefore $(z - h_\beta(\theta \mid \eta_1))(\bar{q}(z) - \bar{q}(\theta))$ is increasing in z on $z < z_2(\theta)$.

(iv) Construction of $z(\cdot)$

Finally let us construct $z(\cdot)$, based on $\bar{z}(\cdot)$, $z_1(\cdot)$ and $z_2(\cdot)$. According to the procedure in Step 2, for $[\theta', \theta''] = [\underline{\theta}_1, \bar{\theta}_3]$, choose $\delta > 0$. We construct $z(\theta)$ as follows:

(i) $z(\theta)$ is differentiable and increasing in θ on Θ with $|z(\theta) - \theta| < \delta\beta$ and $|z'(\theta) - 1| < \delta\beta$

(ii) $z(\theta) \in (\theta, \min\{\bar{z}(\theta), z_1(\theta), z_2(\theta)\})$ on $(\underline{\theta}_1, \bar{\theta}_1)$

(iii) $z(\theta) \in (\max\{\bar{z}(\theta), z_1(\theta)\}, \theta)$ on $(\underline{\theta}_2, \bar{\theta}_2)$

(iv) $z(\theta) \in (\theta, \min\{\bar{z}(\theta), z_2(\theta)\})$ on $(\underline{\theta}_3, \bar{\theta}_3)$

(v) $z(\theta) = \theta$ elsewhere

(vi) $E[(z(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_1] = 0$

(vii) $E[(\bar{q}(\theta) - \bar{q}(z(\theta)))\frac{F(\theta|\eta_2)}{f(\theta|\eta_2)} \mid \eta_2] = 0$.

(i) implies $z(\theta) \in Z(\eta_1)$. We argue that there exists $z(\theta)$ which satisfies (i)-(vii). It is evident that there exists $z(\cdot)$ which satisfies (i)-(v). In addition, since $(z - h_\beta(\theta \mid \eta_1))(\bar{q}(z) - \bar{q}(\theta))$ is increasing in z for $z < z_2(\theta)$, $z(\theta) > \theta$ on Θ_1 and Θ_3 (or $z(\theta) < \theta$ on Θ_2) has the effect on raising (or reducing) $E[(z(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_1]$ away from zero. By making a balance between two effects, $z(\cdot)$ can also satisfy (vi).

Suppose $z(\cdot)$ which satisfies (i)-(vi), but does not satisfy (vii). It is shown that we can construct a new function which satisfies all of (i)-(vii) with small adjustment of

$z(\cdot)$. First we define $\tilde{z}(\cdot, \epsilon)$ ($\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$) as $\tilde{z}(\theta, \epsilon) \equiv \theta + \epsilon_i(z(\theta) - \theta)$ on Θ_i ($i = 1, 2, 3$) and $\tilde{z}(\theta, \epsilon) = \theta$ elsewhere. It is evident that for any $\epsilon_i \in (0, 1]$ ($i = 1, 2, 3$), $\tilde{z}(\cdot, \epsilon)$ satisfies (i)-(v), since $\tilde{z}(\cdot, \epsilon)$ is closer to $\hat{\theta}(\cdot)$ than $z(\cdot)$ in both the distance and the slope. For the convenience of the exposition, define $\Pi(\epsilon_1, \epsilon_2, \epsilon_3)$ as

$$\Pi(\epsilon_1, \epsilon_2, \epsilon_3) \equiv E[(\tilde{z}(\theta, \epsilon) - h_\beta(\theta \mid \eta_1))(\bar{q}(\tilde{z}(\theta, \epsilon)) - \bar{q}(\theta)) \mid \eta_1].$$

It is evident that $\Pi(1, 1, 1) = 0$, since $z(\cdot)$ satisfies (vi), and $\Pi(0, 0, 0) = 0$. $\Pi(\epsilon_1, \epsilon_2, \epsilon_3)$ is continuous for each ϵ_i ($i = 1, 2, 3$), increasing in ϵ_1 and ϵ_3 and decreasing in ϵ_2. Then since $\Pi(1, 0, 0) > 0$ and $\Pi(1, 1, 0) < 0$, there exists $\epsilon_2' \in (0, 1)$ such that $\Pi(1, \epsilon_2', 0) = 0$. Similarly since $\Pi(0, 0, 1) > 0$ and $\Pi(0, 1, 1) < 0$, there exists $\epsilon_2'' \in (0, 1)$ such that $\Pi(0, \epsilon_2'', 1) = 0$. Define $\epsilon' \equiv (1, \epsilon_2', 0)$ and $\epsilon'' \equiv (0, \epsilon_2'', 1)$. It is shown that there exists a function $\epsilon(t)$ on $t \in [0, 1]$ such that $\epsilon(t)$ is continuous and monotonic function with $\epsilon(0) = \epsilon'$ and $\epsilon(1) = \epsilon''$, and $\Pi(\epsilon(t)) = 0$ for any $t \in [0, 1]$. Evidently $\epsilon(t) \neq 0$ for any $t \in [0, 1]$. Suppose the case that $\epsilon_2' < \epsilon_2''$. (The same argument is applied for the case of $\epsilon_2' \geq \epsilon_2''$, and so we omit the argument for the latter case.) We choose arbitrary continuous and monotonic functions $(\epsilon_1(t), \epsilon_2(t))$ with $(\epsilon_1(0), \epsilon_2(0)) = (1, \epsilon_2')$ and $(\epsilon_1(0), \epsilon_2(0)) = (0, \epsilon_2'')$. $\epsilon_2(t)$ is increasing in t. Then for $t \in (0, 1)$,

$$\Pi(\epsilon_1(t), \epsilon_2(t), 0) < \Pi(1, \epsilon_2', 0) = 0 = \Pi(0, \epsilon_2'', 1) < \Pi(\epsilon_1(t), \epsilon_2(t), 1).$$

It implies that there exists $\epsilon_3(t) \in (0, 1)$ such that $\Pi(\epsilon_1(t), \epsilon_2(t), \epsilon_3(t)) = 0$. The continuity of $\Pi(\epsilon), \epsilon_1(t), \epsilon_2(t)$ implies that $\epsilon_3(t)$ is continuous. For $t, t' \in [0, 1]$ such that $t < t'$, and for any $\epsilon_3 \in (0, 1)$, $\Pi(\epsilon_1(t), \epsilon_2(t), \epsilon_3) > \Pi(\epsilon_1(t'), \epsilon_2(t'), \epsilon_3)$, implying that $\epsilon_3(t)$ is increasing in t. For $\epsilon' \equiv (1, \epsilon_2', 0)$,

$$
\begin{aligned}
& E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon')) - \bar{q}(\theta)) \mid \eta_2] \\
= \ & E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon')) - \bar{q}(\theta)) \mid \eta_2] + \frac{\rho_1}{\beta} E[(\tilde{z}(\theta, \epsilon') - h_\beta(\theta \mid \eta_1))(\bar{q}(\tilde{z}(\theta, \epsilon')) - \bar{q}(\theta)) \mid \eta_1] \\
= \ & E[\int_\theta^{\tilde{z}(\theta, \epsilon')} [\{\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_1)} + \frac{\rho_1}{\beta}(z - h_\beta(\theta \mid \eta_1))\} \bar{q}'(z) + \frac{\rho_1}{\beta}(\bar{q}(z) - \bar{q}(\theta))] dz \mid \eta_1] \\
= \ & E[\int_\theta^{\tilde{z}(\theta, \epsilon')} \Psi_1(z, \theta) dz \mid \eta_1] > 0,
\end{aligned}
$$

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since $\Psi_1(z,\theta) > 0$ for any $z \in (\theta, z(\theta))$ and any $\theta \in \Theta_1$ and $\Psi_1(z,\theta) < 0$ for any $z \in (\theta + \epsilon_2'(z(\theta) - \theta), \theta)$ and any $\theta \in \Theta_2$. Similarly for $\epsilon'' \equiv (0, \epsilon_2'', 1)$.

$$E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon'')) - \bar{q}(\theta)) \mid \eta_2]$$
$$= E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon'')) - \bar{q}(\theta)) \mid \eta_2] + \frac{\rho_2}{\beta}E[(\tilde{z}(\theta, \epsilon'') - h_\beta(\theta \mid \eta_1))(\bar{q}(\tilde{z}(\theta, \epsilon'')) - \bar{q}(\theta)) \mid \eta_1]$$
$$= E[\int_\theta^{\tilde{z}(\theta, \epsilon'')} \Psi_2(z, \theta)dz \mid \eta_1] < 0,$$

since $\Psi_2(z, \theta) > 0$ for any $z \in (\theta + \epsilon_2''(z(\theta) - \theta), \theta)$ and any $\theta \in \Theta_2$ and $\Psi_2(z, \theta) < 0$ for any $z \in (\theta, z(\theta))$ and any $\theta \in \Theta_3$. Moreover $E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon)) - \bar{q}(\theta)) \mid \eta_2]$ is continuous for ϵ. Therefore there exists $t \in (0, 1)$ such that

$$E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(\tilde{z}(\theta, \epsilon(t)))) - \bar{q}(\theta)) \mid \eta_2] = 0.$$

This argument implies that there exists $\epsilon \neq 0$ such that both (vi) and (vii) are satisfied under $\tilde{z}(\cdot, \epsilon)$. For this $\tilde{z}(\cdot, \epsilon)$, all conditions (i)-(vii) are satisfied.

Step 4: Finally we check that under $z(\theta)$ which is constructed in Step 3,

$$E[V(\bar{q}(z(\theta))) - \frac{2k - 1}{k}\bar{X}(z(\theta)) - \frac{1 - k}{k}\theta\bar{q}(z(\theta)) \mid \eta]$$
$$> E[V(\bar{q}(\theta)) - \frac{2k - 1}{k}\bar{X}(\theta) - \frac{1 - k}{k}\theta\bar{q}(\theta) \mid \eta].$$

For (λ_1, λ_2) specified in Step 1,

$$E[V(\bar{q}(z(\theta))) - \frac{2k - 1}{k}\bar{X}(z(\theta)) - \frac{1 - k}{k}\theta\bar{q}(z(\theta)) \mid \eta_1]$$
$$- E[V(\bar{q}(\theta)) - \frac{2k - 1}{k}\bar{X}(\theta) - \frac{1 - k}{k}\theta\bar{q}(\theta) \mid \eta_1]$$
$$= E[V(\bar{q}(z(\theta))) - \frac{2k - 1}{k}\bar{X}(z(\theta)) - \frac{1 - k}{k}\theta\bar{q}(z(\theta)) \mid \eta_1]$$
$$+ \frac{(2k - 1)\lambda_2}{k\beta_1}E[(z(\theta) - h_\beta(\theta \mid \eta_1))(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_1]$$
$$+ \frac{2k - 1}{k}\lambda_1[E[\frac{F(\theta \mid \eta_2)}{f(\theta \mid \eta_2)}(\bar{q}(z(\theta)) - \bar{q}(\theta)) \mid \eta_2]$$
$$- E[V(\bar{q}(\theta)) - \frac{2k - 1}{k}\bar{X}(\theta) - \frac{1 - k}{k}\theta\bar{q}(\theta) \mid \eta_1] = E[\int_\theta^{z(\theta)} \Phi(z, \theta)dz \mid \eta_1] > 0.$$

The first equality comes from (vi) and (vii) in Step 3. Therefore P's payoff is improved over the optimal NS. It completes the proof. ∎