Supplement to 'Consulting Collusive Experts'

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1 Introduction

This material includes some arguments which supplement our paper 'Consulting Collusive Experts'. Some proofs, which are omitted in the paper, are also provided in this note. The notation and terms used here are the same as the paper.

2 Attainability of Second-Best Payoff as a PBE of (C3)

Proposition 1 *P can attain any second best allocation as a PBE outcome of the collusion game (C3) following a suitable choice of grand contract.*

Proof of Proposition 1: For a second best allocation $(u_A^{SB}, u_M^{SB}, q^{SB})$, construct the grand contract

$$(X_A(m_A, m_M), X_M(m_A, m_M), q(m_A, m_M); \mathcal{M}_A, \mathcal{M}_M)$$

as follows (where $\mathcal{M}_M = \mathcal{N} \cup \{e_M\}$ and $\mathcal{M}_A = \Theta \cup \{e_A\}$):

(i) $X_M(m_A, m_M) = 0$ for any (m_A, m_M).

(ii) $q(\theta, \eta) = q^{SB}(\theta, \eta)$ and $X_A(\theta, \eta) = \theta q^{SB}(\theta, \eta) + u_A^{SB}(\theta, \eta)$, if $(m_A, m_M) = (\theta, \eta) \in K$, otherwise both are set equal to zero.

(iii) $X_A(e_A, m_M) = q(e_A, m_M) = 0$ for any m_M.

(iv) $(X_A(\theta, e_M), q(\theta, e_M)) = (\hat{X}_A(\theta), \hat{q}(\theta))$, which satisfies the following properties: (a) $\hat{X}_A(\theta) - \theta\hat{q}(\theta) \geq \hat{X}_A(\theta') - \theta\hat{q}(\theta')$ for any $\theta, \theta' \in \Theta$, (b) $\hat{X}_A(\theta) - \theta\hat{q}(\theta) \geq 0$ for any $\theta \in \Theta$ and (c) there exists $\theta' \in \Theta$ such that $\hat{q}(\theta') = q(\theta, \eta)$ and $\hat{X}_A(\theta') > X_A(\theta, \eta)$ for any $(\theta, \eta) \in \Theta \times \mathcal{N}$.[2]

For this grand contract, we will check that the second best allocation is achieved in a PBE of the collusion game. In the Bayesian game induced by this grand contract, both $(m_A(\theta, \eta), m_M(\eta)) = (\theta, \eta)$ and $(m_A(\theta, \eta), m_M(\eta)) = (\theta, e_M)$ are non-cooperative equilibria, regardless of M's beliefs about θ. We claim there exist PBE with the following property: if

[2]For instance, we can choose $(\hat{X}_A(\theta), \hat{q}(\theta))$ such that (i) $\hat{q}(\theta)$ is continuous and strictly decreasing in θ with $\hat{q}(\underline{\theta}) = \max_{(\theta,\eta)\in\Theta\times\mathcal{N}} q(\theta, \eta)$ and $\hat{q}(\bar{\theta}) = \min_{(\theta,\eta)\in\Theta\times\mathcal{N}} q(\theta, \eta)$, and (ii) $\hat{X}_A(\theta) = \theta\hat{q}(\theta) + \int_\theta^{\bar{\theta}} \hat{q}(y)dy + R$ for sufficiently large $R > 0$

a side-contract (SC) is not offered by M then $(m_A(\theta,\eta), m_M(\eta)) = (\theta,\eta)$, while if a non-null SC is offered by M it is rejected by A, resulting in $(m_A(\theta,\eta), m_M(\eta)) = (\theta, e_M)$. In the latter case, A earns $\hat{X}_A(\theta) - \theta\hat{q}(\theta)$. It suffices to check that M does not benefit from offering a non-null side-contract. So consider the following problem:

$$\max E[X_A(\tilde{m}(\theta,\eta)) + X_M(\tilde{m}(\theta,\eta)) - \theta q(\tilde{m}(\theta,\eta)) - \tilde{u}_A(\theta,\eta) \mid \eta]$$

subject to $\tilde{m}(\theta,\eta) \in \Delta(\mathcal{M}_A \times \mathcal{M}_M)$,

$$\tilde{u}_A(\theta,\eta) \geq \tilde{u}_A(\theta^{'},\eta) + (\theta^{'} - \theta) q(\tilde{m}(\theta^{'},\eta))$$

for any $\theta, \theta^{'} \in \Theta(\eta)$ and

$$\tilde{u}_A(\theta,\eta) \geq \hat{X}_A(\theta) - \theta\hat{q}(\theta)$$

for any (θ,η). By construction of $(\hat{X}_A(\theta), \hat{q}(\theta))$ in (iv), $\tilde{m}(\theta,\eta) = (\theta, e_M)$ (meaning the degenerate probability measure concentrated on (θ, e_M)) and $\tilde{u}_A(\theta,\eta) = \hat{X}_A(\theta) - \theta\hat{q}(\theta)$ solve this problem. Then the maximum value of the above problem is equal to zero. Since A receives at least $\hat{X}_A(\theta) - \theta\hat{q}(\theta)$ in the continuation game following a non-null side-contract offer, this maximum value provides an upper bound on M's payoff from offering a non-null side-contract. Hence M never benefits from offering a non-null side-contract. Consequently, there is a PBE of this game in which M never offers any non-null side contract. This implies that M and A play $(m_A(\theta,\eta), m_M(\eta)) = (\theta,\eta)$ and the second-best allocation is achieved, concluding the proof. ■

3 Proof of Results Used in Proving Propositions 4 and 5

We first prove the following result invoked in the proof of Proposition 4 in the Appendix of the paper.

Result 1 *There exists $z(\cdot \mid \eta^{**}) \in Z(\eta^{**})$ which satisfies the following conditions.*

(B-i) $z(\theta \mid \eta^{**}) = \theta$ for any $\theta \notin \Theta_H \cup \Theta_L$

(B-ii) For $\theta \in \Theta_L$, $z(\theta \mid \eta^{**})$ satisfies (a) $z(\theta \mid \eta^{**}) \leq \theta$ with strict inequality for some subinterval of Θ_L of positive measure, and (b) $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) > 0$ for any $z \in [z(\theta \mid \eta^{**}), \theta]$.

(B-iii) For $\theta \in \Theta_H$, $z(\theta \mid \eta^{**})$ satisfies (a) $z(\theta \mid \eta^{**}) \geq \theta$ with strict inequality for some some subinterval of Θ_H of positive measure, (b) $z(\theta \mid \eta^{**}) < h(\theta \mid \eta^{**})$ and (c) $H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) < 0$ for any $z \in [\theta, z(\theta \mid \eta^{**})]$.

(B-iv) $E[(z(\theta \mid \eta^{**}) - h(\theta \mid \eta^{**}))Q^*(z(\theta \mid \eta^{**})) + \int_{z(\theta|\eta^{**})}^{\bar{\theta}(\eta^{**})} Q^*(z)dz \mid \eta^{**}] = 0.$

Proof:

Step A: For any $\eta \in \mathcal{N}$ and any closed interval $[\theta^{'}, \theta^{''}] \subset \Theta(\eta)$ such that $\underline{\theta}(\eta) < \theta^{'} < \theta^{''} < \bar{\theta}(\eta)$, there exists $\delta > 0$ such that $z(\cdot) \in Z(\eta)$ for any $z(\cdot)$ satisfying the following properties:

(i) $z(\theta)$ is increasing and differentiable with $|z(\theta) - \theta| < \delta$ and $|z^{'}(\theta) - 1| < \delta$ for any $\theta \in \Theta(\eta)$

(ii) $z(\theta) = \theta$ for any $\theta \notin [\theta^{'}, \theta^{''}]$.

Proof of Step A

For arbitrary $\eta \in \mathcal{N}$ and arbitrary closed interval $[\theta^{'}, \theta^{''}] \subset \Theta(\eta)$ such that $\underline{\theta}(\eta) < \theta^{'} < \theta^{''} < \bar{\theta}(\eta)$, we choose ϵ_1 and ϵ_2 such that

$$\epsilon_1 \equiv \min_{\theta \in [\theta^{'}, \theta^{''}]} f(\theta \mid \eta)$$

and

$$\epsilon_2 \equiv \max_{\theta \in [\theta^{'}, \theta^{''}]} |f^{'}(\theta \mid \eta)|.$$

From our assumptions that $f(\theta \mid \eta)$ is continuously differentiable and positive on $\Theta(\eta)$, $\epsilon_1 > 0$, and ϵ_2 is positive and bounded above. We choose $\delta > 0$ such that

$$\delta \in (0, \frac{\epsilon_1}{\epsilon_1 + \epsilon_2}).$$

For this δ, it is obvious that there exists $z(\theta)$ which satisfies conditions (i) and (ii) of the statement. Define

$$\Lambda(\theta \mid \eta) \equiv (\theta - z(\theta))f(\theta \mid \eta) + F(\theta \mid \eta).$$

Since $z(\theta)$ is differentiable on $\Theta(\eta)$, $\Lambda(\theta \mid \eta)$ is also so. It is equal to $\Lambda(\theta \mid \eta) = F(\theta \mid \eta)$ on $\theta \notin [\theta^{'}, \theta^{''}]$. For $\theta \in [\theta^{'}, \theta^{''}]$,

$$\begin{aligned} \frac{\partial \Lambda(\theta \mid \eta)}{\partial \theta} &= (2 - z^{'}(\theta))f(\theta \mid \eta) + (\theta - z(\theta))f^{'}(\theta \mid \eta) > (1-\delta)f(\theta \mid \eta) - \delta|f^{'}(\theta \mid \eta)| \\ &\geq (1-\delta)\epsilon_1 - \delta\epsilon_2. \end{aligned}$$

This is positive by the definition of $(\epsilon_1, \epsilon_2, \delta)$. Then $\Lambda(\theta \mid \eta)$ is increasing in θ on $\Theta(\eta)$ with $\Lambda(\underline{\theta}(\eta) \mid \eta) = 0$ and $\Lambda(\bar{\theta}(\eta) \mid \eta) = 1$. Since $z(\theta)$ is increasing in θ by the definition, it is preserved even by ironing rule. Therefore $z(\cdot) \in Z(\eta)$. ■

Step B

For η^{**} and the closed interval $I = [\theta^{'}, \theta^{''}] \subset \Theta(\eta^{**})$ which are selected in Step 1 of the paper's Appendix, we select $\delta > 0$ according to the procedure in Step A. By the continuity of $\frac{F(\theta)}{f(\theta)}$ and $\frac{F(\theta|\eta^{**})}{f(\theta|\eta^{**})}$, the closedness of Θ_L and Θ_H and $\bar{\theta}^L < \underline{\theta}^H$, we can select $\epsilon > 0$ such that

$$\lambda < [\frac{F(\theta)}{f(\theta)} - \epsilon] / \frac{F(\theta \mid \eta^{**})}{f(\theta \mid \eta^{**})} \text{ for } \theta \in \Theta_L \equiv [\underline{\theta}^L, \bar{\theta}^L]$$

$$\lambda > [\frac{F(\theta)}{f(\theta)} + \epsilon] / \frac{F(\theta \mid \eta^{**})}{f(\theta \mid \eta^{**})} \text{ for } \theta \in \Theta_H \equiv [\underline{\theta}^H, \bar{\theta}^H].$$

These conditions are equivalent to

$$H(\theta) - (1-\lambda)\theta - \lambda h(\theta \mid \eta^{**}) > \epsilon \text{ for } \theta \in \Theta_L$$

and

$$H(\theta) - (1-\lambda)\theta - \lambda h(\theta \mid \eta^{**}) < -\epsilon \text{ for } \theta \in \Theta_H.$$

By the continuity of $H(\theta) - (1-\lambda)\theta$ for θ and closedness of Θ_L and Θ_H, there exists $\epsilon_L > 0$ and $\epsilon_H > 0$ such that

$$H(\theta) - (1-\lambda)\theta - \epsilon \leq H(z) - (1-\lambda)z$$

for any $z \in [\theta - \epsilon_L, \theta]$ and any $\theta \in \Theta_L$, and

$$H(\theta) - (1-\lambda)\theta + \epsilon \geq H(z) - (1-\lambda)z$$

for any $z \in [\theta, \theta + \epsilon_H]$ and any $\theta \in \Theta_H$. Equivalently, there exists $\epsilon_L > 0$ and $\epsilon_H > 0$ such that

$$H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) > 0 \text{ for any } \theta \in \Theta_L \text{ and any } z \in [\theta - \epsilon_L, \theta]$$

and

$$H(z) - (1-\lambda)z - \lambda h(\theta \mid \eta^{**}) < 0 \text{ for any } \theta \in \Theta_H \text{ and any } z \in [\theta, \theta + \epsilon_H].$$

Step C

We select $z(\cdot \mid \eta^{**})$ such that

(i) $z(\theta \mid \eta^{**})$ is increasing and differentiable with $|z(\theta \mid \eta^{**}) - \theta| < \min\{\delta, \epsilon_L, \epsilon_H\}$ and $|z_\theta(\theta \mid \eta^{**}) - 1| < \delta$ for any $\theta \in \Theta(\eta^{**})$

(ii) $z(\theta \mid \eta^{**}) = \theta$ for any $\theta \notin \Theta_H \cup \Theta_L$

(iii) For $\theta \in \Theta_L$, $z(\theta \mid \eta^{**}) \leq \theta$ with strict inequality for some subinterval of Θ_L of positive measure.

(iv) For $\theta \in \Theta_H$, $\theta \leq z(\theta \mid \eta^{**})$ with strict inequality for some some subinterval of Θ_H of positive measure, and $z(\theta \mid \eta^{**}) < h(\theta \mid \eta^{**})$.

It is evident that such a $z(\cdot \mid \eta^{**})$ exits. The argument in Step A and B ensures that $z(\cdot \mid \eta^{**})$ is in $Z(\eta^{**})$, and satisfies (B-(ii)) (c) and (B-(iii)) (c). By the construction, it is evident that this satisfies all other conditions in (B(i)-(iii)).

Step D

Suppose $z(\cdot \mid \eta^{**}) \in Z(\eta^{**})$ which satisfies (B(i)-(iii)). Since

$$(z - h(\theta \mid \eta^{**}))Q^*(z) + \int_z^{\bar{\theta}(\eta^{**})} Q^*(y)dy$$

is increasing in z for $z < h(\theta \mid \eta^{**})$, and

$$E[(\theta - h(\theta \mid \eta^{**}))Q^*(\theta) + \int_\theta^{\bar{\theta}(\eta^{**})} Q^*(y)dy \mid \eta^{**}] = 0,$$

the choice of $z(\theta \mid \eta^{**}) \leq \theta$ on Θ_L (or $z(\theta \mid \eta^{**}) \geq \theta$ on Θ_H) reduces (or raises)

$$E[(z(\theta \mid \eta^{**}) - h(\theta \mid \eta^{**}))Q^*(z(\theta \mid \eta^{**})) + \int_{z(\theta|\eta^{**})}^{\bar{\theta}(\eta^{**})} Q^*(z)dz \mid \eta^{**}]$$

away from zero. For any pair of parameters α_H, α_L lying in $[0,1]$, define a function $z_{\alpha_L,\alpha_H}(\theta|\eta^{**})$ which equals $(1-\alpha_L)z(\theta|\eta^{**}) + \alpha_L\theta$ on Θ_L, equals $(1-\alpha_H)z(\theta|\eta^{**}) + \alpha_H\theta$ on Θ_H and equals θ elsewhere. It is easily checked that any such function also is in $Z(\eta^{**})$ and satisfies conditions (B(i)-(iii)). Define

$$\begin{aligned} W(\alpha_L, \alpha_H) \equiv & E[(z_{\alpha_L,\alpha_H}(\theta \mid \eta^{**}) - h(\theta \mid \eta^{**}))Q^*(z_{\alpha_L,\alpha_H}(\theta \mid \eta^{**})) \\ + & \int_{z_{\alpha_L,\alpha_H}(\theta|\eta^{**})}^{\bar{\theta}(\eta^{**})} Q^*(z)dz \mid \eta^{**}]. \end{aligned}$$

Then W is continuously differentiable, strictly increasing in α_L and strictly decreasing in α_H with $W(1,1) = 0$. The Implicit Function Theorem ensures existence of α_L^*, α_H^* both smaller than 1 such that $W(\alpha_L^*, \alpha_H^*) = 0$. Hence the function $z_{\alpha_L^*,\alpha_H^*}(\theta|\eta^{**})$ is in $Z(\eta^{**})$ and satisfies (B(i)-(iv)). ■

Result 2 *For $z(\cdot \mid \eta)$ constructed in Step 2 (in the proof of Proposition 4 in the Appendix of the paper), consider the following allocation (u_A, u_M, q):*

$$q(\theta,\eta) = Q^*(z(\theta \mid \eta))$$

$$u_A(\theta,\eta) = \int_{\theta}^{\bar{\theta}(\eta)} Q^*(z(y \mid \eta))dy + \int_{\bar{\theta}(\eta)}^{\bar{\theta}} Q^*(y)dy$$

$$u_M(\theta,\eta) = X^*(z(\theta \mid \eta)) - \theta Q^*(z(\theta \mid \eta)) - \int_{\theta}^{\bar{\theta}(\eta)} Q^*(z(y \mid \eta))dy - \int_{\bar{\theta}(\eta)}^{\bar{\theta}} Q^*(y)dy.$$

where

$$X^*(z(\theta \mid \eta)) \equiv z(\theta \mid \eta)Q^*(z(\theta \mid \eta)) + \int_{z(\theta|\eta)}^{\bar{\theta}} Q^*(z)dz.$$

Then (u_A, u_M, q) is a EAC feasible allocation.

Proof: The construction of $z(\theta \mid \eta)$ implies that $z(\bar{\theta}(\eta) \mid \eta) \leq \bar{\theta}$ for any $\eta \in \mathcal{N}$. Hence

$$X^*(z(\theta \mid \eta)) - z(\theta \mid \eta)Q^*(z(\theta \mid \eta)) \geq 0$$

for any $(\theta,\eta) \in K$. It is evident that the construction of $z(\theta \mid \eta)$ implies $E[u_M(\theta,\eta) \mid \eta] = 0$.

Since $z(\theta \mid \eta^{**})$ is increasing in θ, there is no pooling region with $\Theta(\pi(\cdot \mid \eta^{**}), \eta^{**}) = \phi$. The coalition incentive constraint is satisfied, since

$$\begin{aligned}
& X(\theta', \eta') - z(\theta \mid \eta)q(\theta', \eta') \\
= \; & X^*(z(\theta' \mid \eta')) - z(\theta \mid \eta)Q^*(z(\theta' \mid \eta')) \\
= \; & \int_{z(\theta'|\eta')}^{\bar{\theta}} Q^*(z)dz + (z(\theta' \mid \eta') - z(\theta \mid \eta))Q^*(z(\theta' \mid \eta')) \\
\leq \; & \int_{z(\theta|\eta)}^{\bar{\theta}} Q^*(z)dz = X(\theta,\eta) - z(\theta \mid \eta)q(\theta,\eta).
\end{aligned}$$

The A's incentive constraint is satisfied, since

$$\begin{aligned}
& u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta) \\
= \; & \int_{\theta'}^{\bar{\theta}} Q^*(z(y \mid \eta))dy + (\theta' - \theta)Q^*(z(\theta' \mid \eta)) \\
\leq \; & \int_{\theta}^{\bar{\theta}} Q^*(z(y \mid \eta))dy = u_A(\theta,\eta).
\end{aligned}$$

The last inequality follows from the fact that $Q^*(z(\theta \mid \eta))$ is non-increasing in θ. These arguments guarantee that (u_A, u_M, q) is a EAC feasible allocation. ■

Now we prove the following result invoked in the proof of Proposition 5 in the Appendix.

(i) $\hat{h}(\theta \mid \eta^*) > \hat{h}(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta}]$ and $\hat{h}(\underline{\theta} \mid \eta^*) = \hat{h}(\underline{\theta} \mid \eta) = \underline{\theta}$ for any $\eta \neq \eta^*$

(ii) Define $G(h \mid \eta) \equiv \int_{\{\theta \mid \hat{h}(\theta \mid \eta) \leq h\}} f(\theta \mid \eta) d\theta$. Then $G(h \mid \eta^*)$ is a mean-preserving spread of $G(h \mid \eta)$ for any $\eta \neq \eta^*$

Proof of (i): Since $\frac{f(\theta \mid \eta^*)}{f(\theta \mid \eta)}$ is strictly decreasing in θ for any $\eta \neq \eta^*$, $\frac{f(\theta' \mid \eta^*)}{f(\theta \mid \eta^*)} > \frac{f(\theta' \mid \eta)}{f(\theta \mid \eta)}$ for $\theta > \theta'$. $\Theta(\eta) = \Theta(\eta^*) = \Theta$ then implies

$$\frac{F(\theta \mid \eta^*)}{f(\theta \mid \eta^*)} = \int_{\underline{\theta}}^{\theta} \frac{f(\theta' \mid \eta^*)}{f(\theta \mid \eta^*)} d\theta' > \int_{\underline{\theta}}^{\theta} \frac{f(\theta' \mid \eta)}{f(\theta \mid \eta)} d\theta' = \frac{F(\theta \mid \eta)}{f(\theta \mid \eta)}.$$

Hence $h(\theta \mid \eta^*) > h(\theta \mid \eta)$ for $\theta \in (\underline{\theta}, \bar{\theta}]$ and $h(\underline{\theta} \mid \eta^*) = h(\underline{\theta} \mid \eta) = \underline{\theta}$. The property of the ironing procedure (explained in later section) ensures that $\hat{h}(\theta \mid \eta^*) > \hat{h}(\theta \mid \eta)$ for any $\theta > \underline{\theta}$ and $\hat{h}(\underline{\theta} \mid \eta^*) = \hat{h}(\underline{\theta} \mid \eta) = \underline{\theta}$ for any $\eta \neq \eta^*$. ■

Proof of (ii): Since

$$\int_{\underline{h}}^{\bar{h}} h dG(h \mid \eta) = \int_{\underline{\theta}}^{\bar{\theta}} \hat{h}(\theta \mid \eta) dF(\theta \mid \eta) = \int_{\underline{\theta}}^{\bar{\theta}} h(\theta \mid \eta) dF(\theta \mid \eta) = \bar{\theta}$$

for each η, the two distributions for η^*, η have the same mean. For every convex function $u(h)$,

$$\begin{aligned}
& \int_{\underline{h}}^{\bar{h}} u(h) dG(h \mid \eta^*) = \int_{\underline{\theta}}^{\bar{\theta}} u(h(\theta \mid \eta^*)) f(\theta \mid \eta^*) d\theta \\
\geq & \int_{\underline{\theta}}^{\bar{\theta}} [u(\hat{h}(\theta \mid \eta)) - u'(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - h(\theta \mid \eta^*))] f(\theta \mid \eta^*) d\theta \\
= & \int_{\underline{\theta}}^{\bar{\theta}} [u(\hat{h}(\theta \mid \eta)) - u'(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - \theta) + \int_{\theta}^{\bar{\theta}} u'(\hat{h}(y \mid \eta)) dy] f(\theta \mid \eta^*) d\theta \\
\geq & \int_{\underline{\theta}}^{\bar{\theta}} [u(\hat{h}(\theta \mid \eta)) - u'(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - \theta) + \int_{\theta}^{\bar{\theta}} u'(\hat{h}(y \mid \eta)) dy] f(\theta \mid \eta) d\theta \\
= & \int_{\underline{\theta}}^{\bar{\theta}} u(\hat{h}(\theta \mid \eta)) f(\theta \mid \eta) d\theta = \int_{\underline{h}}^{\bar{h}} u(h) dG(h \mid \eta)
\end{aligned}$$

The first inequality follows from the convexity of $u(h)$: $u(h) \geq u(h^{'}) - u^{'}(h^{'})(h^{'} - h)$ for any $h, h^{'} \in H$. The second inequality is the result that

$$u(\hat{h}(\theta \mid \eta)) - u^{'}(\hat{h}(\theta \mid \eta))(\hat{h}(\theta \mid \eta) - \theta) + \int_{\theta}^{\bar{\theta}} u^{'}(\hat{h}(y \mid \eta))dy$$

is non-increasing in θ, and $F(\theta \mid \eta)$ first order stochastically dominates $F(\theta \mid \eta^{*})$ (because of the MLRP assumption). These hold with strict inequality if $u(h)$ is strictly convex. According to the definition, $G(h \mid \eta^{*})$ is a mean-preserving spread of $G(h \mid \eta)$ for any $\eta \neq \eta^{*}$. ■

4 Justification for EACP Allocations When Contracts are Offered by Third Party

To overcome the problem highlighted by Celik and Peters (2011) in the context where the side-contract is designed by a third party, we model side-contracts as a two stage game played by M and A. The first stage is a 'participation' stage where they communicate their participation decisions in the side contract, in addition to some auxiliary messages in the event of agreeing to participate. The role of these messages is to allow A to signal information about his type while agreeing to participate, which can help replicate whatever information is communicated by side-contract rejection in a setting where communication concerning participation decisions is dichotomous. A and M observe the messages sent by each other at the end of the first stage. At the second stage, A and M submit type reports, conditional on having agreed to participate at the first stage.

Let (D_A^p, D_M^p) denote the message sets of A and M at the participation stage (or p-stage). $e_A \in D_A^p$ and $e_M \in D_M^p$ are the exit options of A and M respectively. The message sets at this stage may include other auxiliary messages as well.

What occurs at the second stage ('execution' or e-stage) depends on $d^p = (d_A^p, d_M^p)$ chosen at the first stage.

- If $d_A^p \neq e_A$ and $d_M^p \neq e_M$, A and M select $(d_A^e, d_M^e) \in D_A^e(d^p) \times D_M^e(d^p)$ respectively, where the conditional message sets $D_A^e(d^p), D_M^e(d^p)$ are specified by the side contract. The report to P is selected according to $\tilde{m}(d^p, d^e) \in \Delta(\mathcal{M}_A \times \mathcal{M}_M)$, associated with the transfers to A and M, $t_A(d^p, d^e)$ and $t_M(d^p, d^e)$ respectively. These satisfy $t_A(d^p, d^e) + t_M(d^p, d^e) \leq 0$.

- If either $d_A^p = e_A$ or $d_M^p = e_M$, A and M play GC non-cooperatively.

Given GC and η, the third party decides whether to offer a side-contract $SC(\eta)$ or not (i.e., offer a null side-contract NSC). If a non-null side-contract is offered, A and M play a game denoted by $GC \circ SC(\eta)$ with two stages as described above. On the other hand, if the third party offers a null side-contract NSC at the first stage, A and M play GC non-cooperatively based on prior beliefs (denoted by $b_\phi(\eta)$). The third-party's objective is to maximize $E[\alpha u_A(\theta,\eta) + (1-\alpha)u_M(\theta,\eta) \mid \eta]$ in state η.

The refinement PBE(c) introduced in the paper for the case where the side contract is offered by M, can now be extended as follows.

Definition 1 *Following the selection of a grand contract by P, a PBE(c) is a Perfect Bayesian Equilibrium (PBE) of the subsequent game in which side-contracts are designed by a third party, which has the following property. There does not exist some η for which there is a Perfect Bayesian Equilibrium (PBE) of subgame C3 in which (conditional on η) the third-party's payoff is strictly higher, without lowering the payoff of M and any type of A.*

Definition 2 *An allocation (u_A, u_M, q) is EAC feasible if the following is true. When side contracts are designed by a third party assigning welfare weight α to A, there exists a grand contract and a PBE(c) of the subsequent side contract subgame which results in this allocation.*

Proposition 2 *An allocation (u_A, u_M, q) is EAC feasible when side contracts are designed by a third party assigning welfare weight α to A, if and only if it is an EACP(α) allocation satisfying the interim participation constraints $u_A(\theta,\eta) \geq 0$ for all (θ,η) and $E[u_M(\theta,\eta) \mid \eta] \geq 0$ for all η.*

Proof of Proposition 2

Proof of Necessity

For some GC, suppose that allocation (u_A, u_M, q) is achieved in the game with collusion. Suppose the allocation is achieved as the outcome of a PBE(c) of subgame C3 in which a non-null side contract $SC^*(\eta)$ is offered on the equilibrium path in some state η, which is rejected either by some types of A, or by M. We show it can also be achieved as the outcome of a PBE(c) in which a non-null side contract is offered in state η and always accepted by A and M. Let $d^p_A(\theta,\eta)$ and $d^p_M(\eta)$ denote A and M's participation decisions respectively (whether or not they chose the exit option at the first stage). Following rejection by either A or M, they play the grand contract GC based on updated beliefs $b(\cdot \mid d^p_A(\theta,\eta), d^p_M(\eta), \eta)$. Let $d^{p*}_A(\theta,\eta)$ denote A's decision, and $d^{p*}_M(\eta)$ M's participation decisions on the equilibrium path.

Now construct a new side-contract $\tilde{SC}(\eta)$ which differs from $SC^*(\eta)$ by replacing the message space D^p_A for A at the first stage by $D^p_A \times D^p_A$. Similarly M's message space is now $D^p_M \times D^p_M$. The interpretation is that the first component of this message d^p_A is a participation decision, while the second component $\tilde{d}^p_A$ is a 'signal'. This allows a decoupling of the participation decision from sending a signal to the other player which changes beliefs with which they play the grand contract noncooperatively in the event that the side contract is rejected by someone. For example, if A selected $d^p_A = e_A$ in the previous side-contract in order to send a signal about his type θ to M, the same signal can be sent now through the second component of the message, while opting to participate in the choice of the first component (by selecting $d^p_A \neq e_A, \tilde{d}^p_A = e_A$). The first component of the message d^p_A now matters only insofar as it is an exit decision or not; conditional on it not being an exit decision the precise message does not matter. If both decide to participate (i.e., not exit), they move on to the second stage of the game, where the mechanism replicates the allocation resulting on the equilibrium path of the original PBE associated with $SC^*(\eta)$ (i.e., agrees with the second stage mechanism in $SC^*(\eta)$ whenever both agreed to participate in $SC^*(\eta)$, and otherwise assigns the allocation resulting from noncooperative play of GC in the original PBE). If one or both decides not to participate in $\tilde{SC}(\eta)$, they play GC noncooperatively with beliefs based on first stage messages according to $b(\cdot \mid \tilde{d}^p_A(\theta,\eta), \tilde{d}^p_M(\eta), \eta)$. Note that these beliefs do not depend on d^p_A or d^p_M.

It is easily verified that there exists a PBE where the third party offers $\tilde{SC}(\eta)$ in state η, in which A and M always accept the side-contract (i.e., in state (θ,η) they respectively select $d^p_A(\theta,\eta) \neq e_A, d^p_M(\eta) \neq e_M$ while choosing $\tilde{d}^p_A(\theta,\eta)$ equal to $d^p_A(\theta,\eta)$ in the original PBE, and $\tilde{d}^p_M(\eta)$ equal to $d^p_M(\eta)$ in the original PBE). The underlying idea is that since A's first stage report $\tilde{d}^p_A$ now affects beliefs at the second stage in exactly the same way that d^p_A did in the original PBE, it is optimal for A to choose $\tilde{d}^p_A(\theta,\eta)$ equal to $d^p_A(\theta,\eta)$

in the original PBE. Moreover, the first stage d_A^p report now affects only A's participation decision at the second stage, and by construction has no effect on second stage allocations (conditional on participation). So it is optimal for A to decide to participate. The same logic applies to M. Hence the newly constructed strategies and beliefs constitute a PBE. It can also be verified that since the original equilibrium was a PBE(c), so is the newly constructed equilibrium.

Next we show that if allocation (u_A, u_M, q) is realized in a PBE (c) in which the offered side contract is not rejected on the equilibrium path, it must be a EACP(α) allocation. Suppose not: the allocation resulting from some non-null side contract $(\tilde{u}_A^*(\theta,\eta), \tilde{m}^*(\theta,\eta)) \neq (u_A(\theta,\eta), (\theta,\eta))$ solves problem $TP(\eta;\alpha)$ for some η. Define $\tilde{u}_M^*(\theta,\eta) \equiv X(\tilde{m}^*(\theta \mid \eta)) - \theta Q(\tilde{m}^*(\theta \mid \eta)) - \tilde{u}_A^*(\theta,\eta)$. It is evident that

$$E[\alpha\tilde{u}_A^*(\theta,\eta) + (1-\alpha)\tilde{u}_M^*(\theta,\eta) \mid \eta] > E[\alpha u_A(\theta,\eta) + (1-\alpha)u_M(\theta,\eta) \mid \eta],$$

$$\tilde{u}_A^*(\theta,\eta) \geq u_A(\theta,\eta)$$

and

$$E[\tilde{u}_M^*(\theta,\eta) \mid \eta] \geq E[u_M(\theta,\eta) \mid \eta].$$

There exists $m^c(\theta,\eta) \in \Delta(\mathcal{M}_A \times \mathcal{M}_M)$ in GC such that

$$(X_A(m^c(\theta,\eta)) + X_M(m^c(\theta,\eta)), q(m^c(\theta,\eta))) = (X(\tilde{m}^*(\theta \mid \eta)), Q(\tilde{m}^*(\theta \mid \eta))).$$

Now construct a new side-contract $SC(\eta)$ which realizes

$$(\tilde{u}_A^*(\theta,\eta), \tilde{u}_M^*(\theta,\eta), Q(\tilde{m}^*(\theta \mid \eta)))$$

as a PBE outcome, contradicting the hypothesis that (u_A, u_M, q) is realized in a PBE (c). $SC(\eta)$ is specified as follows:

- $D^p \equiv D^{p*}$ where $D^{p*} = (D_A^{p*}, D_M^{p*})$ are A and M's message sets at the participation stage of the original side-contract $SC^*(\eta)$.
- $D_A^e = \Theta(\eta)$ and $D_M^e = \phi$
- A's choice of $d_A^e = \theta \in \Theta(\eta)$ generates the report $m^c(\theta,\eta)$ to P, and side transfers to A and M respectively as follows:

$$t_A(\theta,\eta) = \tilde{u}_A^*(\theta,\eta) - [X_A(m^c(\theta,\eta)) - \theta q(m^c(\theta,\eta))]$$

and

$$t_M(\theta,\eta) = \tilde{u}^*_M(\theta,\eta) - X_M(m^c(\theta,\eta)).$$

Given any (d^p_A, d^p_M) with $d^p_A \neq e_A$ and $d^p_M \neq e_M$ at the participation stage, it is optimal for A to always select $d^e_A = \theta$, since $\theta^{'} = \theta$ maximizes

$$X_A(m^c(\theta^{'},\eta)) - \theta q(m^c(\theta^{'},\eta)) + t_A(\theta^{'},\eta) = \tilde{u}^*_A(\theta^{'},\eta) + (\theta^{'} - \theta)Q(\tilde{m}^*(\theta^{'} \mid \eta)).$$

At the participation stage, A is indifferent among any $d^p_A \in D^p_A \backslash \{e_A\}$ as the optimal response to $d^p_M \neq e_M$, since the outcome in the continuation game does not depend on this choice. Select beliefs consequent on non-participation by either A or M in the same way as in the original equilibrium; then participation continues to be optimal for both. In state η, responses to all other side contract offers are unchanged. In all other states $\eta' \neq \eta$, strategies and beliefs are unchanged. Hence this is a PBE resulting in $(\tilde{u}^*_A(\theta,\eta), \tilde{u}^*_M(\theta,\eta))$, contradicting the PBE(c) property of the equilibrium resulting in (u_A, u_M, q). This completes the proof of necessity.

Proof of Sufficiency

Take an allocation which is EACP(α) and satisfies interim participation constraints. We show it is achievable as a PBE(c) outcome following choice of the following grand contract GC:

$$GC = (X_A(m_A, m_M), X_M(m_A, m_M), q(m_A, m_M) : \mathcal{M}_A, \mathcal{M}_M)$$

where

$$\mathcal{M}_A = K \cup \{e_A\}$$

$$\mathcal{M}_M = \mathcal{N} \cup \{e_M\}$$

$$X_A(m_A, m_M) = X_M(m_A, m_M) = q(m_A, m_M) = 0$$

for (m_A, m_M) such that either $m_A = e_A$ or $m_M = e_M$.

- $(X_A((\theta_A,\eta_A),\eta_M), q((\theta_A,\eta_A),\eta_M)) = (u_A(\theta_A,\eta_M) + \theta_A q(\theta_A,\eta_M), q(\theta_A,\eta_M))$ for $\eta_A = \eta_M$ and $(X_A((\theta_A,\eta_A),\eta_M), q((\theta_A,\eta_A),\eta_M)) = (-T, 0)$ for $\eta_A \neq \eta_M$

- $X_M((\theta_A,\eta_A),\eta_M) = u_M(\theta_A,\eta_A)$ for $\eta_M = \eta_A$ and $X_M((\theta_A,\eta_A),\eta_M) = -T$ for $\eta_M \neq \eta_A$

where $T > 0$ is sufficiently large. The EACP(α) property implies that $u_A(\theta, \eta) \geq u_A(\theta', \eta) + (\theta' - \theta)q(\theta', \eta)$ for any $\theta, \theta' \in \Theta(\eta)$. The interim participation constraints imply that this grand contract has a non-cooperative pure strategy equilibrium

$$(m_A^*(\theta, \eta), m_M^*(\eta)) = ((\theta, \eta), \eta)$$

based on prior beliefs.

For this grand contract, we claim there exists a PBE(c) resulting in $(m_A^*(\theta, \eta), m_M^*(\eta)) = ((\theta, \eta), \eta)$. Let the third party offer a null side contract, following which A and M play truthfully in the GC noncooperatively with prior beliefs. If the third party offers any non-null side contract, all types of A and M reject it and subsequently play truthfully in the noncooperative game with prior beliefs. This is clearly a PBE. That it is a PBE(c) follows from the property that the allocation is EACP(α). ■

5 Procuring an Indivisible Good

Here we provide the details of the case where the good to be procured is indivisible. P procures an indivisible good with quantity q either 0 or 1 from A who produces it at cost θ. P obtains a zero gross benefit if $q = 0$, and a benefit of $V > 0$ if $q = 1$. A is privately informed about the realization of θ. The expert M and A jointly observe the realization of signal $i \in \{L, H\}$ of A's cost. Both A and M have outside option payoffs of 0. $F_i(\theta)$ denotes the distribution of θ conditional on i defined on $[\underline{\theta}, \bar{\theta}]$, which has a density $f_i(\theta)$ which is differentiable and positive on $[\underline{\theta}, \bar{\theta}]$. Hence the support of θ does not vary with the signal, and hazard rates are well-defined and finite-valued throughout the support. $\kappa_i \in (0, 1)$ denotes the probability of signal i, with $\kappa_L + \kappa_H = 1$. P does not observe the signal i, and has a prior $F(\theta) \equiv \kappa_L F_L(\theta) + \kappa_H F_H(\theta)$ with density $f(\theta) \equiv \kappa_L f_L(\theta) + \kappa_H f_H(\theta)$.

Assumption 1 *(i)* $\frac{f_L(\theta)}{f_H(\theta)}$ *is decreasing*

(ii) $H(\theta) \equiv \theta + \frac{F(\theta)}{f(\theta)}$, $h_i(\theta) \equiv \theta + \frac{F_i(\theta)}{f_i(\theta)}$ *and* $l_i(\theta) \equiv \theta - \frac{1-F_i(\theta)}{f_i(\theta)}$ $(i = L, H)$ *are increasing*

(iii) $h_L(\bar{\theta}) > V > \underline{\theta}$

Part (i) represents a monotone likelihood ratio property wherein $i = L$ (resp. $i = H$) is a signal of low (resp. high) cost. (ii) is a standard assumption ensuring monotonicity of (conditional) virtual costs and valuations. These imply $F_H(\theta) < F(\theta) < F_L(\theta)$ and $h_H(\theta) < H(\theta) < h_L(\theta)$ for any $\theta \in (\underline{\theta}, \bar{\theta})$. The second inequality of (iii) ensures gains from trade between P and A; the first one ensures that costless access to M's signal is valuable for P in the absence of collusion. These conditions are satisfied in the following example with a uniform prior $F(\theta) = \theta$ on $[0,1]$ and linear conditional densities: $F_L(\theta) = 2d\theta - (2d-1)\theta^2$, $F_H(\theta) = 2(1-d)\theta + (2d-1)\theta^2$ on $[0,1]$, $\kappa_L = \kappa_H = 1/2$, $d \in (1/2, 1)$ and V between 0 and $1 + \frac{1}{2(1-d)}$. d is interpreted as a parameter of information precision. We shall illustrate our analysis with numerical computations for this example.

The situation where P has no access to M's signal is referred to as the *No Monitor (NM)* case. Here P offers a non-contingent price p^{NM} to maximize $F(p)[V-p]$, which satisfies $V = H(p^{NM})$ if $V < H(\bar{\theta})$, and equals $\bar{\theta}$ otherwise. Let $\Pi^{NM} \equiv F(p^{NM})[V - p^{NM}]$ denote the resulting expected payoff of P. The *second-best allocation* results when there is no collusion whence P can costlessly access M's signal; here P offers A a price p_i^{SB} which maximizes $(V - p_i)F_i(p_i)$. The ordering of virtual cost functions implied by Assumption 1 ensures a lower price elasticity of supply and thus a lower second-best price in the low cost signal state. However, the supply curve is shifted to the right in the low signal state, so the ordering of resulting supply likelihoods between the two states is ambiguous, which turns out to depend on V:

Lemma 1 *(i)* $p_H^{SB} > p^{NM} > p_L^{SB}$ *if* $V < H(\bar{\theta})$*, and* $p_H^{SB} = p^{NM} = \bar{\theta} > p_L^{SB}$ *otherwise*

(ii) There exist V^* *and* V^{**} *such that* $\underline{\theta} < V^* \leq V^{**} < h_H(\bar{\theta})$*, where* $F_L(p_L^{SB}) > F_H(p_H^{SB})$ *for* $V \in (\underline{\theta}, V^*)$ *and* $F_L(p_L^{SB}) < F_H(p_H^{SB})$ *for* $V \in (V^{**}, h_L(\bar{\theta}))$*.*

Proof of Lemma 1: (i) is straightforward. To establish (ii), for any $q \in [0,1]$, define $P_i(q) \in [\underline{\theta}, \bar{\theta}]$ such that $F_i(P_i(q)) = q$ and $C_i(q) \equiv qP_i(q)$. So we may interpret $C_i(q)$ as the 'cost' function in state i. Since $C_i'(F_i(\theta)) = h_i(\theta)$, Assumption 1 (ii) implies $C_i'(q)$ is increasing in q on $[0,1]$. Then $q_i^{SB} \equiv F_i(p_i^{SB})$ satisfies $V = C_i'(q_i^{SB})$ for $V \in (\underline{\theta}, h_i(\bar{\theta}))$. From Assumption 1 (i) and $f_i(\theta) > 0$ on $[\underline{\theta}, \bar{\theta}]$ for $i \in \{L, H\}$, $C_L(q) < C_H(q)$ on $q \in (0,1)$ with $C_L(0) = C_H(0) = 0$ and $C_L(1) = C_H(1) = \bar{\theta}$. Hence there are intervals of small q such that $C_L'(q) < C_H'(q)$ and large q such that $C_L'(q) > C_H'(q)$. This guarantees the existence of V^* and V^{**} with the stated properties. ■

5.1 Delegation to Expert with Ex Ante Collusion

Consider P's option to contract solely with M and delegate the authority to contract with A. With ex ante collusion, M does not commit to responding to P's offer before contracting with A. So after P offers M a contract, the latter offers A a contract. Following A's response, M then responds to P. On the other hand, with interim collusion argued in the later subsection, M makes the participation decision to P's contract before contracting with A.

Given this timing, standard arguments imply that (following any given contract offer) M can confine attention to offering A a take-it-or-leave-it price p_i in state i for delivering the good to P. And similarly P can confine attention to offering M a two part contract X_0, X_1 where X_q is the payment for delivery of output q. There is no added value to P asking M to submit a report of her signal or the outcome of contracting with A, as conditional on the q delivered M would select whichever message would maximize her payment received.

In order to induce M to deliver the good with positive probability, P must offer $X_1 > \underline{\theta}$. Upon observing signal i, M will then decide what price $p_i \in [\underline{\theta}, X_1]$ to offer A, along with participation decision in P's contract in either of the two events where A does or does not accept M's offer. If A accepts, it is optimal for M to agree to participate in P's contract since the optimal price will satisfy $p_i < X_1$. Let $I \in \{0, 1\}$ denote M's participation decision in the event that A does not accept M's offer. Then M selects p_i and I to maximize $F_i(p_i)(X_1 - p_i) + I[1 - F_i(p_i)]X_0$. It follows that $I = 1$ only if $X_0 \geq 0$. If $X_0 < 0$, M will not accept P's offer in the event that A does not accept M's offer. The same outcome is realized if P sets $X_0 = 0$. Hence without loss of generality, $X_0 \geq 0$, and M always accepts P's offer.

The constraint $X_0 \geq 0$ plays a key role in the subsequent analysis. It arises owing to ex ante collusion, whereby M contracts and communicates with A prior to responding to P's offer. In an interim collusion setting this constraint does not arise, and is replaced by interim participation constraints for M, whence X_0 can be negative and yet P's contract could be accepted by M.

Let b denote the delivery *bonus* $X_1 - X_0$. The choice of p_i will be made by M to maximize $F_i(p_i)(b - p_i)$. If $b \leq \underline{\theta}$, it is optimal for M to offer A a price below $\underline{\theta}$, whence the good is never delivered to P. Otherwise there is a unique optimal price $p_i(b)$ which satisfies $\underline{\theta} < p_i(b) < b$. Eventually P earns expected payoff $[\kappa_L F_L(p_L(b)) + \kappa_H F_H(p_H(b))](V - b) - X_0$, which is sought to be maximized by choosing $b > \underline{\theta}, X_0 \geq 0$. Now note that any such payoff would be strictly dominated by the option of not consulting M at all where P directly offers A a price of b. This follows since $b < V$ is necessary for P to earn a positive payoff; hence $[\kappa_L F_L(p_L(b)) + \kappa_H F_H(p_H(b))](V - b) - X_0 < F(b)(V - b) \leq \Pi^{NM}$. We thus obtain:

Proposition 3 *With an indivisible good and ex ante collusion, delegation to the expert is worse for the principal compared to not consulting the expert at all.*

As we shall later see, delegation could dominate the no-monitor outcome under interim collusion. This represents a stark contrast between the two forms of collusion. In delegation with ex ante collusion, M earns rents which cannot be taxed away upfront by P at the time of contracting with M, thereby generating a double marginalization of rents (DMR). Under interim collusion, P may be able to extract some of M's interim rents (in the absence of knowledge of A's type) via an upfront fee, thereby limiting the DMR problem.

5.2 Centralized Contracting with Ex Ante Collusion

Under ex ante collusion, therefore, if at all P obtains an advantage from consulting M, she needs to contract simultaneously with both M and A. M and A can negotiate a side-contract (SC, for short) prior to responding to P's offer. Following private communication of a cost message by A to M, the SC coordinates their respective messages (which include participation decisions and cost reports) sent to P, besides a side payment between A and M. As shown in Section 5.1 of the paper, without loss of generality M has all the bargaining power within the coalition and makes a take-it-or-leave-it SC offer to A. If A refuses it, they play P's mechanism non-cooperatively. As Proposition 1 of the paper shows, P can confine attention to allocation that is ex ante collusion proof (EACP) satisfying interim participation constraints, i.e., for which it is optimal for M to not offer any non-null SC to A, and both M and A agree to participate. We now explain the implied individual and coalition incentive compatibility constraints in the context of an indivisible good.

First, a contract offer to A reduces to a single take-it-or-leave-it price offer p_i when the cost signal is i. Second, in order to deter collusion, P must offer an aggregate payment to M and A which depends only on whether or not the good is produced. Let $X_0 + b, X_0$ denote the aggregate payments when the good is and is not produced respectively. The two prices p_L, p_H combined with X_0, b characterize an allocation entirely. This is associated with a mechanism where M and A are asked to submit reports of the signal i to P. If the two reports happen to match, A is offered the option to produce and deliver the good directly to P in exchange for price p_i, while M is paid X_0 if the good is not delivered, and $b + X_0 - p_i$ if it is delivered. If the two reports do not match, there is no production and both M and A are required to pay a high penalty to P. The key feature distinguishing centralized contracting from delegation is that in the former P makes a contract offer directly to A which is conditioned on reported signals. This provides an outside option to A which M is constrained to match while offering an SC to A. This is an important strategic tool which enables P to manipulate the outcome of collusion between M and A, and reduce the severity of the DMR problem.

Along the equilibrium path where A and M decide to participate, report i truthfully to P, and do not enter into a deviating SC, A produces the good in state i and receives the payment p_i if and only if θ is smaller than p_i. Without loss of generality, A receives no payment in the event of non-production (since any mechanism paying a positive amount to A in the event of non-production is dominated by one that does not). This generates utility to A of $u_A(\theta, i) = \max\{p_i - \theta, 0\}$. M ends up with $X_0 + b - p_i$ in the event that production takes place, and X_0 otherwise.

The allocation p_L, p_H, X_0, b has to satisfy the following constraints. First, in order to ensure that ex post the coalition does not prefer to reject it, the aggregate payment to M and A must be nonnegative in the event that the good is not delivered:

$$X_0 \geq 0. \tag{1}$$

The reason is that if the good is not delivered, A earns no rent; hence rejection of P's contract by the coalition does not entail any payoff consequence for A. If $X_0 < 0$, M would then benefit from rejecting P's contract; hence it is Pareto improving for the coalition to do so.[3] This constraint is distinctive to the ex ante collusion setting, where participation decisions in P's contract are made after M and A have negotiated a side contract.

[3]No analogous non-negativity constraint on aggregate payments $X_0 + b$ corresponding to delivery of the good is imposed here, because the decision to reject P's contract could result in a loss of rents for A. M would then have to compensate A for this loss, and the required compensation may be large enough that it may be optimal for M to instead accept P's contract despite $X_0 + b$ being negative. The issue of coalition incentive compatibility is addressed in more detail below.

Second, in order to induce M to participate ex ante:

$$F_H(p_H)(b - p_H) + X_0 \geq 0 \tag{2}$$

$$F_L(p_L)(b - p_L) + X_0 \geq 0 \tag{3}$$

Individual participation constraints for A are already incorporated into the supply decision represented by a supply likelihood of $F_i(p_i)$ in state i.

Third, M and A should not be tempted to enter a deviating SC. A deviating SC would involve a different set of prices $\tilde{p}_i$ offered to A (in state i) for delivering the good, combined with a lump-sum payment $\tilde{u}_i$. A would then produce if θ is smaller than $\tilde{p}_i$, and M would earn an expected payoff $F_i(\tilde{p}_i)(b - \tilde{p}_i) + X_0 - \tilde{u}_i$. Type θ of A would accept the deviating SC provided

$$\max\{\tilde{p}_i - \theta, 0\} + \tilde{u}_i \geq \max\{p_i - \theta, 0\} \tag{4}$$

Proposition 1 of the paper shows that without loss of generality, attention can be restricted to ex ante collusion-proof (EACP) allocations. Hence collusion-proofness requires $(\tilde{p}_i, \tilde{u}_i) = (p_i, 0)$ to maximize $F_i(\tilde{p}_i)(b - \tilde{p}_i) + X_0 - \tilde{u}_i$ subject to (4) for all types $\theta \in [\underline{\theta}, \bar{\theta}]$.

This condition can be broken down as follows. First, if $p_i > \underline{\theta}$, M should not benefit by deviating to a price $\tilde{p}_i < p_i$. This would necessitate offering a lump-sum payment of $\tilde{u}_i = p_i - \tilde{p}_i$ to ensure that all types of A accept the SC, which would then generate M an interim expected payoff of $F_i(\tilde{p}_i)(b - \tilde{p}_i) + X_0 - p_i + \tilde{p}_i$. A necessary and sufficient condition for such a deviation to not be worthwhile is that

$$b \geq p_i - \frac{1 - F_i(p_i)}{f_i(p_i)} \equiv l_i(p_i) \tag{5}$$

since $l_i(p)$ is increasing in p as per the monotone hazard rate assumption (Assumption 1(ii)). Intuitively, offering a lower price than p_i is similar to M selling the good back to A. Condition (5) which states that the value (b) of the good to M exceeds its virtual value to A, ensures that such a sale is not worthwhile.

Similarly, if $p_i < \bar{\theta}$, M should not want to offer A a higher price $\tilde{p}_i$. Unlike the case of a lower offer price, such a variation cannot be accompanied by a negative lump sum payment $\tilde{u}_i$ to A, owing to the need for A's ex post participation constraint to be satisfied in non-delivery states. Offering $\tilde{p}_i > p_i$ will then generate an interim payoff of $F_i(\tilde{p}_i)(b - \tilde{p}_i) + X_0$. For M to not want to deviate to a higher price, it must be the case that

$$b \leq p_i + \frac{F_i(p_i)}{f_i(p_i)} = h_i(p_i) \tag{6}$$

This condition can be interpreted simply as the value of delivery (b) to M being lower than the virtual cost of A delivering it.

(5, 6) can be combined into the single collusion-proofness condition

$$\max\{\hat{l}_L(p_L), \hat{l}_H(p_H)\} \leq b \leq \min\{\hat{h}_L(p_L), \hat{h}_H(p_H)\}. \tag{7}$$

where $\hat{l}_i(p_i)$ denotes $l_i(p_i)$ if $p_i > \underline{\theta}$ and $-\infty$ otherwise, and $\hat{h}_i(p_i)$ denotes $h_i(p_i)$ if $p_i < \bar{\theta}$ and ∞ otherwise (since the corresponding state i constraint is relevant only when p_i differs from $\underline{\theta}, \bar{\theta}$ respectively). This condition is referred to as coalition incentive constraint henceforth.

Proposition 1 in the paper implies that these conditions are necessary and sufficient for the allocation (p_L, p_H, b, X_0) to be the outcome of a Perfect Bayesian Equilibrium (PBE) of the ex ante collusion contracting game, which is interim-Pareto-undominated for the coalition by any other PBE. Hence, an optimal allocation must maximize

$$[\kappa_H F_H(p_H) + \kappa_L F_L(p_L)](V - b) - X_0 \tag{8}$$

subject to (1, 2, 3, 7). We refer to these constraints as characterizing ex ante collusion (EAC) feasibility.

It is convenient to restate P's profit as

$$U(p_L, p_H) - R(b, X_0; p_L, p_H) \tag{9}$$

where $U(p_L, p_H) \equiv \kappa_H F_H(p_H)(V - p_H) + \kappa_L F_L(p_L)(V - p_L)$ is the expression for expected profit in the second-best setting, from which M's rent $R(b, X_0; p_L, p_H) \equiv \kappa_H F_H(p_H)(b - p_H) + \kappa_L F_L(p_L)(b - p_L) + X_0$ has to be subtracted in the presence of collusion. Note also that given b, p_L, p_H it is optimal to set $X_0 = \max\{0, \max_i\{F_i(p_i)(p_i - b)\}\}$. With this convention we can henceforth represent an EAC allocation by the triple (p_L, p_H, b).

We start the analysis by making some simple but key observations regarding properties of any EAC-feasible allocation in which M is valuable (i.e, where the resulting profit exceeds the maximum profit attainable in NM).

Lemma 2 *In any EAC-feasible allocation in which M is valuable:*

(i) $b < p_i$ *for some* i *and* $X_0 > 0$

(ii) $p_L < p_H$

(iii) $F_L(p_L) > F_H(p_H)$.

Proof of Lemma 2: (i) If $b \geq p_i, i = L, H$, the optimal $X_0 = 0$. P's profit (8) then equals $[\kappa_H F_H(p_H) + \kappa_L F_L(p_L)](V - b)$, which is non-negative only if $V - b \geq 0$. This implies that P's profit is (weakly) dominated by the allocation $\tilde{p}_H = \tilde{p}_L = b$, which in turn is weakly dominated by what P could earn in NM. (ii) The interim participation constraints imply that M will attain a nonnegative rent. Hence P's profit is bounded above by $U(p_L, p_H)$. If $p_L \geq p_H$, the value of $U(p_L, p_H)$ is smaller than the maximum value of $U(\tilde{p}_L, \tilde{p}_H)$ subject to the constraint that $\tilde{p}_L \geq \tilde{p}_H$. The constraint must bind, since the unconstrained solution is represented by second-best prices which violate the constraint. Hence the maximum value of the constrained problem is realized at $\tilde{p}_H = \tilde{p}_L = p^{NM}$. The expected profit of P would then be dominated by the NM allocation where P offers p^{NM} to A in both states. (iii) Parts (i) and (ii) imply that in order to dominate the best NM allocation, an EAC feasible allocation must satisfy $p_H - b > \max\{0, p_L - b\}$. So if (iii) did not hold, $F_H(p_H)(p_H - b) \geq F_L(p_L)(p_L - b)$, and optimal $X_0 = F_H(p_H)(p_H - b)$. Then as $F_H(p_H) \geq F_L(p_L)$ implies $\kappa_H F_L(p_L) + \kappa_H F_H(p_H) \leq F_H(p_H)$, and $V \geq b$ to ensure that P earns non-negative profit, it follows that P's profit equals $[\kappa_L F_L(p_L) + \kappa_H F_H(p_H)](V - b) - F_H(p_H)(p_H - b) \leq F_H(p_H)(V - p_H) \leq F(p_H)(V - p_H) \leq \Pi^{NM}$, a contradiction. ■

Part (i) states that relevant EAC allocations must involve *low-powered incentives* for M in at least one state i, in the sense that ex post M is worse off in state i if the good is delivered than when it is not. This is the very opposite of delegation, where M earns a nonnegative margin on any transaction in every state. In ex ante collusion, the base pay X_0 must be positive in order to compensate for the 'loss' incurred by M when the good is delivered in state i (so as to ensure that M wants to participate at the interim stage corresponding to state i). Conversely, (i) may be viewed as stating that A is offered higher powered incentives than M in some state; this is a 'countervailing incentive' designed to raise A's outside option in bargaining with M over a side contract, so as to counter the DMR problem.

Part (ii) states that the low cost signal results in a lower price offered to A, just as in the second-best setting. The reason is that when the prices offered to A can vary with the cost signal, P's profit rises only if they result in a lower price being offered following a low cost signal. A variation in the opposite direction would directly result in lower profit, besides possibly entailing some rents paid to M. Part (iii) restricts the extent to which the prices can vary across the two states: the price in the low cost state should not be so low that the resulting supply likelihood becomes smaller in that state. Intuitively, larger variations in prices are not worthwhile because they generate high collusion stakes which raise M's rent excessively.

Lemma 2 indicates the problem of finding an optimal EAC allocation can be broken down into two successive stages. At the first stage, for any given pair of prices p_L, p_H satisfying (ii) and (iii), we find an optimal contract b for M to minimize M's rent subject to the coalition incentive constraint (7), and the requirements that $b < p_H$ and $X_0 = \max_i\{F_i(p_i)(p_i - b)\}$. Then at the second stage, prices p_L, p_H are selected to maximize $U(p_L, p_H) - R^*(p_L, p_H)$ subject to $p_L < p_H, F_L(p_L) > F_H(p_H)$, where $R^*(p_L, p_H)$ denotes the minimized rent of M at the first stage.

The next result describes the solution to the first stage problem, i.e., the optimal bonus for any set of prices satisfying (ii) and (iii). Upon substituting for the optimal base pay X_0, the expression for M's expected rent reduces to

$$\tilde{R}(b; p_L, p_H) \equiv \kappa_L F_L(p_L)(b - p_L) + \kappa_H F_H(p_H)(b - p_H) - \min\{F_L(p_L)(b - p_L), F_H(p_H)(b - p_H)\}. \tag{10}$$

Clearly $\tilde{R}$ is non-negative and attains a global minimum of zero at $b = \frac{p_L F_L(p_L) - p_H F_H(p_H)}{F_L(p_L) - F_H(p_H)} \equiv B(p_L, p_H) < p_L < p_H$. This turns out to be feasible (and hence $B(p_L, p_H)$ is optimal) if $B(p_L, p_H) \geq \max\{l_L(p_L), l_H(p_H)\}$, otherwise it is optimal to select the lowest bonus that is feasible, which is $\max\{l_L(p_L), l_H(p_H)\}$.

Lemma 3 *Given p_L, p_H satisfying $p_L < p_H$ and $F_L(p_L) > F_H(p_H)$, the optimal bonus $b(p_L, p_H) = \max\{B(p_L, p_H), l_L(p_L), l_H(p_H)\}$ where $B(p_L, p_H) \equiv \frac{p_L F_L(p_L) - p_H F_H(p_H)}{F_L(p_L) - F_H(p_H)}$.*

Proof of Lemma 3: To start with, note that the restrictions $p_L < p_H$ and $F_L(p_L) > F_H(p_H)$ imply that the prices are interior: $\underline{\theta} < p_i < \bar{\theta}, i = H, L$. Hence the coalition incentive constraint (7) simplifies to $\max_i\{l_i(p_i)\} \leq b \leq \min_i\{h_i(p_i)\}$. Next, note that upon substituting for the optimal base pay X_0, the expression for M's expected rent reduces to

$$\tilde{R}(b; p_L, p_H) \equiv \kappa_L F_L(p_L)(b - p_L) + \kappa_H F_H(p_H)(b - p_H) - \min\{F_L(p_L)(b - p_L), F_H(p_H)(b - p_H)\}. \tag{11}$$

Clearly $\tilde{R}$ is non-negative and attains a global minimum of zero at $b = B(p_L, p_H) < p_L < p_H$. If $B(p_L, p_H) \geq \max\{l_L(p_L), l_H(p_H)\}$, it is feasible to select $b = B(p_L, p_H)$ as the coalition incentive constraint (7) is satisfied (given that $p_i \leq h_i(p_i), i = H, L$), as well as the constraint that $b < p_H$. Hence in this case the optimal bonus equals $B(p_L, p_H)$. If $B(p_L, p_H) < \max\{l_L(p_L), l_H(p_H)\}$, then observe that over the range $b \geq B(p_L, p_H)$, $(b - p_L)F_L(p_L) \geq (b - p_H)F_H(p_H)$, implying that $X_0 = F_H(p_H)(b - p_H)$, or

$$\tilde{R} = \kappa_L[\{F_L(p_L) - F_H(p_H)\}b - p_L F_L(p_L) + p_H F_H(p_H)]. \tag{12}$$

Hence $\tilde{R}$ is strictly increasing in b over the range $b \geq B(p_L, p_H)$, and the optimal bonus in this case equals $\max\{l_L(p_L), l_H(p_H)\}$. ■

Next, we characterize properties of optimal EAC allocations (with p_i^E denoting the corresponding optimal price in state i).

Proposition 4 *With an indivisible good and ex ante collusion:*

(a) There exists $\hat{V}_1 \in (\underline{\theta}, h_H(\bar{\theta}))$ such that if $V \in (\underline{\theta}, \hat{V}_1)$ the second-best profit can be achieved.

(b) M is valuable if $V < H(\bar{\theta})$, but not if $V > \hat{V}_2$ for some $\hat{V}_2 \in (H(\bar{\theta}), h_L(\bar{\theta}))$.

(c) $p_H^E \leq p_H^{SB}$.

(d) $p_L^E \geq p_L^{SB}$ if $l_L(.)$ is convex.

Proof of Proposition 4: (a) By Lemma 1, $F_L(p_L^{SB}) > F_H(p_H^{SB})$ for V close to $\underline{\theta}$. As V approaches $\underline{\theta}$, p_i^{SB} approaches $\underline{\theta}$ for both $i = H, L$, and $B(p_L^{SB}, p_H^{SB})$ approaches $\underline{\theta} > \max_i\{l_i(\underline{\theta})\}$, implying $b(p_L^{SB}, p_H^{SB}) = B(p_L^{SB}, p_H^{SB})$ for V sufficiently close to $\underline{\theta}$. So $(p_L, p_H, b) = (p_L^{SB}, p_H^{SB}, B(p_L^{SB}, p_H^{SB}))$ is EAC feasible, implying the second-best profit can be achieved for V close to $\underline{\theta}$. By Lemma 1 (ii), there exists $V^{**} \in (\underline{\theta}, h_H(\bar{\theta}))$ such that $F_L(p_L^{SB}) \leq F_H(p_H^{SB})$ for $V \geq V^{**}$. Lemma 2 (iii) implies that $F_L(p_L^{SB}) > F_H(p_H^{SB})$ must hold if the second-best profit can be achieved in EAC-feasible allocation. Hence $\hat{V}_1 < h_H(\bar{\theta})$.

(b) $V < H(\bar{\theta})$ implies $p^{NM} < \bar{\theta}$. For any such V, we can find p_L, p_H sufficiently close to p^{NM} satisfying $p_L^{SB} \leq p_L < p^{NM} < p_H \leq p_H^{SB}, F_L(p_L) > F_H(p_H)$ and $\max_i\{l_i(p_i)\} < B(p_L, p_H)$ (since $B(p, p) = p > l_i(p), i = L, H$ for any $p < \bar{\theta}$). The allocation $(p_L, p_H, B(p_L, p_H))$ is then EAC feasible, in which M earns zero rent, and P earns a profit of $U(p_L, p_H) > U(p^{NM}, p^{NM}) = \Pi^{NM}$.

Next we show that M is not valuable at $V = \hat{V} \equiv \kappa_L h_L(\bar{\theta}) + \kappa_H h_H(\bar{\theta}) < h_L(\bar{\theta})$. Suppose otherwise, whence $F_L(p_L^E) > F_H(p_H^E)$ by Lemma 2. Note that $\hat{V} = \bar{\theta} + [\kappa_L \frac{1}{f_L(\bar{\theta})} + \kappa_H \frac{1}{f_H(\bar{\theta})}] > \bar{\theta} + \frac{1}{\kappa_L f_L(\bar{\theta}) + \kappa_H f_H(\bar{\theta})} = H(\bar{\theta})$. Hence $\Pi^{NM}(\hat{V}) = \hat{V} - \bar{\theta} = \kappa_L(h_L(\bar{\theta}) - \bar{\theta}) + \kappa_H(h_H(\bar{\theta}) - \bar{\theta})$. Now $\bar{\theta}$ is the second-best price when V equals $h_i(\bar{\theta})$ in state i, implying $h_i(\bar{\theta}) - \bar{\theta} \geq F_i(p_i^E)(h_i(\bar{\theta}) - p_i^E)$. Hence $\Pi^{NM}(\hat{V}) \geq \kappa_L F_L(p_L^E)(h_L(\bar{\theta}) - p_L^E) + \kappa_H F_H(p_H^E)(h_H(\bar{\theta}) - p_H^E) \geq \kappa_L F_L(p_L^E)(\hat{V} - p_L^E) + \kappa_H F_H(p_H^E)(\hat{V} - p_H^E) = U(p_L^E, p_H^E)$, where the second inequality follows from the definition of $\hat{V}$ and $F_L(p_L^E) > F_H(p_H^E)$. Since P's expected profit in EAC is bounded above by $U(p_L^E, p_H^E)$, we obtain a contradiction. Hence it is optimal to offer $p_i = \bar{\theta}$ for both i at $\hat{V}$. By a standard revealed preference argument, these prices are also optimal at any higher V. Hence M is not valuable at any $V > \hat{V}$.

(c) We first show that M's rent is locally non-decreasing in p_H at (p_L^E, p_H^E). If $B(p_L^E, p_H^E) > \max_i\{l_i(p_i^E)\}$, M earns zero rent which is unaffected by small variations in p_H. So suppose $B(p_L^E, p_H^E) \leq \max_i\{l_i(p_i^E)\}$ in which case $b^E = \max_i\{l_i(p_i^E)\}$ and $R^*(p_L^E, p_H^E) = \kappa_L[\{F_L(p_L^E) - F_H(p_H^E)\}\max_i\{l_i(p_i^E)\} + F_H(p_H^E)p_H^E - F_L(p_L^E)p_L^E] = \kappa_L \max_i \rho_i(p_H^E, p_L^E)$ where $\rho_i(p_H, p_L) \equiv \{F_L(p_L) - F_H(p_H)\}l_i(p_i) + F_H(p_H)p_H - F_L(p_L)p_L$. Now ρ_L is locally nondecreasing in p_H at (p_L^E, p_H^E) because $F_H(p_H)[p_H - l_L(p_L)]$ is increasing in p_H at (p_L^E, p_H^E) (the latter in turn follows from Lemma 2 and (7) which together imply $p_H^E > b^E = \max_i\{l_i(p_i^E)\} \geq l_L(p_L^E)$). And ρ_H is nondecreasing in p_H over the range of p_H which satisfies $F_L(p_L) > F_H(p_H)$ since $l'_H(p_H)[F_L(p_L) - F_H(p_H)] + f_H(p_H)[h_H(p_H) - l_H(p_H)] \geq 0$.

It now follows that if $p_H^E > p_H^{SB}$, a slight lowering of p_H will have a positive first order effect on $U(p_L, p_H)$, without raising M's rent. Hence $p_H^E \leq p_H^{SB}$.

(d) We show that M's rent is locally non-increasing in p_L at (p_L^E, p_H^E) if $l_L(p_L)$ is convex. When $B(p_L^E, p_H^E) > \max_i\{l_i(p_i^E)\}$, M's rent is zero which does not vary locally with p_L. So suppose $B(p_L^E, p_H^E) \leq \max_i\{l_i(p_i^E)\}$ implying that $R^*(p_L^E, p_H^E) = \kappa_L \max_i \rho_i(p_H^E, p_L^E)$. Now $F_L(p_L)[l_H(p_H^E) - p_L]$ is locally non-increasing in p_L at p_L^E, since its partial derivative with respect to p_L at p_L^E equals $f_L(p_L^E)[l_H(p_H^E) - h_L(p_L^E)]$, which is non-positive as (7) implies $l_H(p_H^E) \leq b^E \leq h_L(p_L^E)$. Hence ρ_H is locally nonincreasing in p_L at (p_L^E, p_H^E). The result therefore holds when $\rho_L(p_L^E, p_H^E) < \rho_H(p_L^E, p_H^E)$.

Next consider the case where $\rho_H(p_L^E, p_H^E) \leq \rho_L(p_L^E, p_H^E) = \frac{R^*(p_L^E, p_H^E)}{\kappa_L}$. Since $\frac{\partial \rho_L}{\partial p_L} = l'_L(p_L)[F_L(p_L) - F_H(p_H)] - 1$, the convexity of $l_L(p_L)$ implies the convexity of ρ_L in p_L over the range of p_L which satisfies $F_L(p_L) > F_H(p_H)$ for any fixed value of p_H. Now as p_L approaches p_H^E, $\rho_L(p_L, p_H^E)$ approaches $[F_L(p_H^E) - F_H(p_H^E)][l_L(p_H^E) - p_H^E] < 0$. Since

$\rho_L(p_L^E, p_H^E) \geq 0$, there must exist $\tilde{p}_L \in [p_L^E, p_H^E)$ where $\rho_L(\tilde{p}_L, p_H^E) = 0$ and ρ_L is locally decreasing in p_L. The convexity of $\rho_L(p_L, p_H^E)$ in p_L then implies that $\rho_L(p_L, p_H^E)$ is also locally decreasing in p_L at every p_L which satisfies $p_L \leq \tilde{p}_L$ and $F_L(p_L) > F_H(p_H^E)$. Since $p_L^E \leq \tilde{p}_L$, it follows that ρ_L is locally decreasing in p_L at (p_L^E, p_H^E).

It now follows that if $p_L^E < p_L^{SB}$, a slight increase in p_L will have a positive first-order effect on $U(p_L, p_H)$, without raising M's rent. Hence $p_L^E \geq p_L^{SB}$. ■

Part (a) states that the second-best profit can be achieved by P when V is low enough, while (b) says that consulting M is valuable for low values of V but not for sufficiently high values. Parts (c) and (d) describe how prices offered to A deviate from second-best prices. Provided l_L is convex, a condition satisfied in our example with linear conditional density functions and uniform prior, the dispersion between prices in the two states is narrower than in the second-best. The heuristic reason underlying these results is that collusion costs tend to rise with dispersion in prices p_i across the two states. For sufficiently low values of V, the second-best can be implemented, essentially because the dispersion between second-best prices corresponding to the different cost signals is small enough. The value of consulting M tends to decline as V rises, because this raises price dispersion and hence the rents paid to M.

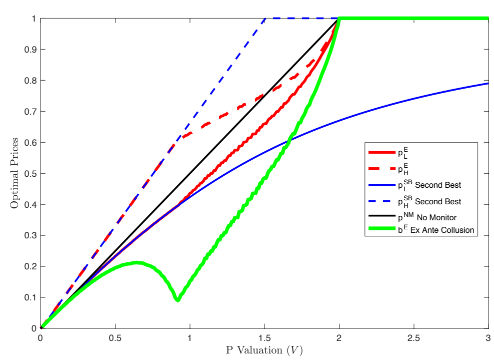

Figure 1: Second-Best, No Monitor and EAC Optimal Prices in Example with $d = 0.99$

This intuitive argument also helps explain why M is valuable for values of V smaller than $H(\bar{\theta})$. Starting with the optimal NM allocation where an interior price $p^{NM} < \bar{\theta}$ is offered, consulting M enables P to vary the price p_i with the cost signal in the direction of the second-best prices ($p_L^{SB} < p_L < p^{NM} < p_H < p_H^{SB}$). When the variation is slight, the induced stakes of collusion are small enough that M can earn no rents, thereby generating a profit improvement for P. Parts (c) and (d) reinforce this intuition, by showing that the distortion in prices compared with the second-best involves lower dispersion (given convexity of l_L).

These results are illustrated in our numerical example with $d = 0.99$. Figure 1 plots optimal prices offered to A in the second-best (SB), no monitor (NM) and ex ante collusion (E) settings, corresponding to different values of V. It also plots the corresponding EAC-optimal bonus values b^E. For low values of V, the second-best is implemented. Over this range price dispersion rises, as in the second-best. For intermediate values of V, M is valuable; over this range price dispersion narrows in contrast to rising dispersion in second-best prices. Eventually the gap between p_L^E and p_H^E is eliminated as V grows further, from which point onwards M ceases to be valuable.

5.3 Contrasting Optimal Solutions in Ex Ante and Interim Collusion Contexts

We now describe how (and when) the solution to EAC differs from interim collusion (INC). The formulation of the INC problem differs from the EAC problem in only one respect: the collusive participation constraint $X_0 \geq 0$ does not apply. An INC allocation can also be represented by the triple (b, p_L, p_H), where base pay X_0 is optimally set equal to $\max_i\{F_i(p_i)(p_i - b)\}$ and is permitted to be negative. Part (i) of Lemma 1 then no longer applies, opening up the possibility of providing high powered incentives with a bonus b larger than $\max_i\{p_i\}$ (as in a delegation setting), and then extracting M's rent upfront with a negative base pay. In particular, delegation to M can no longer be ruled out.

It is easy to check that in INC, part (ii) of Lemma 2 continues to apply (for the same reason), so $p_L < p_H$ is still necessary. However part (iii) need not apply: the likelihood of supply could be higher in the high cost state. The reason is that under interim collusion part (i) of Lemma 2 no longer holds — incentives could be high-powered ($b > p_H$). Part (iii) is then modified as follows (upon using a similar argument as in Lemma 2): an INC allocation where M is valuable must either (i) be low-powered (in the sense that $b < p_H$) and satisfy $F_L(p_L) > F_H(p_H), X_0 > 0$, or (ii) high-powered ($b > p_H$) and satisfy $F_H(p_H) > F_L(p_L), X_0 < 0$. It is evident that (i) is EAC feasible, while (ii) is not. We therefore obtain:

Lemma 4 *The optimal INC allocation differs from the optimal EAC allocation only if the former involves high powered incentives ($b > p_H > p_L$) and $F_H(p_H) > F_L(p_L)$.*

So we now focus on allocations with high-powered incentives where $b > p_H > p_L$ and $F_H(p_H) > F_L(p_L)$. The optimal bonus in ex ante collusion now differs from Lemma 3 as follows.

Lemma 5 *Given p_L, p_H satisfying $p_L < p_H$ and $F_H(p_H) > F_L(p_L)$, the optimal bonus in interim collusion is $b(p_L, p_H) = \min\{B(p_L, p_H), \hat{h}_L(p_L), \hat{h}_H(p_H)\}$. M is valuable only if $b > V$.*

Proof of Lemma 5: Given any pair of prices satisfying $p_L < p_H$ and $F_H(p_H) > F_L(p_L)$, the optimal bonus must minimize M's rent subject to $b > p_H$ and the coalition incentive constraint (7). M earns zero rent at the bonus $B(p_L, p_H) = \frac{p_H F_H(p_H) - p_L F_L(p_L)}{F_H(p_H) - F_L(p_L)}$ which is now larger than p_H. Since the choice of b is restricted to the range $b > p_H$ where $b > \max_i\{\hat{l}_i(p_i)\}$ is automatically satisfied, the bonus $B(p_L, p_H)$ is optimal if $B(p_L, p_H) \leq \min_i\{\hat{h}_i(p_i)\}$. Otherwise, $B(p_L, p_H) > \min_i\{\hat{h}_i(p_i)\}$ and the choice of b is restricted to the range $(p_H, \min_i\{\hat{h}_i(p_i)\}]$. Over this range $b < B(p_L, p_H)$ which implies $F_H(p_H)(b - p_H) < F_L(p_L)(b - p_L)$ and therefore $X_0 = -F_H(p_H)(b - p_H)$. The expression for M's rent is then modified to $\tilde{R}(b; p_L, p_H) = \kappa_L F_L(p_L)(b - p_L) + \kappa_H F_H(p_H)(b - p_H) - F_H(p_H)(b - p_H) = \kappa_L[\{F_L(p_L) - F_H(p_H)\}b + F_H(p_H)p_H - F_L(p_L)p_L]$, which is now *decreasing* in b.

To see that $b > V$ is necessary for M to be valuable, note that since the function $\tilde{R}(b; p_L, p_H)$ is decreasing in b, if $b \leq V$ then $p_H < V$, implying that P's profit is bounded above by $U(p_L, p_H) - \kappa_L[\{F_L(p_L) - F_H(p_H)\}V + F_H(p_H)p_H - F_L(p_L)p_L] = F_H(p_H)(V - p_H) \leq F(p_H)(V - p_H)$, the profit attained in NM upon choosing the price of p_H in both states. ■

The relevant range of bonuses and their effect on M's rent are thus reversed in interim collusion, compared to the EAC setting: the relevant range of b is $(p_H, \min_i\{\hat{h}_i(p_i)\}]$, over which M's rent is decreasing in b. Whenever M earns positive rents in INC, it is optimal for P to make incentives as high-powered as possible, and set the bonus to the maximum level $\min_i\{\hat{h}_i(p_i)\}$ consistent with the coalition incentive constraint. Moreover, the bonus needs to exceed V in order for M to be valuable.

We are now in a position to characterize some features of INC optimal allocations.

Lemma 6 *(i) An INC optimal allocation which is not EAC feasible can be implemented via delegation to M.*[4]

(ii) Second-best profits cannot be achieved by an INC optimal allocation which is not EAC feasible.

(iii) M is valuable in the INC optimal allocation for all $V \in [H(\bar{\theta}), h_L(\bar{\theta}))$.

Proof of Lemma 6: (i) As explained above, an INC optimal allocation which is infeasible in EAC must involve $p_L < p_H, F_H(p_H) > F_L(p_L)$ and in which M is valuable (since any allocation in NM is feasible in EAC). P attains profit $\Pi = [\kappa_H F_H(p_H) + \kappa_L F_L(p_L)](V - b) + F_H(p_H)(b - p_H) = \kappa_L F_L(p_L)(V - b) + F_H(p_H)[\kappa_H V + \kappa_L b - p_H]$. By Lemma 5, it is necessary that $b > V$. To show that this can be attained via INC with delegation, we need to show that if $p_i < \bar{\theta}$ then $b = h_i(p_i)$, while if $p_i = \bar{\theta}$ then $b \geq h_i(\bar{\theta})$.

Suppose first that $p_i < \bar{\theta}$ for either i. Then $\hat{h}_i(p_i) = h_i(p_i) \geq b$. If $i = L$ and $h_L(p_L) > b$, note that Π is strictly decreasing in p_L, so profit can be raised by lowering p_L slightly. Similarly, if $i = H$ and $b < h_H(p_H)$, we have $\kappa_H V + \kappa_L b < h_H(p_H)$, implying $F_H(p_H)[\kappa_H V + \kappa_L b - p_H]$ is locally strictly decreasing in p_H, and profit can be raised by lowering p_H slightly.

Next, suppose $p_i = \bar{\theta}$. If $b < h_i(\bar{\theta})$, the same argument as above applies: profit can be raised by lowering p_i slightly. Hence it must be the case that $b \geq h_i(\bar{\theta})$.

(ii) From (i), an INC optimal allocation which is EAC infeasible satisfies $p_i = p_i(b)$ which maximizes $F_i(p)(b - p)$ with respect to choice of $p \in [\underline{\theta}, \bar{\theta}]$. Since $F_L(p) > F_H(p)$ for all $p \in (\underline{\theta}, \bar{\theta})$, it must be true that $F_L(p_L)(b - p_L) \geq F_H(p_H)(b - p_H)$, with strict inequality if $b < h_L(\bar{\theta})$. Hence $b < h_L(\bar{\theta})$ implies M earns positive rent in state L (as $X_0 = -F_H(p_H)(b - p_H)$), and second-best profits cannot be achieved. And if $b \geq h_L(\bar{\theta})$, it must be the case that $p_L = p_H = \bar{\theta}$, in which case the INC optimal allocation can be attained in NM and therefore also in EAC.

(iii) Consider any $V \geq H(\bar{\theta})$, whence $\Pi^{NM} = V - \bar{\theta}$. The optimal INC profit is bounded below by what can be achieved via delegation in the interim collusion setting. If the bonus is b, the resulting prices will be $p_i(b)$, base pay will be set equal to $-F_H(p_H(b))(b -$

[4] It is evident that the INC optimal allocation which is EAC feasible is not achievable via delegation, since Lemma 2 implies $b \leq p_H$ in EAC feasible allocation, but the delegation to M induces $p_H (< b)$ which maximizes $F_H(p_H)(b - p_H)$.

$p_H(b)$) (using the argument in (ii) above), so the resulting profit will be $\Pi^{IND}(b;V) \equiv [\kappa_L F_L(p_L(b)) + \kappa_H F_H(p_H(b))][V-b] + F_H(p_H(b))[b - p_H(b)]$. $\Pi^{IND}(b;V)$ equals the P's payoff under bonus b in the delegation to M (in the interim collusion setting). The derivative of Π^{IND} with respect to b evaluated at $b = V$ then equals $\kappa_L[F_H(p_H(V)) - F_L(p_L(V))]$. Now observe that by definition of the $p_i(b)$ function, $p_i(V) = p_i^{SB}$. So $V \geq H(\bar{\theta})$ implies $p_H^{SB} = p^{NM} = \bar{\theta}$, so $p_H(V) = \bar{\theta}$. On the other hand, $p_L(V) = p_L^{SB} < \bar{\theta}$ since $V < h_L(\bar{\theta})$, so $F_L(p_L(V)) < 1 = F_H(p_H(V))$. It follows that Π^{IND} is strictly increasing in b when evaluated at $b = V$. Since $\Pi^{IND}(V;V) = V - \bar{\theta} = \Pi^{NM}$, it follows that M adds value in the INC optimal allocation. ■

Result (i) follows from observing that P's profits are decreasing in each price p_i in INC. Raising prices paid to A raises the likelihood of the good being delivered, which lowers P's profit largely as a consequence of paying a bonus exceeding what the good is worth to P (as shown in the previous Lemma). Hence if $h_i(p_i)$ exceeds b, it is profitable to lower p_i slightly while leaving the bonus b unchanged, as this would preserve feasibility of the allocation. This implies that the price offered to A is exactly what would have been chosen in each state by M under delegation. And under delegation M would earn a higher profit in the low cost state compared with the high cost state, owing to A's 'supply curve' being shifted to the right in the former relative to the latter. It is then impossible for P to fully extract M's rents in the low cost state, as M has to be willing to accept the contract in both states. Hence second-best profits cannot be achieved. Part (iii) shows that unlike the ex ante collusion setting, M remains valuable in interim collusion for all large V between $H(\bar{\theta})$ and $h_L(\bar{\theta})$. Intuitively this is because in the absence of collusion in participation and the associated DMR problem, delegation helps P control the stakes of collusion better.

Combining the various results above, we obtain the following Proposition which contrasts optimal solutions in the ex ante and interim collusion settings The solution to ex ante collusion involves low powered incentives, and in particular can never be achieved by delegation. Interim collusion involves a different allocation for large values of V, which is implemented via high-powered incentives (a delivery bonus that exceeds the value of the good to P, combined with delegation). Recall that we consider the range of possible values of V between $\underline{\theta}$ and $h_L(\bar{\theta})$.

Proposition 5 *(i) For sufficiently small values of V, EAC and INC optimal allocations*

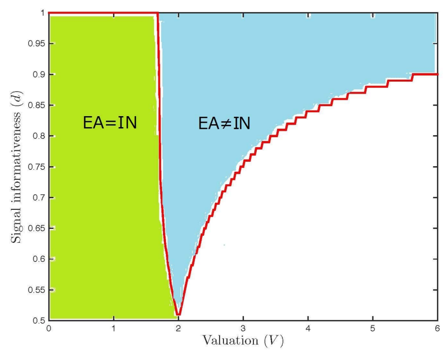

Figure 2: When Interim and Ex Ante Collusion Solutions Differ

coincide. For sufficiently large V, they are different.[5]

(ii) M is valuable in INC for all $V > H(\bar{\theta})$, whereas M is not valuable in EAC for sufficiently large V.

(iii) Whenever the INC optimal allocation differs from the EAC optimal allocation, it can be implemented via delegation to M, with prices $p_i^I \geq p_i^{SB}$ for $i = L, H$ and a bonus $b^I > V$ (with (p_i^I, b^I) corresponding to the INC optimal allocation).[6]

In the context of our numerical example, Figure 2 shows different regions of the two dimensional parameter space (V, d) where the INC optimal and EAC optimal solutions do and do not coincide. The unshaded subregion on the extreme right is excluded by our restriction that $V < h_L(\bar{\theta})$. In the subregion on the left (marked "EA=IN") involving relatively low values of V, the EAC and INC solutions coincide. In the middle subregion (marked "EA $\neq$ IN") they diverge.

Figure 3 plots the pattern of optimal prices in the INC optimal solution, corresponding

[5]By Proposition 4(a), it is evident that EAC and INC optimal allocations coincide for small V such that the second best allocation is EAC feasible. Even when the second best profit is not achievable, they can coincide, since the second best payoff is approximated by the EAC optimal one with sufficiently small V, while the proof of Lemma 6(ii) implies that it is never approximated with the delegation to M in the interim collusion setting.

[6]The result comparing INC optimal prices with second-best prices obtains from observing that prices corresponding to delegation with a bonus of V equal second-best prices, and the optimal bonus must exceed V.

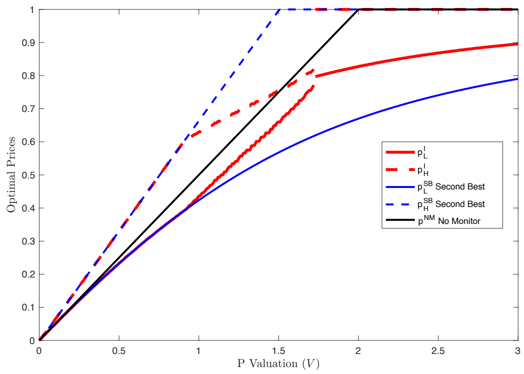

Figure 3: Optimal Prices with Interim Collusion

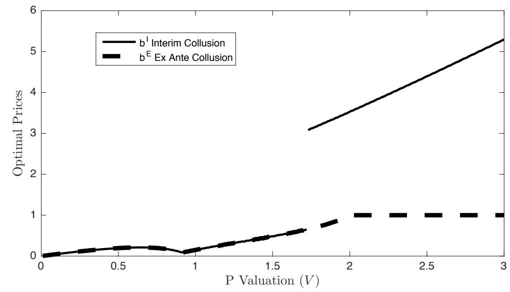

Figure 4: Optimal Bonus in Ex Ante and Interim Collusion

to different values of V (with d set equal to 0.99). For intermediate values of V where the second-best is not attained and the two solutions coincide, the price offered in the high cost signal state is smaller than the corresponding second-best price. As V rises further, the INC solution diverges from the EAC, causing a discontinuous switch in the pricing pattern: the price offered in the high cost signal state jumps up to the second-best price, resulting in locally increasing price dispersion.

Figure 4 plots the optimal bonus against alternative values of V (with d set equal to 0.99). Over the range where the EAC and INC solutions coincide, incentives are low-powered (the bonus is smaller than V). At the threshold where they just begin to diverge, the INC optimal bonus jumps discontinuously upwards while the EAC bonus continues to remain below V. Lemma 3 and 5 imply that when the second best allocation is not achievable, the INC optimal b has a corner solution property that it equals either the left hand side or the right hand side of the coalitional incentive constraints (7), depending on V. As V crosses the threshold, b switches from one to the other, generating a discontinuous change in (p_L, p_H).

Interim collusion is thus characterized by a discontinuous change in contracting strategy as V crosses the threshold, from a 'bureaucracy' (low-powered incentives, centralized contracting and low sensitivity of supplier price to cost information of the expert), to a 'market-like' contract resembling a franchise arrangement (high powered incentives, delegation, revenues earned primarily through franchise fees, and higher sensitivity of price to cost information). The market-based strategy is infeasible in the presence of ex ante collusion, since franchisees can then collude with their suppliers to avoid paying the upfront franchise fee in states of the world where suppliers cannot deliver owing to high cost realizations.

6 Ironing Rule and Related Results

Here we summarize the ironing procedure and its related properties which are frequently used throughout the paper. We specify an ironing rule to construct $\hat{\pi}(x)$ from two functions $\pi(x)$ and $G(x)$, and explain some properties about $\hat{\pi}(x)$. According to Myerson (1981) and Baron and Myerson (1982), the ironing rule is described as follows.

Definition 1 *Suppose that $\pi(x)$ and $G(x)$ defined on $[\underline{x}, \bar{x}]$ have the following properties:*

(i) $\pi(x-) \geq \pi(x+)$ for any $x \in [\underline{x}, \bar{x}]$.

(ii) $G(x)$ is distribution function with $G(\underline{x}) = 0$ and $G(\bar{x}) = 1$. $G(x)$ is strictly increasing and continuously differentiable on $[\underline{x}, \bar{x}]$.

Then $\hat{\pi}(x) \equiv \hat{\pi}(x \mid \pi(\cdot), G(\cdot))$ is constructed from $\pi(x)$ and $G(x)$ as follows.

(i) $\Pi(\phi) = \int_0^{\phi} \pi(h(y))dy$ where $h(\phi)$ satisfies $G(h(\phi)) = \phi$ for $\phi \in [0, 1]$.

(ii) $\underline{\Pi}(\phi)$ is maximum convex function so that $\Pi(\phi) \geq \underline{\Pi}(\phi)$.

(iii) $\hat{\pi}(x)$ satisfies (i) $\hat{\pi}(x) = \underline{\Pi}^{'}(G(x))$ whenever the derivative $\underline{\Pi}^{'}(G(x))$ is defined,[7] *and (ii) $\hat{\pi}(x) = \underline{\Pi}^{'}(G(x-))$ for any $x \in (\underline{x}, \bar{x}]$.*

We provide two lemmata, which show some properties used in the paper.

Lemma 7 *$\hat{\pi}(x) = \hat{\pi}(x \mid \pi(\cdot), G(\cdot))$ constructed from $\pi(x)$ and $G(x)$ satisfies:*

(i) $\hat{\pi}(x)$ is continuous and non-decreasing in x. If $\pi(x)$ is non-decreasing in x, $\hat{\pi}(x) = \pi(x)$.

(ii) $\int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x)$ if $q(x)$ is constant for each interval of x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$ (or $\hat{\pi}(x)$ takes constant value).

(iii) If $\pi(x) > x$ on $(\underline{x}, \bar{x}]$, $\hat{\pi}(x) > \hat{\pi}_\alpha(x)$ on $(\underline{x}, \bar{x}]$ for $\pi_\alpha(x) \equiv (1-\alpha)\pi(x) + \alpha x$ with $\alpha \in (0, 1]$.

[7] Since $\underline{\Pi}(\phi)$ is convex, it is almost everywhere differentiable.

(iv) $\hat{\pi}(\underline{x}) \le \pi(\underline{x})$ *and* $\hat{\pi}(\bar{x}) \ge \pi(\bar{x})$. *If there exists an increasing* $v(x)$ *so that* $v(x) < \pi(x)$ *for any* $x > \underline{x}$, $v(x) < \hat{\pi}(x)$ *for any* $x > \underline{x}$ *and if there exists an increasing* $v(x)$ *so that* $v(x) > \pi(x)$ *for any* $x > \underline{x}$, $v(x) > \hat{\pi}(x)$ *for any* $x > \underline{x}$.

(v) Suppose that $q^*(x)$ *is the solution of the following problem:*

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)]dG(x)$$

subject to $q(x)$ *is non-increasing. Then* $q^*(x)$ *solves*

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

Then

$$\int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \pi(x)q^*(x)]dG(x) = \int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \hat{\pi}(x)q^*(x)]dG(x).$$

Proof of Lemma 7

Proof of (i)

Since $\underline{\Pi}(\phi)$ is convex and $G(x)$ is increasing, $\hat{\pi}(x)$ is non-decreasing. In order to show the continuity of $\hat{\pi}(x)$, we start with proving the statement: *If* $\underline{\Pi}(\phi^{'}) = \Pi(\phi^{'})$ *for some* $\phi^{'} \in [0,1]$, *then*

$$\Pi^{'}(\phi^{'}-) \le \underline{\Pi}^{'}(\phi^{'}-) \le \underline{\Pi}^{'}(\phi^{'}+) \le \Pi^{'}(\phi^{'}+).$$

By the definition of $\underline{\Pi}(\phi)$ which is the maximum convex function such that $\Pi(\phi) \ge \underline{\Pi}(\phi)$ on $[0,1]$, if $\phi^{'} > 0$, for any $\epsilon \in [0, \phi^{'}]$,

$$\Pi(\phi^{'} - \epsilon) - \underline{\Pi}(\phi^{'} - \epsilon) \ge 0 = \Pi(\phi^{'}) - \underline{\Pi}(\phi^{'})$$

or equivalently

$$\frac{\underline{\Pi}(\phi^{'}) - \underline{\Pi}(\phi^{'} - \epsilon)}{\epsilon} \ge \frac{\Pi(\phi^{'}) - \Pi(\phi^{'} - \epsilon)}{\epsilon}.$$

This implies $\underline{\Pi}^{'}(\phi^{'}-) \ge \Pi^{'}(\phi^{'}-)$. Similarly, if $1 > \phi^{'}$, for any $\epsilon \in [0, 1-\phi^{'}]$,

$$\Pi(\phi^{'} + \epsilon) - \underline{\Pi}(\phi^{'} + \epsilon) \ge 0 = \Pi(\phi^{'}) - \underline{\Pi}(\phi^{'})$$

or equivalently

$$\frac{\underline{\Pi}(\phi^{'} + \epsilon) - \underline{\Pi}(\phi^{'})}{\epsilon} \le \frac{\Pi(\phi^{'} + \epsilon) - \Pi(\phi^{'})}{\epsilon}.$$

This implies $\underline{\Pi}^{'}(\phi^{'}+) \leq \Pi^{'}(\phi^{'}+)$. The convexity of $\underline{\Pi}(\phi)$ implies $\underline{\Pi}^{'}(\phi^{'}-) \leq \underline{\Pi}^{'}(\phi^{'}+)$. It completes the proof of the above statement.

By the definition, $\hat{\pi}(x)$ is left-continuous. To show the continuity of $\hat{\pi}(x)$, suppose otherwise that there exists x so that $\hat{\pi}(x) < \hat{\pi}(x+)$. It means that $\underline{\Pi}^{'}(G(x-)) < \underline{\Pi}^{'}(G(x+))$. Then $\underline{\Pi}(G(x)) = \Pi(G(x))$, since otherwise we can find a higher convex function than $\underline{\Pi}(\phi)$. By the definition of $\pi(x)$, $\pi(x-) = \Pi^{'}(G(x-))$ and $\pi(x+) = \Pi^{'}(G(x+))$. Then using the statement proven above,

$$\pi(x-) = \Pi^{'}(G(x-)) \leq \underline{\Pi}^{'}(G(x-)) < \underline{\Pi}^{'}(G(x+)) \leq \Pi^{'}(G(x+)) = \pi(x+)$$

This is contradiction since we assume that $\pi(x-) \geq \pi(x+)$. Therefore $\hat{\pi}(x)$ is continuous.

Suppose that $\pi(x)$ is non-decreasing in x. With $\Pi(\phi) = \int_0^\phi \pi(h(y))dy$, $\Pi^{'}(\phi) = \pi(h(\phi))$. Then $\Pi(\phi)$ is convex and $\Pi(\phi) = \underline{\Pi}(\phi)$, implying $\pi(x) = \hat{\pi}(x)$.

Proof of (ii)

Define I by

$$I \equiv \{x \in [\underline{x}, \bar{x}] \mid \Pi(G(x)) > \underline{\Pi}(G(x))\}.$$

For any $x \in I$, there exists $d(x)$ and $u(x)$ such as

$$\Pi(G(x^{'})) > \underline{\Pi}(G(x^{'}))$$

on $x^{'} \in (d(x), u(x))$, $\Pi(G(d(x))) = \underline{\Pi}(G(d(x)))$ and $\Pi(G(u(x))) = \underline{\Pi}(G(u(x)))$. Then $\underline{\Pi}(\phi^{'})$ is a linear function of $\phi^{'}$ on $[G(d(x)), G(u(x))]$ and $\hat{\pi}(x^{'})$ is constant on $x^{'} \in [d(x), u(x)]$. Then since $q(x^{'})$ is constant on $x^{'} \in [d(x), u(x)]$,

$$\int_{[d(x),u(x)]} q(x^{'})d\Pi(G(x^{'})) = \int_{[d(x),u(x)]} q(x^{'})d\underline{\Pi}(G(x^{'})).$$

Therefore it implies that

$$\int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) = \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)).$$

Since $\underline{\Pi}(\phi)$ is convex, it is almost everywhere differentiable with $\underline{\Pi}^{'}(G(x)) = \hat{\pi}(x)$ almost everywhere. This means that

$$\int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) = \int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x).$$

It is concluded that

$$\int_{\underline{x}}^{\bar{x}} q(x)\hat{\pi}(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)\pi(x)dG(x).$$

Proof of (iii)

Since the linear combination of two convex functions is convex, $(1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^\phi h(y)dy$ is convex function. Defining $\Pi_\alpha(\phi)$ by

$$\Pi_\alpha(\phi) \equiv \int_0^\phi \pi_\alpha(h(y))dy = (1-\alpha)\Pi(\phi) + \alpha \int_0^\phi h(y)dy.$$

Since

$$\Pi_\alpha(\phi) \geq (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^\phi h(y)dy,$$

$\underline{\Pi}_\alpha(\phi)$, which is the maximum convex function such that $\Pi_\alpha(\phi) \geq \underline{\Pi}_\alpha(\phi)$, satisfies

$$\Pi_\alpha(\phi) \geq \underline{\Pi}_\alpha(\phi) \geq (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^\phi h(y)dy.$$

Here our proof is composed of the analysis of two cases: (a) the region of x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$ and (b) the region of x such that $\Pi(G(x)) = \underline{\Pi}(G(x))$.

(a) For arbitrary x such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, there exists $d(x)$ and $u(x)$ such as

$$\Pi(G(x^{'})) > \underline{\Pi}(G(x^{'}))$$

on $x^{'} \in (d(x), u(x))$, $\Pi(G(d(x))) = \underline{\Pi}(G(d(x)))$ and $\Pi(G(u(x))) = \underline{\Pi}(G(u(x)))$. At $\phi = G(d(x))$ and $\phi = G(u(x))$,

$$\Pi_\alpha(\phi) = (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^\phi h(y)dy.$$

It implies that

$$\underline{\Pi}_\alpha(\phi) = (1-\alpha)\underline{\Pi}(\phi) + \alpha \int_0^\phi h(y)dy$$

at $\phi = G(d(x))$ and $\phi = G(u(x))$. Then since (i) of this lemma implies that $\underline{\Pi}_\alpha^{'}(\phi)$ and $\underline{\Pi}(\phi)$ are differentiable with respect to ϕ for any $\phi \in [0,1]$, the derivatives of both sides of the above equation with respect to ϕ, if evaluated at $G(u(x))$, have the following relationship:

$$\underline{\Pi}_\alpha^{'}(G(u(x))) \leq (1-\alpha)\underline{\Pi}^{'}(G(u(x))) + \alpha u(x) = (1-\alpha)\hat{\pi}(u(x)) + \alpha u(x).$$

Since $\hat{\pi}(u(x)) = \pi(u(x)) > u(x)$ (by $u(x) > \underline{x}$) and $\hat{\pi}_\alpha(u(x)) = \underline{\Pi}_\alpha^{'}(G(u(x)))$,

$$\hat{\pi}_\alpha(u(x)) < \hat{\pi}(u(x))$$

for any $\alpha \in (0,1]$. For any $x^{'} \in (d(x), u(x))$, $\hat{\pi}(x^{'}) = \hat{\pi}(u(x))$ and $\hat{\pi}_{\alpha}(x^{'}) \leq \hat{\pi}_{\alpha}(u(x))$ (since $\hat{\pi}_{\alpha}(x)$ is non-decreasing in x). Therefore

$$\hat{\pi}_{\alpha}(x^{'}) < \hat{\pi}(x^{'})$$

for any $x^{'} \in (d(x), u(x))$.

(b) For any $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$,

$$\Pi_{\alpha}(G(x)) = (1-\alpha)\underline{\Pi}(G(x)) + \alpha \int_{0}^{G(x)} h(y)dy.$$

It implies

$$\underline{\Pi}_{\alpha}(G(x)) = (1-\alpha)\underline{\Pi}(G(x)) + \alpha \int_{0}^{G(x)} h(y)dy$$

and

$$\hat{\pi}_{\alpha}(x) = \underline{\Pi}_{\alpha}^{'}(G(x)) = (1-\alpha)\hat{\pi}(x) + \alpha x < \hat{\pi}(x)$$

for any $\alpha \in (0,1]$, since $\hat{\pi}(x) = \pi(x) > x$ for $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$.

The argument in (a) and (b) implies the statement of (iii).

Proof of (iv)

(a) $\hat{\pi}(\underline{x}) \leq \pi(\underline{x})$ and $\hat{\pi}(\bar{x}) \geq \pi(\bar{x})$ are obtained from $\Pi^{'}(\phi = 0) \geq \underline{\Pi}^{'}(\phi = 0)$, $\Pi^{'}(\phi = 1) \leq \underline{\Pi}^{'}(\phi = 1)$ and $\Pi^{'}(G(x)) = \pi(x)$.

(b) The case of $v(x) < \pi(x)$: For $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$, $\hat{\pi}(x) = \pi(x) > v(x)$. For $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, and for $u(x)$ that is defined in the proof of (iii), $\hat{\pi}(x) = \Pi^{'}(G(u(x))) = \pi(u(x)) > v(u(x)) \geq v(x)$. It implies $\hat{\pi}(x) > v(x)$ for any $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$. Therefore $\hat{\pi}(x) > v(x)$ for any $x > \underline{x}$.

(c) The case of $v(x) > \pi(x)$: For $x > \underline{x}$ such that $\Pi(G(x)) = \underline{\Pi}(G(x))$, $\hat{\pi}(x) = \pi(x) < v(x)$. For $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$, and for $d(x)$ that is defined in the proof of (iii), $\hat{\pi}(x) = \Pi^{'}(G(d(x))) = \pi(d(x)) \leq v(d(x)) < v(x)$. It implies $\hat{\pi}(x) < v(x)$ for any $x > \underline{x}$ such that $\Pi(G(x)) > \underline{\Pi}(G(x))$. Therefore $\hat{\pi}(x) < v(x)$ for any $x > \underline{x}$.

Proof of (v)

Step 1:

For any non-increasing $q(x)$,

$$\int_{\underline{x}}^{\bar{x}} \pi(x)q(x)dG(x) = \int_{\underline{x}}^{\bar{x}} q(x)d\Pi(G(x)) \geq \int_{\underline{x}}^{\bar{x}} q(x)d\underline{\Pi}(G(x)) = \int_{\underline{x}}^{\bar{x}} \hat{\pi}(x)q(x)dG(x)$$

Proof of Step 1

Since $\Pi(G(x))$ and $\underline{\Pi}(G(x))$ are continuous, applying the integration by parts,

$$\int_{\underline{x}}^{\bar{x}} q(x) d\Pi(G(x)) + \int_{\underline{x}}^{\bar{x}} \Pi(G(x)) dq(x) = \Pi(1)q(\bar{x}) - \Pi(0)q(\underline{x})$$

and

$$\int_{\underline{x}}^{\bar{x}} q(x) d\underline{\Pi}(G(x)) + \int_{\underline{x}}^{\bar{x}} \underline{\Pi}(G(x)) dq(x) = \underline{\Pi}(1)q(\bar{x}) - \underline{\Pi}(0)q(\underline{x}).$$

With $\Pi(1) = \underline{\Pi}(1)$ and $\Pi(0) = \underline{\Pi}(0)$,

$$\begin{aligned} & \int_{\underline{x}}^{\bar{x}} q(x) d\Pi(G(x)) - \int_{\underline{x}}^{\bar{x}} q(x) d\underline{\Pi}(G(x)) \\ = & \int_{\underline{x}}^{\bar{x}} (\underline{\Pi}(G(x)) - \Pi(G(x))) dq(x) \geq 0 \end{aligned}$$

Step 2:

$$\int_{[\underline{x},\bar{x}]} [V(q^{**}(x)) - \pi(x)q^{**}(x)] dG(x) = \int_{[\underline{x},\bar{x}]} [V(q^{**}(x)) - \hat{\pi}(x)q^{**}(x)] dG(x)$$

for $q^{**}(x) \in \arg\max_q V(q) - \hat{\pi}(x)q$.

Proof of Step 2:

By the definition, $q^{**}(x)$ is constant for each interval of x where $\hat{\pi}(x)$ is constant. Then by (ii) of the lemma,

$$\int_{\underline{x}}^{\bar{x}} \pi(x) q^{**}(x) dG(x) = \int_{\underline{x}}^{\bar{x}} \hat{\pi}(x) q^{**}(x) dG(x).$$

This completes the proof of Step 2.

Step 3:

By Step 1, for any non-decreasing $q(x)$,

$$\int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)] dG(x) \leq \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)] dG(x).$$

By Step 2, if $q^*(x)$ is the solution of

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \pi(x)q(x)] dG(x)$$

subject to $q(x)$ is non-increasing, then $q^*(x)$ solves

$$\max \int_{\underline{x}}^{\bar{x}} [V(q(x)) - \hat{\pi}(x)q(x)]dG(x).$$

Then

$$\int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \pi(x)q^*(x)]dG(x) = \int_{\underline{x}}^{\bar{x}} [V(q^*(x)) - \hat{\pi}(x)q^*(x)]dG(x).$$

It completes the proof of (v). ■

Lemma 8 *$\hat{h}(\theta \mid \eta)$ is non-increasing and continuous in θ on $\Theta(\eta)$ with $\hat{h}(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$ and $\hat{h}(\theta \mid \eta) > \theta$ for $\theta > \underline{\theta}(\eta)$.*

Proof of Lemma 8

Since $h(\theta \mid \eta)$ is continuous, Lemma 7(i) implies that $\hat{h}(\theta \mid \eta)$ is continuous and non-decreasing in θ. Since $\theta < h(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$, Lemma 7(iv) implies that $\theta < \hat{h}(\theta \mid \eta)$ for $\theta > \underline{\theta}(\eta)$. By the continuity of $\hat{h}(\theta \mid \eta)$, $\underline{\theta}(\eta) \leq \hat{h}(\underline{\theta}(\eta) \mid \eta)$. Lemma 7(iv) also implies $\hat{h}(\underline{\theta}(\eta) \mid \eta) \leq h(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$. Therefore $\hat{h}(\underline{\theta}(\eta) \mid \eta) = \underline{\theta}(\eta)$. ■

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