1. (5+10+10=25 marks) An economy has a population of poor entrepreneurs of varying ability $a \in [0,1]$ that seek to borrow in order to finance an investment project. All entrepreneurs have zero wealth of their own, and zero outside option. An entrepreneur of ability $a$ needs to borrow $(1-a)$ to start the project, which is successful with probability $e$ and a failure otherwise. The project generates $Q > 1$ if successful and nothing otherwise. There is only one lender $L$ who they can approach for funds. Thie lender can lend any amount at a constant opportunity cost of $\rho$. Loans are subject to limited liability. The probability of success $e$ is chosen privately by the borrower after getting a loan, at a personal cost of $\frac{e^2}{2k}$. $L$ can observe any borrowers ability before deciding whether to lend, but not their choice of $e$. The parameters satisfy $4\rho > kQ^2 > 2\rho$ and $kQ < 1$. All parties are risk-neutral.

(a) Suppose $L$ acts as a monopolist. For entrepreneurs of any given ability type $a$, will $L$ agree to lend to them and if so at what interest rate?

Effort $e$ of borrower given repayment obligation of $R \geq 0$ in the event of success maximizes $e(Q - R) - \frac{e^2}{2k}$, so $e = k(Q - R) \leq kQ < 1$. Hence expected payoff of lender in lending $1-a$ with a repayment obligation $R$ is $\Pi_L(R,a) = k(Q - R)R - \rho(1-a)$. Any such loan will generate a non-negative payoff to the borrower (since the borrower always has the option of selecting $e = 0$) so the participation constraint of the borrower can be ignored. The value of $R$ that maximizes $\Pi_L$ is $R = \frac{Q}{2}$, whereupon $L$ earns a payoff of $\Pi_L(\frac{Q}{2},a) = \frac{kQ^2}{2} - \rho(1-a)$, which is non-negative if and only if $a \geq 1 - \frac{kQ^2}{4\rho} \equiv a \in (0, \frac{1}{2})$ since by hypothesis $4\rho > kQ^2 > 2\rho$. Hence all entrepreneurs with $a$ below $\underline{a}$ will not get a loan. Those with $a$ above $\underline{a}$ will receive a loan of $(1-a)$ at an interest rate of $r_M(a) = \frac{Q}{2(1-a)} - 1$.

(b) Suppose the government imposes a cap of $\frac{Q}{4} - 1$ on the interest rate that lenders can charge. What will the impact of this be on loan outcomes, borrower welfare and social (utilitarian) welfare for each ability type?
Note that the interest rate is increasing in $a$, since the repayment amount is independent of $a$ while the loan size is decreasing in $a$. The lowest interest rate is the one charged to type $a$. This is $r_M(a) = \frac{Q}{kQ^2/2\rho} \in \left( \frac{Q}{2}, Q \right)$ given the parameter restrictions. Hence the interest rate cap is binding for all agents who get a loan. Note that for the agent of type $a$ the monopoly lender just broke even with a repayment obligation of $\frac{Q}{2}$, so will lose money in the presence of a cap. So type $a$ will not get a loan with the cap.

To check which types above $a$ will still receive a loan, we calculate the maximum profit the lender can make in the presence of the cap with type $a$. Since the monopolist’s profit function is quadratic in $R$ and the cap is binding for all $a > a$, it will be optimal for the monopolist to set $R(a) = (1 - a)\frac{Q}{2}$. He will then earn a payoff of $\Pi_L(R(a), a) = (1 - a)\rho\left[\frac{(3+a)kQ^2}{16\rho} - 1\right] < (1 - a)\rho\left[\frac{kQ^2}{4\rho} - 1\right] < 0$. Hence the lender will lend to nobody in the presence of the cap: the market will shut down. Both lender and borrower’s payoffs (for types above $a$) will be lower: they will lose the payoffs they were attaining in the monopoly allocation, and there will be a fall in utilitarian welfare (equal to the sum of these payoffs).

(c) Describe how your answer to (a) and (b) would change if the credit market involved Bertrand competition between $L$ and another lender in the market with the same opportunity cost of lending.

With Bertrand competition, the lenders will break-even, implying that for any type $a$ the repayment obligation will be $R_c(a)$ which is the smaller root of $k(Q-R)R-(1-a)\rho = 0$. This equation has real roots if $a \geq \underline{a}$. Hence the range of types that will get a loan will be the same as under monopoly: $a \geq \underline{a}$. For any such $a$, $$R_c(a) = \frac{Q}{2} - \frac{1}{2}\left[Q^2 - \frac{(1-a)\rho}{k}\right]^{\frac{1}{2}}$$

The corresponding interest rate is $r_c(a)$ where $$1 + r_c(a) = \frac{Q}{2(1 - a)} - \frac{1}{2(1 - a)}\left[Q^2 - \frac{(1-a)\rho}{k}\right]^{\frac{1}{2}}$$

Now $1 + r_c(a)$ tends to $\frac{\rho}{4kQ} < \frac{Q}{4}$ as $a$ approaches 1, and it tends to $Q(\frac{1}{2} - \frac{3}{4}) < \frac{Q}{4}$ as $a$ approaches $\underline{a}$. Check that $r_c(a)$ is increasing in $a$. Hence the interest cap is not binding in the presence of Bertrand competition, so it has no effect at all in this case.
2. (2+8+10+5=25 marks) Consider an OLG economy where agents can never borrow, but can lend at a fixed interest rate $r$. A generation $t$ parent inherits financial wealth $W_t \geq 0$, earns a fixed wage $\omega > 0$, ending with lifetime wealth of $W_t + \omega$. She decides on a financial bequest $b_t \geq 0$ to leave to her child, whereupon the child will inherit a financial wealth of $W_{t+1} = (1 + r)b_t$. The parent has a ‘paternalistic’ bequest motive, and seeks to maximize $U(c_t) + V(W_{t+1})$ where $c_t$ denotes the generation $t$ parent’s consumption (constrained to be nonnegative), and $U, V$ are strictly increasing, strictly concave, smooth functions satisfying Inada conditions.

(a) Show that $W_{t+1}$ is increasing in $W_t$.

The problem faced by a parent is select $b_t \geq 0$ to maximize $U(\omega + W_t - b_t) + V((1+r)b_t)$. Owing to concavity of $U$, the marginal cost of a bequest to the parent $U''(\omega + W_t - b_t)$ is decreasing in $W_t$. Hence the optimal bequest is nondecreasing in $W_t$, which implies $W_{t+1} = (1 + r)b_t$ is nondecreasing in $W_t$.

(b) Suppose $U$ has constant elasticity $\sigma$, and $V = \delta U$, where $\delta \in (0,1)$. Calculate explicitly $W_{t+1}$ as a function of $W_t, \omega, r, \sigma$. Provide a condition on the parameters which ensures that $W_t$ converges to a unique limit, irrespective of $W_0$. Interpret this condition.

The problem is to maximize $\frac{1}{\sigma}[(\omega + W_t - \frac{W_{t+1}}{1+r})^\sigma + \delta(W_{t+1})^\sigma]$ where $\sigma < 1$ and $\sigma \neq 0$ (the case of $\sigma = 0$ corresponds to log utility). The first order condition implies $W_{t+1} = \frac{\theta}{1+\frac{r}{1+r}}(\omega + W_t)$ where $\theta \equiv [\delta(1+r)]^{\frac{1}{1-\sigma}}$.

The condition for convergence to a unique limit is $\frac{\theta}{1+\frac{r}{1+r}} \in (0,1)$, which reduces to $\delta < (1+r)(1 + \frac{1}{r})^{1-\sigma}$. Parental altruism cannot be too large, relative to the interest rate. Otherwise family wealths will grow unboundedly.

(c) Extend the model to allow parents to make educational investments in their children, in addition to leaving them a financial bequest. The economy has two occupations (skilled and unskilled), with wages in each occupation that depend on the proportion of skilled households in the economy. There is a given training cost $x > 0$ necessary to enter the skilled occupation.
(while no training is required for the unskilled occupation). Provide a definition of a dynamic competitive equilibrium with perfect foresight in this economy and associated steady states.

For parents to want to invest in their children’s education they should care about their earnings as well as bequests. A reasonable specification of the parental utility is $U(w_t + W_t) + V(w_{t+1} + W_{t+1})$. The wealth parent in any given generation $t$ has two components: their skill $e_t \in \{0, 1\}$ and financial wealth $W_t$.

A competitive equilibrium starting from a given joint distribution $G_0$ over $(e_0, W_0)$ is a sequence of joint distributions $G_t, t \geq 1$ over $(e_t, W_t)$ with corresponding proportions of skilled individuals $\lambda_t, t \geq 0$ and skilled and unskilled wages $w_{st} = F_s(\lambda_t, 1 - \lambda_t), w_{ut} = F_u(\lambda_t, 1 - \lambda_t)$ such that:

(i) a parent in generation $t$ with assets $e_t, W_t$ selects assets of his child $e_{t+1} \in \{0, 1\}, W_{t+1} \geq 0$ to maximize $U(e_t w_{st} + (1 - e_t) w_{ut} + W_t - xe_{t+1} - \frac{W_{t+1}}{1+r}) + V(e_{t+1} w_{s,t+1} + (1 - e_{t+1}) w_{u,t+1} + W_{t+1})$

(ii) these decisions generate the distribution $G_{t+1},$ given $G_t$.

A steady state is a stationary distribution $G^*$ over assets, i.e., if $G_t = G^*$, then $G_{t+1} = G^*$.

(d) Would you conjecture that in this economy every steady state will necessarily be characterized by persistent inequality? Provide some intuition or reasons for your answer.

It is possible for there to be steady states with perfect equality of total wealth, even if both occupations are essential. Skilled individuals must of course earn more than unskilled individuals ($w_s > w_u$), so as to to provide incentives for skill accumulation. This could be offset by unskilled individuals inheriting more than skilled individuals ($W_s < W_u$), so that $w_s + W_s = w_u + W_u$. All parents would have the same wealth and thus incur the same sacrifice for improving the wealth of their children. They would have to be indifferent between leaving them wealth in the form of human capital and in the form of financial assets. So it must be the case that $(1 + r)x = w_s - w_u$, i.e., the rate of return on education must be equal to $r$ the return on financial assets. This in turn ties down what the proportion
of skilled individuals must be in such an equal steady state. The usual logic for persistence of inequality therefore does not apply in the presence of financial assets.