Ec721 PROBLEM SET 2 SOLUTIONS

1. There are two individuals A and B whose incomes are perfectly negatively correlated: with probability half, A earns $y + \Delta$ (where $\Delta > 0$) while B earns $y - \Delta$, while with probability half their incomes are reversed. Income shocks are independent across dates $t = 1, 2, ...$ Each is risk-averse, with a VNM utility $u(c)$ which is strictly increasing and strictly concave in consumption c. They enter into an informal risk-sharing arrangement where the person earning the higher income promises to transfer $t \in [0, \Delta]$ to the other. They have a common discount factor $\delta \in (0, 1)$.

(a) How does the ex ante utility of the two individuals vary with t? Whats the unconstrained optimal transfer?

The ex ante utility is $\frac{1}{2}[u(y + \Delta - t) + u(y - \Delta + t)]$ *which is increasing in* t *over* $[0, \Delta]$ *owing to strict concavity of* u. *Hence the unconstrained optimal transfer is* $t^* = \Delta$.

(b) Write down the incentive constraint which characterizes values of t that can be sustained as a subgame perfect Nash Equilibrium.

$$u(y + \Delta) - u(y + \Delta - t) \leq \frac{\delta}{2(1 - \delta)}[u(y + \Delta - t) + u(y - \Delta + t) - \{u(y + \Delta) + u(y - \Delta)\}]$$

(c) Provide conditions under which: (i) the first-best transfer can be sustained; (ii) no transfer can be sustained.

(i) The first-best $t^* = \Delta$ *can be sustained if* $\delta \geq \delta^*$, *where* δ^* *solves*

$$u(y + \Delta) - u(y) = \frac{\delta^*}{2(1 - \delta^*)}[2u(y) - \{u(y + \Delta) + u(y - \Delta)\}]$$

(ii) Rewrite the incentive constraint as follows (using D *to denote* $\frac{\delta}{2(1-\delta)}$*):*

$$L(t) \equiv (1 + D)u(y + \Delta - t) + Du(y - \Delta + t) \geq (1 + D)u(y + \Delta) + Du(y - \Delta)$$

Note that $L(t)$ *is strictly concave, while the right-hand-side is independent of* t. *Moreover, at* $t = 0$ *the two sides are equal. Hence no positive* t *can be sustained if* $L'(0) \leq 0$, *or* $Du'(y - \Delta) \leq (1 + D)u'(y + \Delta)$. *Thus if* $\underline{D}$ *satisfies* $\frac{D}{1+D} = \frac{u'(y+\Delta)}{u'(y-\Delta)}$, *then* $D \leq \underline{D}$ *implies no positive* t *can be sustained.*

(d) Suppose neither conditions for (i) or (ii) in (c) hold. Show that the set of transfers that can be sustained is an interval of the form $[0, \hat{t}]$ where $\hat{t} \in (0, \Delta)$. What is the optimal transfer that can be sustained, and show how this can be computed in the case where $u(c) = \log c$.

Since $L(t)$ *is strictly concave, the set of transfers that can be sustained is convex, hence an interval. If neither (i) or (ii) hold, this interval must take the form* $[0, \hat{t}]$ *where* $\hat{t} < \Delta$. *Hence the optimal transfer is* $\hat{t}$, *which satisfies* $L(\hat{t}) = u(y + \Delta) + D[u(y + \Delta) + u(y - \Delta)]$. *In the case of log utility, this equation takes the form*

$$(y + \Delta - \hat{t})(y - \Delta + \hat{t})^{\frac{D}{1+D}} = (y + \Delta)(y - \Delta)^{\frac{D}{1+D}}.$$

2. A farmer plants two crops r and h at $t = 0$, spending x_r, x_h on corresponding inputs respectively. At $t = 1$ the crops are harvested, but their returns are uncertain. There are two states g, b with probabilities π_g, π_b respectively. In state g, the farmer earns $A_g f(x_r)$, while in state b earnings are $A_b f(x_h)$. The function f is strictly increasing, strictly concave, twice differentiable and satisfies $f'(0) = \infty$. Moreover, $A_g > A_b$ and $\pi_g A_g > \pi_b A_b > 0$. The farmer has liquid wealth m at $t = 0$. The farmer's seeks to maximize $u(c_0) + \delta[\pi_g u(c_g) + \pi_b u(c_b)]$ where c_0, c_g, c_b denote consumption at $t = 0$, and states g, b respectively at $t = 1$. The utility u function is strictly increasing, strictly concave and satisfies Inada conditions. The discount factor $\delta = \frac{1}{1+r}$ where $r > 0$.

(a) Suppose the farmer can borrow and lend without any restriction at interest rate r, and can also purchase any amount of insurance I by paying a premium of $\frac{\pi_b}{\pi_g} I$ in state g to receive a payout I in state b. Derive the farmer's optimal choices of saving and inputs x_r, x_h, and show that the farmer's income is perfectly smoothed. What would be the effects of a small capital grant G offered by the government at the time of planting? Or of a relief payment K in state b?

Given any G, K, the farmer selects s, x_r, x_h to maximize (where $R \equiv 1 + r$):

$$u(m - s - x_r - x_h + G) + \frac{1}{R}[\pi_g u(Rs + A_g f(x_r) - \frac{\pi_b}{\pi_g} I) + \pi_b u(Rs + A_b f)x_h) + I + K)$$

FOC with respect to s, I, x_r, x_h gives:

$$u'(c_0) = \pi_g u'(c_g) + \pi_b u'(c_b) \quad (1)$$

$$u'(c_g) = u'(c_b) \quad (2)$$

$$u'(c_0) = \frac{1}{R}\pi_g A_g f'(x_r) u'(c_g) = \frac{1}{R}\pi_b A_b f'(x_h) u'(c_b) \quad (3)$$

Equations (1, 2) imply $c_0 = c_g = c_b$, and (3) then implies input choices are productively efficient $x_r = x_r^*, x_h = x_h^*$ where $\pi_g A_g f'(x_r^*) = R = \pi_b A_b f'(x_h^*)$. Hence the capital grant or relief payment will have no effect on crop inputs.

(b) Now suppose there is a borrowing constraint whereby savings have to be non-negative, while there is no constraint on insurance purchases. Show that the farmer's optimal response will imply $c_g = c_b \geq c_0$. Compare input choices with that in (a) above. What would the effects of a capital grant at $t = 0$, or relief payments in state b be in this situation?

The FOC with respect to insurance purchase will imply $c_g = c_b = c_1$ say, while the borrowing constraint implies $u'(c_0) \geq \pi_g u'(c_g) + \pi_b u'(c_b) = u'(c_1)$. FOC (3) continues to hold, hence we now have $\pi_b A_b f'(x_h) = \pi_g A_g f'(x_r) = R\frac{u'(c_0)}{u'(c_1)} \geq R$, implying the farmer cuts back on inputs for both crops: $x_r \leq x_r^*, x_h \leq x_h^*$. A capital grant will lower $\frac{u'(c_0)}{u'(c_1)}$ and will thus lead to more planting of both crops. The relief payment will have the opposite effect.

(c) Consider the converse situation to (b): there is no borrowing constraint, but the farmer cannot purchase any insurance. Show that in the absence of any capital grants or relief payments $c_g > c_0 > c_b$, and $x_r < x_r^*, x_h > x_h^*$ where x_h^*, x_r^* denote input choices in (a).

Now (1) and (3) hold, but not (2). If $c_b \geq c_g$, it must be the case that $x_h > x_r$ since $A_b < A_g$ and there are no capital grants or relief payments. Hence concavity of f implies $f'(x_h) < f'(x_r)$. We obtain a contradiction with (3) which implies $\pi_g A_g f'(x_r) u'(c_g) = \pi_b A_b f'(x_h) u'(c_b)$, since $\pi_g A_g > \pi_b A_b$ and $u'(c_g) \geq u'(c_b)$ owing to concavity of u. Therefore $c_b < c_g$. The FOC (1) then implies $c_g > c_0 > c_b$. Hence $\frac{u'(c_0)}{u'(c_g)} > 1 > \frac{u'(c_0)}{u'(c_b)}$. The input planting FOCs (3) then implies the farmer underinvests in crop r and overinvests in crop h: $x_r < x_r^*, x_h > x_h^*$.

(d) Suppose the farmer can neither save or borrow, nor purchase any insurance. If $u(c) = \log c$ and $f(x) = x^\alpha$ where $\alpha \in (0, 1)$, describe the effects of increasing relief payments in state b on crop input decisions.

With neither opportunity to save, borrow or insure, only (3) holds, which with log utility takes the form

$$\frac{1}{m - x_r - x_h + G} = \frac{\pi_g A_g f'(x_r)}{R A_g f(x_r)} = \frac{\pi_b A_b f'(x_h)}{R(A_b f(x_h) + K)} \tag{4}$$

which given $f(x) = x^\alpha$ simplifies to

$$m - x_r - x_h + G = \frac{R x_r}{\alpha \pi_g} = \frac{R}{\alpha \pi_b}[x_h + \frac{K}{A_b} x_h^{1-\alpha}] \tag{5}$$

The second inequality here implies we can solve for x_r as a function of x_h and K:

$$x_r = \frac{\pi_g}{\pi_b}[x_h + \frac{K}{A_b} x_h^{1-\alpha}] \tag{6}$$

and then solve for x_h from

$$m + G - \frac{\pi_g}{\pi_b}[x_h + \frac{K}{A_b} x_h^{1-\alpha}] - x_h = \frac{R}{\alpha \pi_b}[x_h + \frac{K}{A_b} x_h^{1-\alpha}] \tag{7}$$

There is a unique solution for x_h in (7) since the left-hand-side is decreasing in x_h while the right-hand-side is increasing in x_h. An increase in K raises the right-hand-side and lowers the left-hand-side, hence x_h will be decreasing in K. Moreover, (7) can be rewritten as

$$m + G - x_h = [\frac{\pi_g}{\pi_b} + \frac{R}{\alpha \pi_b}][x_h + \frac{K}{A_b} x_h^{1-\alpha}] \tag{8}$$

Since the left-hand-side increases as K increases, it follows that $[x_h + \frac{K}{A_b} x_h^{1-\alpha}]$ increases. From (6) it now follows that x_r increases. Hence the relief payments move planting decisions closer to their productively efficient levels.