EC 717, SOLUTIONS TO PROBLEM SET NO. 3

1. (a) The c.d.f. of v is $F(v) = \frac{v}{v_T}$ for $v \in [0, v_T]$, and the density is $f(v) = \frac{1}{v_T}$. So the virtual valuation of v is $v - \frac{1-F(v)}{f(v)} = 2v - v_T$. Hence the monopolist's problem at T is to select probability $\alpha(v)$ of selling to a customer of type v which is nonincreasing in v and which maximizes

$$\int_0^{v_T} \alpha(v)[2v - v_T]\frac{1}{v_T}dv.$$

The solution is $\alpha(v) = 1$ if and only if $v > \frac{v_T}{2}$, i.e., the monopolist sets a constant price of $\frac{v_T}{2}$, whence half the population of remaining buyers buy and the other half do not. The monopolist earns profit of $\frac{v_T}{4}$, conditional on unit mass of this population (with valuations below v_T). The unconditional profit equals the size of the population v_T multiplied with this, to yield a period T profit $\Pi_T(v_T) = \frac{{v_T}^2}{4}$.

(b) Here

$$\Pi_{T-1}(v_{T-1}) = \underset{p_{T-1}}{\text{Max}}[(v_{T-1} - \alpha_{T-1}p_{T-1})p_{T-1} + \delta\Pi_T(\alpha_{T-1}p_{T-1})]$$

First-order conditions for this yield

$$v_{T-1} - 2\alpha_{T-1}p_{T-1} + \delta\alpha_{T-1}\Pi_T'(\alpha_{T-1}p_{T-1}) = 0$$

or $p_{T-1} = \beta_{T-1}v_{T-1}$ where

$$\beta_{T-1} = \frac{1}{\alpha_{T-1}[2 - \frac{\delta}{2}\alpha_{T-1}]}.$$

Corresponding profits are

$$\Pi_{T-1}(v_{T-1}) = v_{T-1}^2[\frac{\frac{1}{\alpha_{T-1}} - \frac{\delta}{4}}{\{2 - \frac{\delta}{2}\alpha_{T-1}\}^2}].$$

Type $\alpha_{T-1}p_{T-1}$ is indifferent between buying at p_{T-1} and waiting till T when the price will be $\frac{\alpha_{T-1}p_{T-1}}{2}$:

$$\alpha_{T-1}p_{T-1} - p_{T-1} = \delta[\alpha_{T-1}p_{T-1} - \frac{\alpha_{T-1}p_{T-1}}{2}]$$

which implies

$$\alpha_{T-1} = \frac{2}{2-\delta}.$$

Check that $\alpha_{T-1}\beta_{T-1} = \frac{2-\delta}{4-3\delta}$ which is smaller than one as $\delta < 1$.

As δ tends to 1, α_{T-1} tends to 2, β_{T-1} tends to $\frac{1}{2}$, and $\alpha_{T-1}\beta_{T-1}$ tends to 1. Hence at $T-1$, the seller charges a price tending to $\frac{v_{T-1}}{2}$, almost no one buys. Then at T, the seller again charges a price approximately $\frac{v_{T-1}}{2}$, and half the population buys then.

(c) Suppose $\Pi_{t+1}(v_{t+1}) = \frac{\kappa_{t+1}}{2}v_{t+1}^2$. Then

$$\Pi_t(v_t) = \underset{p_t}{\text{Max}}[(v_t - \alpha_t p_t)p_t + \delta\Pi_{t+1}(\alpha_t p_t)]$$

and the first-order conditions yield

$$v_t - 2\alpha_t p_t + \delta\alpha_t\Pi'_{t+1}(\alpha_t p_t) = 0$$

or

$$p_t = \frac{v_t}{2\alpha_t - \delta\kappa_{t+1}\alpha_t^2}$$

implying

$$\beta_t = \frac{1}{2\alpha_t - \delta\kappa_{t+1}\alpha_t^2}. \tag{1}$$

It is now evident upon inserting $p_t = \beta_t v_t$ into the expression for Π_t above that

$$\Pi_t(v_t) = [v_t - \alpha_t\beta_t v_t]\beta_t v_t + \delta\frac{\kappa_{t+1}}{2}(\alpha_t\beta_t v_t)^2$$

so Π_t is also quadratic in v_t, and we can calculate κ_t as a function of κ_{t+1} and other parameters.

Next, type $\alpha_t p_t$ must be indifferent between the current price p_t and the price next period $\alpha_t\beta_{t+1}p_t$:

$$\alpha_t p_t - p_t = \delta(\alpha_t p_t - \beta_{t+1}\alpha_t p_t)$$

generating the following difference equation

$$\alpha_t = \frac{1}{1 - \delta\{1 - \frac{1}{2\alpha_{t+1} - \delta\kappa_{t+2}\alpha_{t+1}^2}\}}. \tag{2}$$

With α_t generated by this difference equation, β_t can be obtained from equation (1) above.

2. Consider the following variation on the incentive scheme following the node $x_{it}|h_{t-1}$: at date t the agent gets v utils more; following this node at date $t+1$ the agent gets $\frac{v}{\delta}$ utils less irrespective of report j; and at all other nodes the transfers are unchanged. This does not affect the agent's expected utility of reporting i at t, nor the relative payoffs from different reports at $t+1$ at nodes following the report of i at t. Hence all incentive and participation constraints of the agent are satisfied. This variation causes a change in expected cost to the principal (conditional on being at the node $x_{it}|h_{t-1}$) of

$$\frac{1}{u'(x_{it}|h_{t-1})} - \sum_j f_j \frac{1}{u'(x_{j,t+1}|h_{t-1} \cup \{i\})}$$

which yields the first-order condition provided.

3. If 1 owns the asset,

$$\begin{aligned} B_1 &= \frac{1}{6}[2\{v(\{1,2,3\},a|x) - v(\{2,3\},\emptyset|x)\} + v(\{1,2\},\{a\}|x) + v(\{1,3\},\{a\}|x) \\ &\quad +2\{v(1,a|x) - v(\emptyset|x)\}] \\ &= \frac{1}{6}[2\{r_1(x_1) + r_2(x_2) + v\} + r_1(x_1) + r_2(x_2) + r_1(x_1) + v + 2r_1(x_1)] \\ &= r_1(x_1) + \frac{1}{2}r_2(x_2) + \frac{v}{2} \end{aligned}$$

Hence in this case x_1 will be chosen to maximize $\alpha_1 x_1 - \frac{x_1^2}{2}$, implying $x_1 = \alpha_1$.

$$\begin{aligned} B_2 &= \frac{1}{6}[2\{v(\{1,2,3\},a|x) - v(\{1,3\},\{a\}|x)\} + v(\{1,2\},\{a\}|x) - v(\{1\},\{a\}|x)] \\ &= \frac{1}{6}[2\{r_1(x_1) + r_2(x_2) + v - r_1(x_1) - v\} + r_1(x_1) + r_2(x_2) - r_1(x_1)] \\ &= \frac{1}{2}r_2(x_2) \end{aligned}$$

x_2 will be chosen to maximize $\frac{1}{2}\alpha_2 x_2 - \frac{x_2^2}{2}$, so $x_2 = \frac{1}{2}\alpha_2$.

If $\{1,2,3\}$ collectively own the asset, then

$$\begin{aligned} B_1 &= \frac{1}{6}[2\{v(\{1,2,3\},a|x) - v(\{2,3\},\{a\}|x)\} + v(\{1,2\},\{a\}|x) + v(\{1,3\},\{a\}|x)] \\ &= \frac{1}{6}[2r_1(x_1) + r_1(x_1) + r_2(x_2)r_1(x_1)] \\ &= \frac{2}{3}r_1(x_1) + \frac{1}{6}r_2(x_2) + \frac{v}{6} \end{aligned}$$

Hence in this case x_1 will be chosen to maximize $\frac{2}{3}\alpha_1 x_1 - \frac{x_1^2}{2}$, implying $x_1 = \frac{2}{3}\alpha_1$.

$$\begin{aligned} B_2 &= \frac{1}{6}[2\{r_1(x_1) + r_2(x_2) + v - r_1(x_1) - v\} + r_1(x_1) + r_2(x_2) + r_2(x_2) + v] \\ &= \frac{2}{3}r_2(x_2) + \frac{1}{6}r_1(x_1) + \frac{v}{6} \end{aligned}$$

x_2 will be chosen to maximize $\frac{2}{3}\alpha_2 x_2 - \frac{x_2^2}{2}$, so $x_2 = \frac{2}{3}\alpha_2$.

If 3 owns the asset:

$$\begin{aligned} B_1 &= \frac{1}{6}[2\{r_1 + r_2 + v - r_2 - v\} + r_1] \\ &= \frac{r_1}{2} \end{aligned}$$

implying $x_1 = \frac{1}{2}\alpha_1$, while

$$\begin{aligned} B_2 &= \frac{1}{6}[2\{r_1 + r_2 + v - r_2 - v\} + r_2] \\ &= \frac{r_2}{2} \end{aligned}$$

implying $x_2 = \frac{1}{2}\alpha_2$.

Since there is underinvestment in this model, it follows that welfare is lowest when 3 owns the asset. 1 invests more when (s)he owns the asset privately, while 2 invests less, compared with collective ownership. So the relative welfare of these two ownership structures depends on the relative magnitudes of α_1 and α_2.