1. (a) The c.d.f. of $v$ is $F(v) = \frac{v}{v_T}$ for $v \in [0, v_T]$, and the density is $f(v) = \frac{1}{v_T}$. So the virtual valuation of $v$ is $v - \frac{1 - F(v)}{f(v)} = 2v - v_T$. Hence the monopolist’s problem at $T$ is to select probability $\alpha(v)$ of selling to a customer of type $v$ which is nonincreasing in $v$ and which maximizes

$$\int_0^{v_T} \alpha(v)[2v - v_T] \frac{1}{v_T} \, dv.$$ 

The solution is $\alpha(v) = 1$ if and only if $v > \frac{v_T}{2}$, i.e., the monopolist sets a constant price of $\frac{v_T}{2}$, whence half the population of remaining buyers buy and the other half do not. The monopolist earns profit of $\frac{v_T^2}{4}$, conditional on unit mass of this population (with valuations below $v_T$). The unconditional profit equals the size of the population $v_T$ multiplied with this, to yield a period $T$ profit $\Pi_T(v_T) = \frac{v_T^2}{4}$.

(b) Here

$$\Pi_{T-1}(v_{T-1}) = \max_{p_{T-1}} [(v_{T-1} - \alpha_{T-1}p_{T-1})p_{T-1} + \delta\Pi_T(\alpha_{T-1}p_{T-1})].$$

First-order conditions for this yield

$$v_{T-1} - 2\alpha_{T-1}p_{T-1} + \delta\alpha_{T-1}\Pi'_T(\alpha_{T-1}p_{T-1}) = 0$$

or $p_{T-1} = \beta_{T-1}v_{T-1}$ where

$$\beta_{T-1} = \frac{1}{\alpha_{T-1}[2 - \frac{\delta}{2}\alpha_{T-1}]}.$$

Corresponding profits are

$$\Pi_{T-1}(v_{T-1}) = v_{T-1}^2\left[\frac{1}{\alpha_{T-1}} - \frac{\delta}{4}\left\{2 - \frac{\delta}{2}\alpha_{T-1}\right\}^2\right].$$

Type $\alpha_{T-1}p_{T-1}$ is indifferent between buying at $p_{T-1}$ and waiting till $T$ when the price will be $\frac{\alpha_{T-1}p_{T-1}}{2}$:

$$\alpha_{T-1}p_{T-1} - p_{T-1} = \delta(\alpha_{T-1}p_{T-1} - \frac{\alpha_{T-1}p_{T-1}}{2})$$

which implies

$$\alpha_{T-1} = \frac{2}{2 - \delta}.$$
Check that $\alpha_{T-1} \beta_{T-1} = \frac{2-\delta}{4-3\delta}$ which is smaller than one as $\delta < 1$.

As $\delta$ tends to 1, $\alpha_{T-1}$ tends to 2, $\beta_{T-1}$ tends to $\frac{1}{2}$, and $\alpha_{T-1} \beta_{T-1}$ tends to 1. Hence at $T-1$, the seller charges a price tending to $\frac{v_{T-1}}{2}$, almost no one buys. Then at $T$, the seller again charges a price approximately $\frac{v_{T-1}}{2}$, and half the population buys then.

(c) Suppose $\Pi_{t+1}(v_{t+1}) = \frac{\kappa_{t+1}^2}{2} v_{t+1}^2$. Then

$$\Pi_t(v_t) = \max_{p_t} \left[ (v_t - \alpha_t p_t) p_t + \delta \Pi_{t+1}(\alpha_t p_t) \right]$$

and the first-order conditions yield

$$v_t - 2\alpha_t p_t + \delta \alpha_t \Pi_{t+1}'(\alpha_t p_t) = 0$$

or

$$p_t = \frac{v_t}{2\alpha_t - \delta \kappa_{t+1} \alpha_t^2}$$

implying

$$\beta_t = \frac{1}{2\alpha_t - \delta \kappa_{t+1} \alpha_t^2}. \quad (1)$$

It is now evident upon inserting $p_t = \beta_t v_t$ into the expression for $\Pi_t$ above that

$$\Pi_t(v_t) = [v_t - \alpha_t \beta_t v_t] \beta_t v_t + \delta \frac{\kappa_{t+1}}{2} (\alpha_t \beta_t v_t)^2$$

so $\Pi_t$ is also quadratic in $v_t$, and we can calculate $\kappa_t$ as a function of $\kappa_{t+1}$ and other parameters.

Next, type $\alpha_t p_t$ must be indifferent between the current price $p_t$ and the price next period $\alpha_t \beta_{t+1} p_t$:

$$\alpha_t p_t - p_t = \delta (\alpha_t p_t - \beta_{t+1} \alpha_t p_t)$$

generating the following difference equation

$$\alpha_t = \frac{1}{1 - \delta \left\{ 1 - \frac{1}{2\alpha_{t+1} - \delta \kappa_{t+1} + 2\alpha_{t+1}} \right\}}. \quad (2)$$

With $\alpha_t$ generated by this difference equation, $\beta_t$ can be obtained from equation (1) above.
2. Consider the following variation on the incentive scheme following the node \( x_{it}|h_{t-1} \): at date \( t \) the agent gets \( v \) utils more; following this node at date \( t+1 \) the agent gets \( \frac{v}{3} \) utils less irrespective of report \( j \); and at all other nodes the transfers are unchanged. This does not affect the agent’s expected utility of reporting \( i \) at \( t \), nor the relative payoffs from different reports at \( t+1 \) at nodes following the report of \( i \) at \( t \). Hence all incentive and participation constraints of the agent are satisfied. This variation causes a change in expected cost to the principal (conditional on being at the node \( x_{it}|h_{t-1} \)) of

\[
\frac{1}{u(x_{it}|h_{t-1})} - \sum_j f_j \frac{1}{u(x_{j,t+1}|h_{t-1} \cup \{i\})}
\]

which yields the first-order condition provided.

3. If 1 owns the asset,

\[
B_1 = \frac{1}{6} \left[ 2\{v(1,2,3),a|x) - v(2,3,\emptyset|x) \} + v(1,2),\{a|x) + v(1,3),\{a|x) \\
+ 2\{v(1,a|x) - v(\emptyset|x) \} \right]
\]

\[
= \frac{1}{6} \left[ 2\{r_1(x_1) + r_2(x_2) + v \} + r_1(x_1) + r_2(x_2) + r_1(x_1) + v + 2r_1(x_1) \right]
\]

\[
= r_1(x_1) + \frac{1}{2} r_2(x_2) + \frac{v}{2}
\]

Hence in this case \( x_1 \) will be chosen to maximize \( \alpha_1 x_1 - \frac{x_1^2}{2} \), implying \( x_1 = \alpha_1 \).

\[
B_2 = \frac{1}{6} \left[ 2\{v(1,2,3),a|x) - v(1,3),\{a|x) \} + v(1,2),\{a|x) - v(1),\{a|x) \\
+ 2\{v(1,a|x) - v(\emptyset|x) \} \right]
\]

\[
= \frac{1}{6} \left[ 2\{r_1(x_1) + r_2(x_2) + v - r_1(x_1) - v \} + r_1(x_1) + r_2(x_2) - r_1(x_1) \right]
\]

\[
= \frac{1}{2} r_2(x_2)
\]

\( x_2 \) will be chosen to maximize \( \frac{1}{2} \alpha_2 x_2 - \frac{x_2^2}{2} \), so \( x_2 = \frac{1}{2} \alpha_2 \).

If \{1,2,3\} collectively own the asset, then

\[
B_1 = \frac{1}{6} \left[ 2\{v(1,2,3),a|x) - v(2,3),\{a|x) \} + v(1,2),\{a|x) + v(1,3),\{a|x) \\
+ 2\{v(1,a|x) - v(\emptyset|x) \} \right]
\]

\[
= \frac{1}{6} \left[ 2r_1(x_1) + r_1(x_1) + r_2(x_2) r_1(x_1) \right]
\]

\[
= \frac{2}{3} r_1(x_1) + \frac{1}{6} r_2(x_2) + \frac{v}{6}
\]
Hence in this case $x_1$ will be chosen to maximize $\frac{2}{3}\alpha_1 x_1 - \frac{x_1^2}{2}$, implying $x_1 = \frac{2}{3}\alpha_1$.

$$B_2 = \frac{1}{6}[2\{r_1(x_1) + r_2(x_2) + v - r_1(x_1) - v\} + r_1(x_1) + r_2(x_2) + r_2(x_2) + v]$$

$$= \frac{2}{3}r_2(x_2) + \frac{1}{6}r_1(x_1) + \frac{v}{6}$$

$x_2$ will be chosen to maximize $\frac{2}{3}\alpha_2 x_2 - \frac{x_2^2}{2}$, so $x_2 = \frac{2}{3}\alpha_2$.

If 3 owns the asset:

$$B_1 = \frac{1}{6}[2\{r_1 + r_2 + v - r_2 - v\} + r_1]$$

$$= \frac{r_1}{2}$$

implying $x_1 = \frac{1}{2}\alpha_1$, while

$$B_2 = \frac{1}{6}[2\{r_1 + r_2 + v - r_2 - v\} + r_2]$$

$$= \frac{r_2}{2}$$

implying $x_2 = \frac{1}{2}\alpha_2$.

Since there is underinvestment in this model, it follows that welfare is lowest when 3 owns the asset. 1 invests more when (s)he owns the asset privately, while 2 invests less, compared with collective ownership. So the relative welfare of these two ownership structures depends on the relative magnitudes of $\alpha_1$ and $\alpha_2$. 
