Boston University
PhD Topics course in Contract Theory
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SOLUTIONS TO EC717 MIDTERM EXAM, FALL 2015

1. *Consider a monopolist selling a divisible good which is produced at a constant per unit cost* c *to a population of customers with utility function* $\theta q - t$*, where* $q \in [0, 1]$ *denotes the quantity sold and* t *the corresponding payment.* θ *is known privately by each consumer, distributed in the population over support* $[\underline{\theta}, \bar{\theta}]$ *according to a distribution function* F *and associated positive density* f *(with an inverse hazard rate* $\frac{1-F}{f}$ *which is nonincreasing). Each consumer has a zero outside option utility. The monopolist can choose any nonlinear pricing mechanism. The parameters satisfy* $\bar{\theta} > c > \underline{\theta}$*.*

(a) *Provide necessary and sufficient conditions for the mechanism to be incentive compatible.*

$\theta q(\theta) - t(\theta) = \underline{\theta} q(\underline{\theta}) - t(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)dx$, and $q(.)$ nondecreasing.

(b) *Express the optimization problem solely in terms of choice of* $q(\theta)$*, the function expressing the quantity purchased by each type.*

It is optimal to set $t(\underline{\theta}) = \underline{\theta} q(\underline{\theta})$, so $t(\theta) = \theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(x)dx$ and the objective function reduces to

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta - \frac{1 - F(\theta)}{f(\theta)} - c] q(\theta) d\theta$$

(c) *Show that it is optimal for the monopolist to set a constant per unit price, and let each consumer decide whether and how much to buy. Calculate the optimal price in terms of the parameters of the model.*

The objective function above has to be maximized pointwise, since $\frac{1-F}{f}$ nonincreasing implies that the constraint that $q(.)$ be nondecreasing will not be binding. Pointwise optimization yields $q(\theta) = 1$ if $\theta - \frac{1-F(\theta)}{f(\theta)} > c$ and 0 otherwise. This can be achieved by setting a constant unit price $p^* = \theta^*$ which solves $\theta^* - \frac{1-F(\theta^*)}{f(\theta^*)} = c$. Since $\bar{\theta} > c > \underline{\theta} - \frac{1}{f(\underline{\theta})}$ there exists a unique θ^* solving this.

2. *Suppose there is a buyer B and seller S of an indivisible object, with payoffs* $d\theta_B - t$ *and* $t - d\theta_S$*, where* $d \in \{0, 1\}$ *denotes whether trade takes place, and* t *is a transfer from B to S. B and S are privately informed about* θ_B *and* θ_S *respectively, which are drawn independently from* $[0, 1]$ *according to distribution functions* F_B, F_S *that are common knowledge among them. Let* $d^*(\theta_B, \theta_S)$ *denote the ex post efficient trading rule.*

(a) *Derive* $d^*(\theta_B, \theta_S)$ *the ex post efficient trading rule.*

$d^* = 1$ if $\theta_B > \theta_S$ and 0 if the inequality is reversed.

(b) *Show that any payment rule* $t(\theta_B, \theta_S)$ *implements* $d^*(\theta_B, \theta_S)$ *in dominant strategies if and only if there exist real valued functions* $B(\theta_S), S(\theta_B)$ *such that*

$$t(\theta_B, \theta_S) = \theta_S d^*(\theta_B, \theta_S) + B(\theta_S) = \theta_B d^*(\theta_B, \theta_S) + S(\theta_B) \quad (1)$$

These are the Groves-Clarke transfers. DSIC follows since the problem of selecting θ to maximize (for any given θ_B, θ_S):

$$\theta_B d^*(\theta, \theta_S) - t(\theta, \theta_S) = (\theta_B - \theta_S) d^*(\theta, \theta_S) - B(\theta_S)$$

has $\theta = \theta_B$ as a solution. A similar argument ensures S would have a dominant strategy to report truthfully.

To establish the converse, fix any θ_S, and consider the buyer's incentives. If θ_B, θ'_B are both smaller than θ_S, the dominant strategy incentive compatibility condition requires θ_B not to want to report θ'_B, and vice versa, assuming S reports θ_S. Since reporting either θ_B, θ'_B leads to no trade given report θ_S by S, it follows that $t(\theta_B, \theta_S) = t(\theta'_B, \theta_S)$.

In other words, the payment made by B conditional on no trade and a report θ_S by S must be a function only of θ_S. Let this function be denoted by $B(\theta_S)$.

Now consider θ_B, θ'_B both bigger than θ_S. Then reporting either θ_B, θ'_B will lead to trade. A similar argument as above ensures that the payment made by B conditional on trade and a report θ_S by S must be a function only of θ_S. Let this function be denoted $B_1(\theta_S)$.

Next consider $\theta'_B < \theta_S < \theta_B$. B's incentive compatibility condition (θ_B not to want to report θ'_B, and vice versa, assuming S reports θ_S) now requires $\theta_B \geq B_1(\theta_S) - B(\theta_S) \geq \theta'_B$. Letting θ_B approach θ_S from above, and θ'_B approach θ_S from below, it follows that $\theta_S \geq B_1(\theta_S) - B(\theta_S) \geq \theta_S$, or $B_1(\theta_S) = \theta_S + B(\theta_S)$. In the event of trade, the payment made by B must exceed the payment made in the event of no trade by θ_S. Hence the payment made by B must take the form $\theta_S d^*(\theta_B, \theta_S) + B(\theta_S)$.

A symmetric argument establishes the structure of the payments needed for S's incentives.

An alternative approach to the 'only if' part which is calculus-based goes part of the way. Any DSIC transfer rule $\hat{t}(\theta_B, \theta_S)$ must satisfy the first-order condition

$$\frac{\partial V_B(\theta = \theta_B, \theta_S | \theta_B)}{\partial \theta} = 0$$

where $V_B(\theta, \theta_S | \theta_B) \equiv \theta_B d^*(\theta, \theta_S) - \hat{t}(\theta, \theta_S)$ at any differentiability point (θ_B, θ_S). Since d^* is almost everywhere differentiable, $\hat{t}$ must be almost everywhere differentiable. Hence any two DSIC mechanisms must have the same partial derivatives w.r.t. θ_B almost everywhere, and must differ by some function $B(\theta_S)$ which does not depend on the value of θ_B. However this argument applies only 'almost everywhere', not everywhere. Hence this argument is not complete.

(c) *Show that $B(\theta_B) = S(\theta_S) = k$ for some constant k.*

Take any pair θ_B, θ'_B with $\theta'_B > \theta_B$. Take any $\theta_S > \theta'_B$. Then $t(\theta_B, \theta_S) = t(\theta'_B, \theta_S) = B(\theta_S)$. Also $t(\theta_B, \theta_S) = S(\theta_B)$ and $t(\theta'_B, \theta_S) = S(\theta'_B)$. Hence S is independent of θ_B.

A similar argument shows B is independent of θ_S). It follows that $t(\theta_B, \theta_S) = k = B(\theta_S) = S(\theta_B)$ for some constant whenever there is no trade.

(d) *Use the above results to show there cannot exist any payment rule which implements the efficient trading rule in dominant strategies.*

Equation (1) implies that whenever $\theta_B > \theta_S$, $t(\theta_B, \theta_S) = \theta_S + k = \theta_B + k$, which is a contradiction.

(e) *Does there exist a payment rule which implements the efficient trading rule as a Bayesian equilibrium? Can you find such a payment rule?*

Yes, the d'Aspremont-Gerard-Varet mechanism

$$t(\theta_B, \theta_S) = T_B(\theta_B) - T_S(\theta_S)$$

where

$$T_B(\theta_B) = E_{\theta_S}[\theta_S d^*(\theta_B, \theta_S)], T_S(\theta_S) = E_{\theta_B}[\theta_B d^*(\theta_B, \theta_S)]$$

implements the efficient trading rule as a Bayesian equilibrium.