1. You have been appointed arbitrator for negotiations for sale of an indivisible input produced by a selling firm $S$ to a buying firm $B$ with ex ante uncertain costs and valuations. They respectively have payoffs $t - q\theta_s$ and $q\theta_b - t$ if $q \in \{0, 1\}$ denotes whether or not a sale takes place, and $t$ is the expected monetary transfer from $B$ to $S$. Parameters $\theta_s$ and $\theta_b$ are drawn independently from intervals $[\theta_b, \bar{\theta}_b], [\theta_b, \bar{\theta}_b]$ respectively accordingly to cdf’s $F_s, F_b$ with positive densities $f_s, f_b$ which are common knowledge. Each firm will privately observe its own cost or valuation parameter; neither you or the other firm will know the realization of this parameter. Is it possible to design a mechanism which is ex post efficient, Bayesian incentive compatible and ex ante individually rational (i.e., both parties will be willing to participate if they have to commit to participation before observing their respective costs/valuations)? If not, provide a proof. If yes, construct a mechanism with the required properties.

**ANSWER:** Ex post efficiency requires sale to occur if and only if $\theta_b > \theta_s$. If $t(\theta_b, \theta_s)$ denotes the transfer rule, then Bayesian IC requires

\[
E_{\theta_s}[t(\theta_b, \theta_s)] = \theta_b F_s(\theta_b) - \int_{\theta_b}^{\theta_b} F_s(x)dx + V_b
\]

\[
E_{\theta_b}[t(\theta_b, \theta_s)] = \theta_s [1 - F_b(\theta_s)] - \int_{\theta_s}^{\bar{\theta}_s} [1 - F_b(x)]dx + \bar{V}_s
\]

where $V_b, \bar{V}_s$ denote expected payoffs of the buyer with the lowest valuation, and of the seller with the highest valuation.

Define

\[
T_b(\theta_b) \equiv \theta_b F_s(\theta_b) - \int_{\theta_b}^{\theta_b} F_s(x)dx
\]

\[
T_s(\theta_s) \equiv \theta_s [1 - F_b(\theta_s)] - \int_{\theta_s}^{\bar{\theta}_s} [1 - F_b(x)]dx
\]

and construct $t^*(\theta_b, \theta_s; K) = T_b(\theta_b) + T_s(\theta_s) + K$ where $K$ is a constant to be chosen. The expected value of these transfers differs from that of $t$ for each type by a constant, so it is
also BIC. It suffices to show existence of $K$ that ensures ex ante IR for both agents, i.e.:

\[
E[\theta_b F_s(\theta_b) - T_b(\theta_b) - T_s(\theta_s) - K] \geq 0
\]

\[
E[T_s(\theta_s) + T_b(\theta_b) + K - \theta_s\{1 - F_b(\theta_s)\}] \geq 0
\]

These reduce to the condition that

\[
E[\theta_b F_s(\theta_b) - T_b(\theta_b) - T_s(\theta_s)] \geq K \geq E[\theta_s\{1 - F_b(\theta_s)\} - T_s(\theta_s) - T_b(\theta_b)]
\]

(1)

A necessary and sufficient condition for existence of $K$ which satisfies (1) is that

\[
E[\theta_b F_s(\theta_b)] \geq E[\theta_s\{1 - F_b(\theta_s)\}]
\]

(2)

Using $I_M$ to denote the indicator function of event $M$, the LHS of (2) is equal to $E[\theta_b I_{\{\theta_b > \theta_s\}}]$ and the RHS equals $E[\theta_s I_{\{\theta_s > \theta_b\}}]$. Hence (2) reduces to

\[
E[(\theta_b - \theta_s) I_{\{\theta_b > \theta_s\}}] \geq 0
\]

which is obviously true.

2. You want to sell an indivisible object which you personally do not value. There are two potential bidders, with independent private values. Bidder 1’s value is drawn uniformly over $[0, 1]$, while bidder 2’s value is drawn uniformly over $[\alpha, 1 - \alpha]$, where $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Derive the optimal expected revenue maximizing sealed-bid auction in which bidders have incentives to report their true valuations. Describe the way that the bidding rules in the optimal auction ‘favor’ one bidder over another, and try to explain why.

**ANSWER:** Bidder 1’s virtual valuation is $\theta_1 - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} = 2\theta_1 - 1$, for $\theta_1 \in [0, 1]$. Bidder 2’s virtual valuation is $\theta_2 - \frac{1 - F_2(\theta_2)}{f_2(\theta_2)} = 2\theta_2 - 1 + \alpha$ for $\theta_2 \in [\alpha, 1 - \alpha]$. Hence the optimal assignments are probabilities $\{p_i(\theta_1, \theta_2)\}_{i=1,2}$ which maximize $[(2\theta_1 - 1)p_1 + (2\theta_2 - 1 + \alpha)p_2]$ subject to $0 \leq p_1 + p_2 \leq 1$. [With a uniform distribution for valuations of each bidder, virtual valuations are monotone increasing in true valuations, so the solution of this problem will be monotone in true valuations, which ensures that the solution will be globally incentive compatible.] The solution to this is $p_1(\theta_1, \theta_2) = 1$, i.e., bidder 1 wins the object if and
only if \( \theta_1 > \max\{\frac{1}{2}, \theta_2 + \frac{\alpha}{2}\} \). Now note that \( \theta_2 + \frac{\alpha}{2} \geq \frac{3\alpha}{2} \geq \frac{1}{2} \) since \( \theta_2 \geq \alpha \) and \( \alpha \geq \frac{1}{3} \). Hence we can replace this condition by \( \theta_1 > \theta_2 + \frac{\alpha}{2} \). If this condition is not satisfied, then bidder 2 wins the object (since bidder 2’s virtual valuation is always non-negative given that \( \alpha > 1 - \alpha \)). Hence \( p_2(\theta_1, \theta_2) = 1 - p_1(\theta_1, \theta_2) \).

The following transfer payments can be checked to implement this allocation (in dominant strategies): If bidder 1 wins, she pays \( \theta_2 + \frac{\alpha}{2} \). If bidder 2 wins, she pays \( \theta_1 - \frac{\alpha}{2} \). Hence the second-price auction is modified to favor the second bidder, whose valuation is less uncertain. Bidder 2 when she wins pays less than the losing bid, while bidder 2 when she wins pays more than the losing bid. As a result, bidder 2 wins with a higher probability. The intuitive reason for this is that awarding the object to bidder 1 entails more information rents owing to the greater uncertainty of her valuation.

3. A Principal \( P \) wishes to procure an indivisible object that can be produced either by firm 1 or firm 2. Firm \( i \)'s cost of production is drawn from a uniform distribution over the support \( [0, i] \), \( i = 1, 2 \); each firm privately observes its own cost realization. \( P \) wishes to contract with one of the two firms (called the prime contractor), and delegate to that firm the decision whether it will produce the object itself, or subcontract it to the other firm. Specifically, \( P \) will offer a take-it-or-leave-it contract to the prime contractor, which the latter will have to respond with a yes-no decision to \( P \) before making a take-it-or-leave-it subcontract offer to the other firm (but after it has observed its own cost realization). \( P \) will not observe the subcontract, nor who ultimately produced the object, nor the money transferred between the two firms. All parties are risk neutral. A firm’s payoff is defined by the difference between its cost of production, while \( P \) wishes to minimize the expected cost of procuring the object.

(a) Derive the outcome (in terms of production assignments, transfers and expected cost incurred by \( P \)) of appointing agent \( i \) the prime contractor. Based on this, which agent should be appointed the prime contractor?

(b) Would \( P \) do better to personally contract with the two agents (i.e., design a procurement auction), instead of appointing one of them to act as a prime contractor? If so, explain
how much \( P \) would gain, and how the production assignments would differ.

**ANSWER:**

3. (a) If agent 1 becomes the prime contractor, he will have to decide whether to subcontract to agent 2 or produce it himself. He will offer a price \( P_1 \) to agent 2 to minimize his expected cost \((1 - \frac{P_1}{2})\theta_1 + P_1 \frac{P_1}{2}\) of delivering the good to the principal. The solution is \( P_1 = \frac{\theta_1}{2} \).

The expected cost for type \( \theta_1 \) is then \( C_1(\theta_1) = (1 - \frac{\theta_1}{4})\theta_1 + \frac{\theta_1 \theta_1}{2} = \theta_1 - \frac{\theta_1^2}{8} \). To ensure every type of Agent 1 will be willing to deliver the good, the principal will have to pay him \( C_1(1) = \frac{7}{8} \).

Conversely, if agent 2 is the prime contractor, then type \( \theta_2 \) of this agent will offer a price \( P_2 \) to minimize his expected delivery cost \((1 - P_2)\theta_2 + \frac{P_2^2}{2}\), so will offer \( P_2 = \frac{\theta_2}{2} \). The expected cost \( C_2(\theta_2) = (1 - \frac{\theta_2}{2})\theta_2 + \frac{\theta_2^2}{4} = \theta_2 - \frac{\theta_2^2}{4} \). Hence the principal will have to pay agent 2 an amount \( C_2(2) = 1 \) to guarantee delivery of the good.

Hence it is better to appoint agent 1 as the prime contractor.

(b) If \( P \) were to contract directly with the two suppliers she would assign the production to agent 1 if and only if the virtual cost of agent 1 which is \( h_1(\theta_1) = 2\theta_1 \) is less than the virtual cost of agent 2 which is \( h_2(\theta_2) = 2\theta_2 \). Hence agent 1 produces the object if and only if \( \theta_1 < \theta_2 \). The expected cost for \( P \) will be

\[
E \min[2\theta_1, 2\theta_2] = \int_0^1 \left[ \int_0^{\theta_1} \theta_2 \, d\theta_2 + \int_{\theta_1}^2 \theta_1 \, d\theta_2 \right] \, d\theta_1 = \int_0^1 (2\theta_1 - \frac{\theta_1^2}{2}) \, d\theta_1 = \frac{5}{6}.
\]

So \( P \) would save \( \frac{7}{8} - \frac{5}{6} = \frac{1}{24} \) by contracting directly with the suppliers.

An alternate way of calculating expected cost with direct procurement is to calculate expected payments to the two agents. For agent 1, the probability of assigning production is \( q_1(\theta_1) = 1 - \frac{\theta_1}{2} \). Hence expected payment to type \( \theta_1 \) of agent 1 is

\[
T_1(\theta_1) = \theta_1 q_1(\theta_1) + \int_{\theta_1}^1 q_1(y) \, dy = \frac{3}{4} - \frac{\theta_1^2}{4}.
\]
For agent 2, the probability of being assigned production is $q_2(\theta_2) = 1 - \min\{\theta_2, 1\}$. Hence expected payment to type $\theta_2$ of agent 2 is

$$T_2(\theta_2) = \theta_2 q_2(\theta_2) + \int_{\theta_2}^{1} q_2(y) dy = \frac{1}{2} - \frac{\theta_2^2}{2}$$

if $\theta_2 < 1$ and 0 otherwise. The expected payment to the two agents combined then equals

$$\int_0^1 \left( \frac{3}{4} - \frac{\theta_1^2}{4} \right) d\theta_1 + \frac{1}{2} \int_0^1 \frac{1}{2} - \frac{\theta_2^2}{2} d\theta_2 = \frac{5}{6}.$$