EC 717, SOLUTIONS TO PROBLEM SET NO. 1

1. A monopolist wishes to sell a good produced at constant unit cost c to a large population of consumers with heterogenous preferences: a consumer of type θ has a payoff $\theta \log q - t$ for consuming q units of the good if $q > 0$ (and a zero payoff for consuming nothing), and paying t dollars for it. θ is distributed uniformly on $[0,1]$. The monopolist cannot identify the type of any given consumer. Each customer has an outside option of 0.

(a) *If $q(\theta)$ denotes the quantity sold to type θ, find a condition on this function $q(.)$ that ensures that it is IC (incentive compatible, i.e., there exists some pricing rule $t(q)$ for which $q(\theta)$ is the optimal purchase of type θ)).*

$q(.)$ has to be nondecreasing; this and

$$V(\theta) \equiv \theta \log q(\theta) - t(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \log q(x)dx \tag{1}$$

are necessary and sufficient for incentive compatibility (where $\underline{\theta}$ is the lowest type that purchases a positive quantity of the good).

(b) *For any such IC $q(.)$, what is the associated set of payments (i.e., $t(\theta)$) that customers (of type θ) make to the monopolist?*

From (1), we obtain:

$$t(\theta) = \theta \log q(\theta) - \int_{\underline{\theta}}^{\theta} \log q(x)dx - V(\underline{\theta}) \tag{2}$$

(c) *Obtain an expression for total profit of the monopolist as a function only of the selling strategy $q(.)$.*

$$\begin{aligned}\Pi &= \textstyle\int_{\underline{\theta}}^{1} [t(\theta) - cq(\theta)]d\theta \\ &= \textstyle\int_{\underline{\theta}}^{1} [\theta \log q(\theta) - \int_{\underline{\theta}}^{\theta} \log q(x)dx - cq(\theta)]d\theta - V(\underline{\theta}) \\ &= \textstyle\int_{\underline{\theta}}^{1} [(2\theta - 1)\log q(\theta) - cq(\theta)]d\theta - V(\underline{\theta}).\end{aligned}$$

The problem did not specify the outside option of the consumers. This renders it somewhat tricky. The standard assumption is that it is zero. Note however that this is not the autarkic payoff of consumers: with $q = t = 0$ any consumer with $\theta > 0$ gets a payoff of $-\infty$. If we use $-\infty$ as the outside option of every consumer then the PC reduces to $-V(\underline{\theta}) \geq \infty$, and the problem has no solution: consumers will be willing to pay infinite amounts for even a tiny amount of the good.

So a more sensible formulation would stipulate an outside option of zero for all consumers, i.e., the payoff function of consumers is $\theta \log q - t$ only if $q > 0$, and $-t$ if $q = 0$. The PC will then be $V(\underline{\theta}) \geq 0$. Note that with this formulation, agents must consume at least one unit of the good if they consume it at all. Hence we must also impose $q(\theta) \geq 1$ apart from the constraint that $q(.)$ is nondecreasing.

With this formulation, at the optimum the monopolist will set $V(\underline{\theta}) = 0$. The monopolist's problem is then to select $\underline{\theta} \in [0, 1]$ and $q(\theta) \geq 1$ over $[\underline{\theta}, 1]$ to maximize

$$\Pi = \int_{\underline{\theta}}^{1} [(2\theta - 1) \log q(\theta) - cq(\theta)] d\theta \tag{3}$$

subject to the constraint that $q(.)$ is nondecreasing.

(d) *Calculate the optimal selling strategy $q^*(\theta)$, and the pricing function $t(q)$ which implements it (in the sense of (a) above).*

Ignoring the monotonicity constraint on $q(.)$, point-wise optimization yields $q(\theta) = \frac{2\theta-1}{c}$ provided the associated profit $(2\theta - 1) \log q(\theta) - cq(\theta) \geq 0$. The latter condition is satisfied if and only if $\log \frac{2\theta-1}{c} \geq 1 = \log e$, or $\theta \geq \frac{1+ce}{2}$. Hence the solution is $\underline{\theta} = \frac{1+ce}{2}$, and provided this is less than one (which will be the case if c is small enough): $q^*(\theta) = \frac{2\theta-1}{c}$ for $\theta \in [\frac{1+ce}{2}, 1]$. This is nondecreasing, so the monotonicity constraint does not bind.

The solution therefore involves partial pooling: all $\theta \in [0, \frac{1+ce}{2})$ do not consume any of the good, while it is separating over $[\frac{1+ce}{2}, 1]$. Hence the payments are differentiable over $[\frac{1+ce}{2}, 1]$. Over this range, the payments will satisfy (using (2) above:) $t'(\theta) = \frac{\theta q^{*'}(\theta)}{q^*(\theta)} = \frac{2\theta}{2\theta-1} = 1 + \frac{1}{2\theta-1}$. Integrating, we obtain

$$t(\theta) = \int_{\frac{1+ce}{2}}^{\theta} [1 + \frac{1}{2x - 1}] dx + t(\frac{1 + ce}{2}).$$

Since $t(\frac{1+ce}{2}) = \frac{1+ce}{2} \log q^*(\frac{1+ce}{2}) = \frac{1+ce}{2}$ we obtain

$$\begin{aligned} t(\theta) &= \int_{\frac{1+ce}{2}}^{\theta} [1 + \tfrac{1}{2x-1}]dx + \tfrac{1+ce}{2} \\ &= \theta + \tfrac{1}{2} \log \tfrac{2\theta - 1}{ce}. \end{aligned}$$

Finally to obtain the pricing rule $t(q)$ we invert $\theta(q) = \frac{cq+1}{2}$ to obtain

$$t(q) = \frac{cq+1}{2} + \frac{1}{2} \log \frac{q}{e}$$

for $q \geq e$. The minimum quantity sold is e, purchased by type $\frac{1+ce}{2}$.

2. A risk-neutral principal P hires an agent A, who chooses an effort $a \geq 0$, which results in gross profit $x = a + \epsilon$ for P, where ϵ is uniformly distributed on $[0,1]$. A's payoff equals $\frac{w^{1-\rho}}{1-\rho} - \frac{a^2}{2}$, where w denotes a non-negative wage paid by P, and $\rho > 0, \neq 1$ is a parameter of risk-aversion. A has an outside option payoff of $\bar{U}$ which is non-negative if $\rho < 1$ and negative if $\rho > 1$.

(a) *If a is contractible, characterize the first-best wage and effort levels.*

The first-best problem is to select a and fixed wage w to maximize $[a + \frac{1}{2} - w]$ subject to

$$\frac{w^{1-\rho}}{1-\rho} \geq \frac{a^2}{2} + \bar{U} \qquad (PC)$$

Consider first the case where $\rho < 1$. Then the LHS of (PC) goes from 0 to ∞ as w goes from 0 to ∞, while the RHS is positive. Since the LHS is increasing in w, there is a unique solution:

$$w(a) = \{(1-\rho)[\bar{U} + \frac{a^2}{2}]\}^{\frac{1}{1-\rho}} \qquad (1)$$

and

$$w'(a) = a\{(1-\rho)[\bar{U} + \frac{a^2}{2}]\}^{\frac{\rho}{1-\rho}}$$

which is increasing in a and going from 0 to ∞ as a goes from 0 to ∞. Hence the first-best action is given by the unique solution to $w'(a^*) = 1$, and the first-best wage is $w(a^*)$ as given by (1) above.

Next consider the case where $\rho > 1$. Now $\bar{U} \leq 0$. If $a > [-2\bar{U}]^{\frac{1}{2}}$ then the RHS of (PC) is positive while the LHS is negative for all $w \geq 0$. Hence such actions are not implementable. The set of implementable actions is $a \in [0, \bar{a}]$ where $\bar{a} \equiv [-2\bar{U}]^{\frac{1}{2}}$. In this case also we obtain the same expression for $w(a)$ and a^*, as $w'(a)$ increases from 0 to ∞ as a goes from 0 to $\bar{a}$.

(b) *If a is not contractible, but the profit x is contractible, and $\rho \in (0,1)$, find a condition on the parameters of the problem which ensure that the first-best profit can be achieved by P. If $\bar{U} = 0$, when is this condition satisfied?*

Now there is an effort incentive constraint to be satisfied. If the first-best is to be implemented, the risk-aversion of the agent requires the agent to be paid a constant wage $w(a^*)$ for $x \in [a^*, a^* + 1]$. If there is a payment schedule that renders selection of a^* optimal for the agent, the following will also implement a^*: P pays the agent 0 if $x < a^*$ and $w(a^*)$ otherwise. Then the agent will not want to deviate from a^* to any higher a. For $a < a^*$, the agent's expected payoff is

$$\Pi(a) = [1 + a - a^*][\frac{(a^*)^2}{2} + \bar{U}] - \frac{a^2}{2}$$

since by construction $\frac{(w^*)^{1-\rho}}{1-\rho} = \frac{(a^*)^2}{2} + \bar{U}$. Hence $\Pi'(a) = \bar{U} + \frac{(a^*)^2}{2} - a$. A necessary and sufficient condition for a^* to be optimal for the agent is that $\bar{U} + \frac{(a^*)^2}{2} - a^* \geq 0$, or

$$\bar{U} \geq a^* - \frac{(a^*)^2}{2}. \tag{2}$$

This however is not a condition on the parameters of the problem, as a^* is endogenous. In the case where $\bar{U} = 0$, it is easy to check that $a^* = [\frac{2}{1-\rho}]^{\frac{\rho}{1+\rho}}$. Since the function $g(a) = a - \frac{a^2}{2}$ is negative if and only if $a > 2$, the first-best is implementable if and only if $a^* > 2$, or

$$[\frac{1}{1-\rho}]^{\rho} > 2. \tag{3}$$

This condition is not satisfied for ρ close to zero, but is satisfied for ρ close to one.

(c) *If $\rho > 1$ what can you say about implementability of the first-best profit when a is not contractible? How would you interpret these results?*

In that case the first-best can be implemented as $u(w)$ goes to $-\infty$ as w goes to 0. Hence the payment rule given above will not tempt the agent to deviate to a lower effort, as that will result in a zero wage with positive probability, i.e., a utility of $-\infty$.

What matters critically is the steepness of marginal utility at the minimum payment, i.e., the utility consequences for the agent when the limit of his liability is exercised. In case (c) it is steep enough to ensure implementability. In case (b), condition (3) is required when $\bar{U} = 0$, and this requires the parameter of risk aversion to be large enough.