Ec717a: Introduction

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Ec 717a, 2020: Lecture 1
In Ec703 you have seen some examples of useful applications of mechanism design theory to auctions, public goods and bargaining (e.g., possibility of attaining efficient allocations, revenue maximizing auctions)

Some common assumptions of those models:

- agent $i$ are privately informed about their *one dimensional* valuation $v_i$ of a good
- values $v_1, \ldots, v_n$ are independently distributed (*private values*)
- prior beliefs of P(principal) and other agents over $v_i$ are *common knowledge*
Introduction to Ec717a

Background, contd.

- Equilibrium concept: noncooperative (Bayesian) equilibrium (ignore possibility of collusion among agents)

- Implementation notion: ‘partial’ implementation (there is a Bayesian equilibrium resulting in a desired allocation): ignores the possibility that other equilibria may also exist

- Ignore issues of complexity or communication costs, which may necessitate ‘simpler’ mechanisms

- Commitment: implicitly assume that P can commit to following through with implementation of the mechanism
Extensions: Various Directions

- Subsequently there has been much effort devoted to extending the theory in various directions:
  - multidimensional valuations, multiple goods
  - ‘full’ implementation (all equilibria must result in desired allocations)
  - correlated valuations (interdependent values)
  - possible non-robustness to various details
  - incorporate costs of complexity, communication
  - prospect of collusion

- Other concerns (addressed later in this course): incorporate weaker commitment power for P (relevant especially in dynamic settings)
Structure of This Course

- Part 1 (two weeks): Interdependent Valuations, Robust Design
- Part 2 (two weeks): Complexity, Communication Costs
- Part 3 (two weeks): Delegation, Collusion
- Part 4 (two weeks): Dynamic Relational Contracts
Part 1: I shall focus on extensions to interdependent values, and robustness issues (esp. with respect to the common prior assumption).

Ignore substantial literature on:
- multidimensional mechanism design (difficult, technical)
- full implementation (easy, but involves making mechanism more complex by augmenting message spaces)
Suppose P is auctioning off an indivisible good (eg spectrum license) to n risk neutral bidders with independent private values $v_1, \ldots, v_n$

$i$’s payoff $v_id_i - p_i$, where $d_i \in \{0, 1\}$ denotes whether the good is allocated to $i$, and $p_i$ is net amount paid by $i$

Common prior beliefs: $v_i$ distributed with $C^1$ positive density $f_i$ on $[\underline{v}_i, \bar{v}_i]$ (ignore ties)

- P's objective: efficiency, i.e., award the good to the bidder with the highest valuation: $d_i^*(v_1, \ldots, v_n) = 1$ if $v_i > \max_{j \neq i} \{v_j\}$, and 0 otherwise

- P has not have own reserve value for the good; does not care about revenues raised;

- zero outside option payoff for all bidders (need to ensure voluntary participation)
Designing Efficient Auctions in Private Value Settings: The Problem

- **Problem of (Partial) Bayesian Implementation (given beliefs \{f_i\}):** Does there exist a sealed-bid auction \((d_i(b_1, \ldots, b_n), p_i(b_1, \ldots, b_n))\) which yields a Bayesian equilibrium given beliefs \{f_i\} which results in an efficient outcome in all states \((v_1, \ldots, v_n)\)?

- If yes:
  - how sensitive is this equilibrium to the beliefs?
  - how sensitive is the auction design to these beliefs?

These questions are concerns about the robustness of the mechanism to changes in beliefs.
Designing Efficient Auctions in Private Value Settings: The Problem

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- **If yes:**
  - how sensitive is this equilibrium to the beliefs?
  - how sensitive is the auction design to these beliefs?

- **The first question is about existence of an implementing mechanism, the subsequent ones are concerns about its robustness to changes in beliefs**
Robustness Criteria

- **Definitions:**
  - The Bayesian equilibrium \( \{b_i(v_i)\}_i \) is **robust** with respect to prior beliefs if it is an equilibrium for every possible set of beliefs \( \{\tilde{f}_i\}_i \).

- **Observation**
  - With private values, a Bayesian equilibrium is robust if and only if it is a dominant strategy equilibrium. (Proof of only if part is straightforward: consider degenerate beliefs concentrated on any state, repeat for all states)

- Hence the problem of (partial) robust Bayesian implementation reduces to problem of (partial) dominant strategy implementation: does there exist an auction \( \{d_i(.), p_i(.)\}_i \) which has a dominant strategy equilibrium that results in efficient outcomes in all states?
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Dominant Strategy Implementation of Efficient Outcomes with Private Values

Vickrey (second-price) auction: awards the good to the highest bidder \( d_i = 1 \) iff \( b_i > \max_{j \neq i} \{b_j\} \), who is required to pay the second highest bid \( p_i = \max_{j \neq i} \{b_j\} \) and others pay nothing.

Each bidder has a dominant strategy: bid truthfully \( b_i = v_i \), which results in an efficient allocation.
Dominant Strategy Implementation of Efficient Outcomes with Private Values

- Vickrey (second-price) auction: awards the good to the highest bidder \( d_i = 1 \text{ iff } b_i > \max_{j \neq i}\{b_j\} \), who is required to pay the second highest bid \( p_i = \max_{j \neq i}\{b_j\} \) and others pay nothing.

- Each bidder has a dominant strategy: bid truthfully \( b_i = v_i \), which results in an efficient allocation.

- However, the argument uses the private values assumption (each bidder knows own valuation, does not vary with other’s valuation).
Suppose \( v_i = v + \epsilon_i \) where \( v \) is an unknown common value component distributed according to some density \( f \) on \([v, \bar{v}]\), and \( \epsilon_i \)'s are independent private value components.

Bidder \( i \)'s information: signal \( s_i = v + \delta_i \) of the common value, with independent noise \( \delta_i \).

Bidder \( i \)'s valuation depends on own signal \( s_i \) as well as of others \( s_{-i} \), but observes only \( s_i \).

Reformulate state of the world: \((s_1, \ldots, s_n)\), where \( i \) is privately informed about \( s_i \), and has valuation \( v_i(s_i, s_{-i}) \) where \( \frac{\partial v_i}{\partial s_i} > 0 \).

In this context, also true that \( \frac{\partial v_i}{\partial s_j} > 0 \), but DM do not impose this (I shall, to simplify).
For now, assume common prior beliefs over \((s_1, \ldots, s_n)\), and these are **not** independent (i’s beliefs over \(s_{-i}\) will depend on \(s_i\))

Bayesian equilibrium bidding strategies in a sealed bid auction given these beliefs: \(b = b_i(s_i)\) maximizes

\[
E_{s_{-i}|s_i}[v_i(s_i, s_{-i})d_i(b, b_{-i}(s_{-i})) - p_i(b, b_{-i}(s_{-i}))]
\]

Can no longer define dominant strategy in the usual ‘belief-free’ manner: because \(i\) does not ‘know’ own true value of the good, and would learn from the bids of the others

\(i\) cannot disregard the information of other bidders; beliefs over this information is needed in order to formulate \(i\)’s objective
We can extend the definition of dominant strategy equilibrium, however, by requiring optimality of bidding strategies in every state of the world (rather than irrespective of bids of others)

\{b_i(s_i)\}_i \text{ is an Ex Post equilibrium (EPE) if } b = b_i(s_i) \text{ maximizes } \\
\n\nu_i(s_i, s_{-i})d_i(b, b_{-i}(s_{-i})) - p_i(b, b_{-i}(s_{-i})) \\

\text{ for every possible state } (s_i, s_{-i})

Observation \text{ With interdependent values, a Bayesian equilibrium is robust if and only if it is an EPE.}
Dasgupta-Maskin (2000) show answer is yes, provided the following (Monotonicity (M)) assumption holds:

\[
\frac{\partial v_i}{\partial s_i} > \frac{\partial v_j}{\partial s_i} (i \neq j) \text{ whenever } v_i = v_j = \max_k \{v_k\}
\]

This condition is also necessary (if we have $<$ instead, an efficient EPE equilibrium does not exist)

A **generalized Vickrey** auction, in which bidders submit bid (conditional valuation) functions $b_i(v_{-i})$ where $v_{-i}$ is the vector of bids of others
Generalized Vickrey Auction

- Illustrate in the case of two bidders: bidder $i = 1, 2$ submits $b_i(v_j)$, which must satisfy $|\frac{\partial b_i}{\partial v_j}| < 1$

- $P$ calculates fixed point $(v_1^0, v_2^0) = (b_1(v_2^0), b_2(v_1^0))$ if one exists, otherwise does not allocate the good

- $M$ ensures fixed point, if it exists, is unique

- The good is awarded to the bidder with a higher valuation: e.g., $1$ wins if $v_1^0 > v_2^0$, and pays $v_1^*$ which is a fixed point of $b_2(.)$, i.e., $v_1^* = b_2(v_1^*)$.

- Generalizes second price auction in the following sense: if $1$ were constrained to a constant bid, $v_1^*$ is the minimum bid at which $1$ would win the good
Truthful Bidding is an EPE: Proof

- Suppose 2 bids truthfully: \( b_2(v_1(s_1, s_2)) = v_2(s_1, s_2) \) for all \((s_1, s_2)\)

- Take any state \((s_1, s_2)\), suppose 1 knows the state: show that it is optimal for 1 to also submit a truthful bid \( b_1(v_2(s_1, s_2)) = v_1(s_1, s_2) \)

- **Observation 1:** If both bid truthfully, good will be allocated efficiently

- **Observation 2:** Conditional on winning, bidder 1’s payoff is \( v_1(s_1, s_2) - v_1^* \), independent of what he bid (depends only on \( v_1^* \) which depends on 2’s strategy)

- Hence it suffices to show that truthful bidding will result in 1 winning if and only if \( v_1(s_1, s_2) - v_1^* \) is positive
Truthful Bidding is an EPE: Proof (contd)

- $v_1(s_1, s_2) - v_1^*$ is positive iff $v_1(s_1, s_2) > v_1^*$, iff

  $$b_2(v_1(s_1, s_2)) - b_2(v_1^*) < v_1(s_1, s_2) - v_1^*$$

  (given restriction on slope of bid functions)

- Since $b_2(v_1^*) = v_1^*$, and 2 bids truthfully this is equivalent to

  $$b_2(v_1(s_1, s_2)) = v_2(s_1, s_2) < v_1(s_1, s_2)$$

  which is the outcome of truthful bidding by 1 when 1 wins