1. A monopolist wishes to sell a good produced at constant unit cost $c \in (0, 1)$ to a large population of consumers with heterogeneous preferences: a consumer of type $\theta$ has a payoff $\theta \log(q + 1) - t$ for consuming $q \geq 0$ units of the good and paying $t$ dollars for it. $\theta$ is distributed uniformly on $[0, 1]$. The monopolist cannot identify the type of any given consumer. Each customer has an outside option of 0.

(a) If $q(\theta)$ denotes the quantity sold to type $\theta$, find a condition on this function $q(.)$ that ensures that it is IC (incentive compatible, i.e., there exists some pricing rule $t(q)$ for which $q(\theta)$ is the optimal purchase of type $\theta$)).

$q(.)$ has to be nondecreasing. The reason is that this condition and

$$V(\theta) \equiv \theta \log[1 + q(\theta)] - t(\theta) = V(0) + \int_0^\theta \log[1 + q(x)]dx$$

are necessary and sufficient for incentive compatibility. Given any $q(\theta)$ that is non-decreasing, there exists $V(0)$ and function $t(\theta)$ for which (1) holds.

(b) For any such IC $q(.)$, what is the associated set of payments (i.e., $t(\theta)$) that customers (of type $\theta$) make to the monopolist?

From (1), we obtain:

$$t(\theta) = \theta \log[1 + q(\theta)] - \int_0^\theta \log[1 + q(x)]dx - V(0)$$

(c) Obtain an expression for total profit of the monopolist as a function only of the selling strategy $q(.)$ and payoff of consumer of type 0.

$$\Pi = \int_0^1 [t(\theta) - cq(\theta)]d\theta$$

$$= \int_0^1 \{\theta \log[1 + q(\theta)] - \int_0^\theta \log[1 + q(x)]dx - cq(\theta)\}d\theta - V(0)$$

$$= \int_0^1 \{(2\theta - 1) \log[1 + q(\theta)] - cq(\theta)\}d\theta - V(0).$$
(d) Calculate the optimal selling strategy $q^*(\theta)$, and find the corresponding schedule of payments $t^*(\theta)$.

At the optimum the monopolist will set $V(\theta) = 0$. The monopolist’s problem is then to select $q(\theta) \geq 0$ over $[0, 1]$ to maximize

$$\Pi = \int_0^1 [(2\theta - 1) \log[1 + q(\theta)] - cq(\theta)] d\theta$$

subject to the constraint that $q(.)$ is nondecreasing.

Ignoring the monotonicity constraint on $q(.)$, point-wise optimization yields $q^*(\theta) = \frac{2\theta - 1}{c} - 1$ provided this is non-negative; otherwise $q^*(\theta) = 0$. Hence the solution is $q^*(\theta) = 0$ for all $\theta < \frac{1+c}{2} < 1$, and

$$q^*(\theta) = \frac{2\theta - 1}{c} - 1 \quad \text{if} \quad \theta \in \left[\frac{1+c}{2}, 1\right]$$

This is nondecreasing, so the monotonicity constraint does not bind, and this is the correct solution.

The corresponding schedule of optimal payments is obtained from (2): $t^*(\theta)$ equals zero if $\theta < \frac{1+c}{2}$, and

$$t^*(\theta) = \theta \log\left(\frac{2\theta - 1}{c}\right) - \int_{\frac{1+c}{2}}^{\theta} \log\left(\frac{2x - 1}{c}\right) dx = \frac{1}{2} \log\left(\frac{2\theta - 1}{c}\right) + \theta - \frac{c+1}{2}$$

if $\theta \in \left[\frac{1+c}{2}, 1\right]$.

(e) Find the payment rule $t(q)$ that implements this outcome, i.e., where a consumer of type $\theta$ selects $q^*(\theta)$ to maximize $\theta \log[1 + q] - t(q)$ and $t^*(\theta) = t(q^*(\theta))$. Does the optimal nonlinear pricing rule involve unit price discounts or premia for high $q$ purchases?

We see that $t^*(\theta) = \frac{1}{2} \log[1 + q^*(\theta)] + \frac{c}{2}q^*(\theta)$ for $\theta \in \left[\frac{1+c}{2}, 1\right]$, so the payment rule $t(q) = \frac{1}{2}\{\log[1 + q] + cq\}$ ought to implement the optimal allocation. We check this directly: maximization of $\theta \log[1 + q] - \frac{1}{2}\{\log[1 + q] + cq\}$ does indeed yield $q^*(\theta)$. The per unit price is then $p(q) = \frac{t(q)}{q} = \frac{1}{q} + \frac{\log[1+q]}{2q}$ which is decreasing in $q$. Hence we have discounts for high $q$ purchases.
2. A risk-neutral principal P hires an agent A, who chooses an effort \( a \geq 0 \), which results in gross profit \( x = a + \epsilon \) for P, where \( \epsilon \) is uniformly distributed on \([0, 1]\). Hence the support of the distribution of \( x \) varies with \( a \). A’s payoff equals \( \frac{w^{1-\rho}}{1-\rho} - \frac{a^2}{2} \), where \( w \) denotes a non-negative wage paid by P, and \( \rho > 0, \neq 1 \) is a parameter of risk-aversion. A has an outside option payoff of \( \bar{U} \) which is non-negative if \( \rho < 1 \) and negative if \( \rho > 1 \).

(a) If \( a \) is contractible, characterize the first-best wage and effort levels. Be careful to distinguish between the cases where \( \rho \) is smaller and where it is larger than 1.

The first-best problem is to select \( a \) and fixed wage \( w \) to maximize \([a + \frac{1}{2} - w]\) subject to

\[
\frac{w^{1-\rho}}{1-\rho} \geq \frac{a^2}{2} + \bar{U} \quad (PC)
\]

Consider first the case where \( \rho < 1 \). Then the LHS of (PC) goes from 0 to \( \infty \) as \( w \) goes from 0 to \( \infty \), while the RHS is positive. Since the LHS is increasing in \( w \), there is a unique solution:

\[
w(a) = \{(1 - \rho)(\bar{U} + \frac{a^2}{2})\}^{\frac{1}{1-\rho}} \quad (1)
\]

and

\[
w'(a) = a\{(1 - \rho)(\bar{U} + \frac{a^2}{2})\}^{\frac{\rho}{1-\rho}}
\]

which is increasing in \( a \) and going from 0 to \( \infty \) as \( a \) goes from 0 to \( \infty \). Hence the first-best action is given by the unique solution to \( w'(a^*) = 1 \), and the first-best wage is \( w(a^*) \) as given by (1) above.

Next consider the case where \( \rho > 1 \). Now \( \bar{U} \leq 0 \). If \( a > [-2\bar{U}]^{\frac{1}{2}} \) then the RHS of (PC) is positive while the LHS is negative for all \( w \geq 0 \). Hence such actions are not implementable. The set of implementable actions is \( a \in [0, \bar{a}] \) where \( \bar{a} \equiv [-2\bar{U}]^{\frac{1}{2}} \). In this case also we obtain the same expression for \( w(a) \) and \( a^* \), as \( w'(a) \) increases from 0 to \( \infty \) as \( a \) goes from 0 to \( \bar{a} \).

(b) If \( a \) is not contractible, but the profit \( x \) is contractible, and \( \rho \in (0, 1) \), find a condition on the parameters of the problem which ensure that the first-best profit can be achieved by P. If \( \bar{U} = 0 \), when is this condition satisfied?
Now there is an effort incentive constraint to be satisfied. If the first-best is to be implemented, the risk-aversion of the agent requires the agent to be paid a constant wage $w(a^*)$ for $x \in [a^*, a^* + 1]$. If there is a payment schedule that renders selection of $a^*$ optimal for the agent, the following will also implement $a^*$: P pays the agent 0 if $x < a^*$ and $w(a^*)$ otherwise. Then the agent will not want to deviate from $a^*$ to any higher $a$. For $a < a^*$, the agent’s expected payoff is

$$
\Pi(a) = [1 + a - a^*][\frac{(a^*)^2}{2} + \bar{U}] - \frac{a^2}{2}
$$

since by construction $\frac{(w^*)^{1-\rho}}{1-\rho} = \frac{(a^*)^2}{2} + \bar{U}$. Hence $\Pi'(a) = \bar{U} + \frac{(a^*)^2}{2} - a$. A necessary and sufficient condition for $a^*$ to be optimal for the agent is that $\bar{U} + \frac{(a^*)^2}{2} - a^* \geq 0$, or

$$
\bar{U} \geq a^* - \frac{(a^*)^2}{2}.
$$

(2)

This however is not a condition on the parameters of the problem, as $a^*$ is endogenous. In the case where $\bar{U} = 0$, it is easy to check that $a^* = \frac{2}{1-\rho} \frac{\rho}{\bar{U}}$. Since the function $g(a) = a - \frac{a^2}{2}$ is negative if and only if $a > 2$, the first-best is implementable if and only if $a^* > 2$, or

$$
[\frac{1}{1-\rho}]^\rho > 2.
$$

(3)

This condition is not satisfied for $\rho$ close to zero, but is satisfied for $\rho$ close to one.

(c) If $\rho > 1$ what can you say about implementability of the first-best profit when $a$ is not contractible? How would you interpret these results?

In that case the first-best can be implemented as $u(w)$ goes to $-\infty$ as $w$ goes to 0. Hence the payment rule given above will not tempt the agent to deviate to a lower effort, as that will result in a zero wage with positive probability, i.e., a utility of $-\infty$.

What matters critically is the steepness of marginal utility at the minimum payment, i.e., the utility consequences for the agent when the limit of his liability is exercised. In case (c) it is steep enough to ensure implementability. In case (b), condition (3) is required when $\bar{U} = 0$, and this requires the parameter of risk aversion to be large enough.