1. A monopolist wishes to sell a good produced at constant unit cost $c \in (0, 1)$ to a large population of consumers with heterogeneous preferences: a consumer of type $\theta$ has a payoff $\theta \log(q + 1) - t$ for consuming $q \geq 0$ units of the good and paying $t$ dollars for it. $\theta$ is distributed uniformly on $[0, 1]$. The monopolist cannot identify the type of any given consumer. Each customer has an outside option of 0.

(a) If $q(\theta)$ denotes the quantity sold to type $\theta$, find a condition on this function $q(.)$ that ensures that it is IC (incentive compatible, i.e., there exists some pricing rule $t(q)$ for which $q(\theta)$ is the optimal purchase of type $\theta$).

(b) For any such IC $q(.)$, what is the associated set of payments (i.e., $t(\theta)$) that customers (of type $\theta$) make to the monopolist?

(c) Obtain an expression for total profit of the monopolist as a function only of the selling strategy $q(.)$ and payoff of consumer of type 0.

(d) Calculate the optimal selling strategy $q^*(\theta)$, and find the corresponding schedule of payments $t^*(\theta)$.

(e) Find the payment rule $t(q)$ that implements this outcome, i.e., where a consumer of type $\theta$ selects $q^*(\theta)$ to maximize $\theta \log(1 + q) - t(q)$ and $t^*(\theta) = t(q^*(\theta))$. Does the optimal nonlinear pricing rule involve unit price discounts or premia for high $q$ purchases?

2. A risk-neutral principal P hires an agent A, who chooses an effort $a \geq 0$, which results in gross profit $x = a + \epsilon$ for P, where $\epsilon$ is uniformly distributed on $[0, 1]$. Hence the support of the distribution of $x$ varies with $a$. A’s payoff equals $w^{1-\rho} - \frac{2^2}{2}$, where $w$ denotes a non-negative wage paid by P, and $\rho > 0, \neq 1$ is a parameter of risk-aversion. A has an outside option payoff of $\bar{U}$ which is non-negative if $\rho < 1$ and negative if $\rho > 1$. 


(a) If \( a \) is contractible, characterize the first-best wage and effort levels. Be careful to distinguish between the cases where \( \rho \) is smaller and where it is larger than 1.

(b) If \( a \) is not contractible, but the profit \( x \) is contractible, and \( \rho \in (0, 1) \), find a condition on the parameters of the problem which ensure that the first-best profit can be achieved by P. If \( \bar{U} = 0 \), when is this condition satisfied?

(c) If \( \rho > 1 \) what can you say about implementability of the first-best profit when \( a \) is not contractible? Provide an intuitive explanation of these results.