Mechanism Design: Single Agent, Continuous Types

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Extension to Single Agent, Continuous Types

- Now consider how the analysis extends when there is a continuum of types for the agent, instead of just two
- Most P-A models use a continuum of types, so it is useful to learn how to handle models with a continuum of types more generally
- Non-standard from the standpoint of classic optimization (Kuhn-Tucker) methods, for reasons that will become apparent

The Problem

- Customer of type θ has utility $\theta V(q) - T$, where $\theta \in [\underline{\theta}, \bar{\theta}]$
- Distribution of types given by a cdf F over $[\underline{\theta}, \bar{\theta}]$, assume this has a positive density f
- P does not know the type of any customer, knows only the distribution F
- P's profit
$$\int_{\underline{\theta}}^{\bar{\theta}} [T(\theta) - cq(\theta)] dF(\theta)$$
- The Revelation Principle continues to apply, hence we can confine attention to revelation mechanisms $(T(\theta), q(\theta))$ which satisfy:

$$\theta V(q(\theta)) - T(\theta) \geq \theta V(q(\theta')) - T(\theta'), \forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}] \qquad (IC)$$

$$\theta V(q(\theta)) - T(\theta) \geq 0, \forall \theta \in [\underline{\theta}, \bar{\theta}] \qquad (PC)$$

Problem with a Kuhn-Tucker Approach

- There are a continuum of constraints!
- Can prune some of them, e.g., observe that PC for all types $\theta > \underline{\theta}$ are redundant, owing to the ICs (Check!)
- But we still have ICs for all possible pairs of types
- Maximization problem (for P) with optimization constraints (A's best responses)
- Could try to use calculus first order conditions for the optimization constraint, but:
  - these are necessary but may not be sufficient (depending on curvature of $T(.)$ and $q(.)$)
  - apply only if $T(.)$ and $q(.)$ are differentiable
  - may be too restrictive to confine attention to smooth functions of suitable curvature

The Mirrlees-Myerson Approach

- Use an alternative approach based on indirect utility function of the consumer:

$$U(\theta) = \max_{\theta' \in [\underline{\theta}, \bar{\theta}]} [\theta V(q(\theta')) - T(\theta')]$$

- Recall the Envelope Theorem, which states that U **is differentiable with** $U'(\theta) = V(q(\theta))$**, for almost all values of** θ (without imposing any conditions on the $T(.), q(.)$ functions at all!)
- Integrate this condition to obtain

$$U(\theta) = \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta} + U(\underline{\theta}), \forall \theta \in [\underline{\theta}, \bar{\theta}]$$

- IC states that $U(\theta) = \theta V(q(\theta)) - T(\theta)$ for all θ, hence:

$$\theta V(q(\theta)) - T(\theta) = \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta} + [\underline{\theta} V(q(\underline{\theta})) - T(\underline{\theta})], \forall \theta \in [\underline{\theta}, \bar{\theta}]$$

Mirrlees-Myerson Approach, contd.

- So Envelope Theorem yields an expression for the revenue function that must accompany any chosen $q(.)$:

$$T(\theta) = \theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta} - [\underline{\theta} V(q(\underline{\theta})) - T(\underline{\theta})], \forall \theta \in [\underline{\theta}, \bar{\theta}]$$

- Restriction on what P can extract from type θ: type θ must be left with an 'informational rent' ($IR(\theta)$):

$$IR(\theta) = \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta} + IR(\underline{\theta})$$

- This is a necessary condition; what about sufficiency?

Mirrlees-Myerson Approach: Representation of IC

Proposition

$T(.), q(.)$ satisfies IC if and only if the following two conditions are satisfied:

(a)

$$T(\theta) = \theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta} - [\underline{\theta} V(q(\underline{\theta})) - T(\underline{\theta})], \forall \theta \in [\underline{\theta}, \bar{\theta}$$

(b) $q(.)$ is nondecreasing.

Necessity of (a) already sketched above, of (b) is the same as in the two type case (check!)

Proof of Sufficiency

- Take any $\theta, \theta' > \theta$. (Similar argument, reversed, works for $<$ case). To show that (a) and (b) imply

$$U(\theta') \equiv \theta' V(q(\theta')) - T(\theta') \geq \theta' V(q(\theta)) - T(\theta)$$

- Using (a) and then (b):

$$\begin{aligned} U(\theta') &= \int_{\theta}^{\theta'} V(q(\tilde{\theta}))d\tilde{\theta} + U(\theta) \\ &\geq \int_{\theta}^{\theta'} V(q(\theta))d\tilde{\theta} + U(\theta) \\ &= [\theta' - \theta]V(q(\theta)) + [\theta V(q(\theta)) - T(\theta)] \\ &= \theta' V(q(\theta)) - T(\theta) \end{aligned}$$

Problem Restatement

Observing from condition (a) that PC for all types $\theta > \underline{\theta}$ are redundant, the problem can be restated as:

Select $T(.), q(.)$ to maximize

$$\int_{\underline{\theta}}^{\bar{\theta}} [T(\theta) - cq(\theta)] dF(\theta)$$

subject to conditions (a), (b), and PC only for the lowest type $\underline{\theta}$:

$$T(\theta) = \theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta} - [\underline{\theta} V(q(\underline{\theta})) - T(\underline{\theta})], \forall \theta \in [\underline{\theta}, \bar{\theta}]$$

$$q(.) \quad \text{is nondecreasing}$$

$$\underline{\theta} V(q(\underline{\theta})) - T(\underline{\theta}) \geq 0.$$

Two Stage Approach

- As in the two type case, we proceed in two stages: (1) fix a non-decreasing $q(.)$ and find optimal transfers and resulting maximized revenue, then (2) select optimal $q(.)$
- **Stage One Problem:** Observe that it is optimal to set surplus of the lowest type to zero: $T(\underline{\theta}) = \underline{\theta} V(q(\underline{\theta}))$.
- Then (a) tells you how to set the transfers for all other types:

$$T(\theta) = \theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta}$$

- Hence given arbitrary non-decreasing $q(.)$, resulting profit is

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta V(q(\theta)) - \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta}))d\tilde{\theta} - cq(\theta)]d\theta$$

Second Stage Analysis

- Expression for expected profit reduces to first-best profit, minus expected information rents:

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta V(q(\theta)) - cq(\theta)] dF(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} \{ \int_{\underline{\theta}}^{\theta} V(q(\tilde{\theta})) d\tilde{\theta} \} dF(\theta)$$

- Integrate by parts the expression for expected informational rents:

$$\begin{aligned}
E[IR] &= IR(\bar{\theta})F(\bar{\theta}) - IR(\underline{\theta})F(\underline{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta) IR'(\theta) d\theta \\
&= IR(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} F(\theta) V(q(\theta)) d\theta \\
&= \int_{\underline{\theta}}^{\bar{\theta}} [1 - F(\theta)] V(q(\theta)) d\theta \\
&= \int_{\underline{\theta}}^{\bar{\theta}} [\frac{1 - F(\theta)}{f(\theta)}] V(q(\theta)) dF(\theta)
\end{aligned}$$

Second Stage Analysis

- Expression for expected 'second-best' profit reduces to

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta - \frac{1 - F(\theta)}{f(\theta)}] V(q(\theta) - cq(\theta) d\theta$$

- Interpret $[\frac{1-F(\theta)}{f(\theta)}]V(q(\theta))$ as the revenue lost by P owing to lack of information of true type of each customer
- Second-best problem is just like the first-best problem, except that the customers type θ is replaced by her 'virtual' type $\theta - \frac{1-F(\theta)}{f(\theta)}$
- Virtual type of the agent depends only on the cdf F, specifically on the 'hazard rate' $\frac{1-F(\theta)}{f(\theta)}$ of this distribution

Second Stage Analysis, contd.

- Second stage problem reduces to choosing $q(.)$ to maximize expected second-best profit

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta - \frac{1-F(\theta)}{f(\theta)}] V(q(\theta) - cq(\theta) d\theta$$

subject to $q(.)$ is non-decreasing

- Ignore the monotonicity constraint to start with, then check whether it will be binding: choose $q(.)$ to point-wise maximize second -best profit:

$$[\theta - \frac{1-F(\theta)}{f(\theta)}] V'(q^*(\theta)) = c$$

Second Stage Analysis, contd.

- q^* will be non-decreasing if and only if virtual type $v(\theta) \equiv \theta - \frac{1-F(\theta)}{f(\theta)}$ is non-decreasing
- Sufficient condition for this: hazard rate $\frac{1-F(\theta)}{f(\theta)}$ is non-increasing
- Holds for uniform distribution, and many others (normal, logistic, chi-squared, exponential, Laplace...) where density is not falling 'too fast' anywhere
- While Myerson shows what to do when the monotonicity constraint does bind, we shall ignore this complication and focus hereafter on distributions with a monotone hazard rate

Properties of the Solution

- Second best solution: equate marginal utility of the 'virtual' type to cost c; *closed form solution!!*
- Virtual type equals true type only for highest type $\bar{\theta}$: second-best quantity equals first-best
- For all other types, second-best quantity is smaller
- Just like the two-type case
- Inefficiency takes the form of underproduction, for all but the highest type
- Payoff implications: lowest type gets zero surplus as before, all others with positive quantity get positive surplus/IR