4. Let $Z(p)$ denote the aggregate excess demand function. Let $y = \sum y_j$ be the aggregate profit vector. Then $p$ is a Walrasian equilibrium price vector if and only if $Z(p) \leq 0$, $\forall y \in Y$.

(i) Small households are identical, $Z(p)$ satisfies WARP, since $Z(p)$ does. Let $p$ and $p'$ be Walrasian equilibrium price vectors, and take $p'' = \alpha p + (1-\alpha)p'$, $Z'' = Z(p'')$. Then by Walras' law:

$$p''Z'' = \alpha p'Z' + (1-\alpha)p''Z'' = 0$$

Either $p''Z'' \leq 0$ or $p'Z' \leq 0$, both. Wlog, suppose $p''Z'' \leq 0$. Then $Z'' \neq 0$ implies by virtue of WARP, that $p''Z'' > 0$. This contradicts the fact $p''Y = \alpha p'Y + (1-\alpha)p''Y \leq 0$, $\forall y \in Y$, so $Z'' \neq 0$. Hence $Z'' = 0$, so $p''$ is a Walrasian equilibrium price vector.

(ii) To show $Z(p)$, the aggregate production vector $Z(p)$ in Walrasian equilibrium is unique, suppose otherwise: $\{p, Z(p) = Z\}$ and $\{p', Z'(p) = Z\}$ are two equilibria, with $Z \neq Z'$. Since $Z \leq Y$ and $p'$ is an equilibrium, $p'Z \leq 0$. Hence $Z' \neq Z$ and WARP imply $p'Z > 0$, contradicting the fact that $p$ is an equilibrium and $Z' \leq Y$. 
(i) Let set of types be \( i = 1, \ldots, I \), where \( i \) is denoted by \( X_i \in \mathbb{R}^k, x_i \) which is strictly convex & continuous, and endowment \( w_i \geq 0 \). Let the economy be replicated \( n \) times, \( r = 1, \ldots, n \). Suppose there exists a core allocation in the replica economy \( x_{ir}; i = 1, \ldots, I; r = 1, \ldots, n \). \( x_{ir} \neq x_{iv} \) for some \( i, r, r' \). Reorder agents so that

\[
x_{i1} \preceq x_{i2} \preceq \cdots \preceq x_{in}
\]

and construct a coalition of worst-off agents of each type \( i, i = 1, \ldots, n \). This coalition can block the proposed allocation, since there exists \( x_{ir} = \bar{x}_i = \frac{1}{n} \sum_{r=1}^{n} x_{ir} \), which is feasible for this coalition (as \( \frac{1}{n} \sum_{r=1}^{n} \sum_{i=1}^{I} x_{ir} \leq \sum_{i=1}^{I} w_i \)).

and strict convexity of \( x_i \) implies \( \bar{x}_i \succ x_{ii} \) for all \( i \).

(ii)

Consider this Edgeworth box with two types 1, 2 of traders, where trader 1 has straight-line indifference curves \( I^1, I''^1 \) etc., while trader 2 has strictly convex indifference curves \( I^2 \). Let \( A \) be a Walrasian equilibrium.

Take \( B \) and \( C \) such that \( AB = AC \). Shift on the same indifference curve \( I''^1 \) for 1 as \( w \). Suppose this economy is replicated twice and consider the allocation where both types of trader 2 consume point \( A \), while one type of trader 1 consumes \( B \) and the other \( C \).

This violates equal treatment, but is a Walrasian allocation, and hence must lie in the core.
Define \( Y_i(\phi) = Z_i(\phi, q(\phi)) : \Delta^1 \to \mathbb{R}^+ \) which is continuous and satisfies \( Y_i(\phi) = 0 \), all \( \phi \). Define the \( f : \Delta^1 \to \mathbb{R}^+ \) as follows:

\[
f_i(\phi) = \frac{\phi_i + \max \{0, Y_i(\phi)\}}{1 + \sum_j \max \{0, Y_j(\phi)\}}
\]

and by construction \( f \) maps \( \Delta^1 \) to itself and is continuous. By Brouwer's theorem there exists \( p^* \in \Delta^1 \) s.t. \( f_i(p^*) = \phi_i^* \), all \( i \). Then

\[0 = p^* Y_i(p^*) = \sum_i f_i(p^*) Y_i(p^*) = \frac{1}{K(p)} \sum_i [p_i^* Y_i(p^*) + \phi_i^* \max \{0, Y_i(p^*)\}]
\]

where \( K(p) = 1 + \sum_j \max \{0, Y_j(p^*)\} \cdot \frac{p_i^* \max \{0, Y_i(p^*)\}}{K(p^*)} \)

Hence \( Y_i(p^*) \leq 0 \), all \( i \), and \( p^* \) is a temporary equilibrium price vector.

Household's maximization problem:

\[
\max_{\{z_{ri}\}} \quad \sum_i \pi_i \sigma_i \left( \omega_i + \sum_k g_{ri} z_{ri} \right)
\]

s.t. \( \sum_k g_{ri} z_{ri} \leq 0 \), \( z_{ri} \geq 0 \)

where \( g_{ri} \) represents price of asset \( i \). An equilibrium for this economy is \( \{z_{ri}^*, g_{ri}^*\} \) such that \( \{z_{ri}^*\}_i \) solves (9) with \( g_{ri} = g_{ri}^* \) and \( \sum z_{ri}^* = 0 \).

Note that the feasible set in (9) is always nonempty (i.e., select \( g_{ri} = 0 \), all \( i \)), and is convex, while the objective function is strictly quasiconcave once there exists a unique solution to (9) for all \( g \in \mathbb{R}^k \) such that \( g_{ri} > 0 \), all \( i \) (since the feasible set is then compact, while \( V \) is continuous). So we can define the excess demand function \( z_{ri}(g) \) for any
\( q \gg 0 \). Since \( w_i \gg 0 \) all \( i, s \), this function is continuous at all \( q \gg 0 \).

(To \( q_{\nu} \rightarrow q', z_n \rightarrow z \) with \( z_n = \frac{2}{\nu} \nu \). If \( z + z(\nu) \), let \( z' = \frac{2}{\nu} + z \), so

\( z' \leq 0 \) and is strictly higher \( z' \to 0 \). By strict monotonicity of \( V' (\cdot i) > 0, \nu > 0, \) all \( i, s \), \( q z' = 0 \). Hence \( z', > 0 \) for some \( k \), unless \( z' = 0 \). In either case, we can redefine \( z_n \), slightly, so we can find \( z'' \) arbitrarily close
to \( z' \) such that \( z'' \) is strictly preferred to \( z \), and \( q z'' < 0 \). Then \( q_n z'' < 0 \)
in large \( n \), and we have a contradiction.)

If is homogeneous of degree 0 in \( q \), and satisfies Walras' law (since it is strictly monotone), is bounded below (by \( -B \)). It suffices to show

\[ \lim_{n \to \infty} \max_k \{ z_k(\nu_n) \} \to \infty \text{ if } \nu_n \to q \text{ with } q_k = 0 \text{, some } k. \]

Not, \( z(\nu_n) \) is bounded, and so has a convergent subsequence \( z(\nu_{n_k}) \to z_{xy}. \)

The same argument as above to show that \( z \) maximizes \( V' \) at \( a \) initial vector.

But with \( q_k = 0 \) some \( k \), every household can purchase more of the \( k \)

But, so there cannot exist a solution to \((*)\).
A Pareto optimal allocation is one which maximizes

$$
\sum_i \lambda_i \left[ u_i(x_{oi}) + \sum_o \pi_o u_i(x_{oi}) \right]
$$

subject to

$$
\sum_i x_{oi} \leq W_0 = \sum_i w_{oi}
$$

$$
\sum_i x_{oi} \leq W_o
$$

and

$$
x_{oi} \geq 0, \quad x_{oi} \geq 0, \quad \text{all } i, o.
$$

in some set of positive welfare weights \( \lambda_i > 0 \). Since \( W_0 > 0, \ W_o > 0 \) all \( i \), there exists a feasible solution with \( x_{oi} = \frac{1}{\lambda_i} W_0 > 0, \ x_{oi} = \frac{1}{\lambda_i} W_o > 0 \) which is interior. So \( u_i'(0) = 0 \) implies the optimal solution must be interior. Since \( u_i \) is strictly concave & differentiable, while the feasible set is convex, the following first-order conditions are necessary and sufficient to describe a P.O. allocation:

$$
\lambda_i u'_i(x_{oi}) = \theta_o \quad (2)
$$

$$
\lambda_i u'_i(x_{oi}) = \frac{\theta_o}{\pi_o} \quad (3)
$$

$$
\sum_i x_{oi} = W_0, \quad \sum_i x_{oi} = W_o \quad (4)
$$

or equivalently

$$
\lambda_i u'_i(x_{oi}) = \lambda_j u'_j(x_{oj}), \ \text{all } i, j \quad (2')
$$

$$
\lambda_i u'_i(x_{oi}) = \lambda_j u'_j(x_{oj}), \ \text{all } i, j \quad (3')
$$

in conjunction with (4).
(i) Let \( x \) and \( x' \) be two states with \( W_x \geq W_{x'} \iff x_{si} \geq x'_{si} \), some \( i \).

(\text{by (4))} \iff \lambda_i u_i'(x_{si}) \leq \lambda_i u_i'(x'_{si}), \text{ some } i \iff \lambda_j u_j'(x_{sj}) \leq \lambda_j u_j'(x'_{sj}), \text{ all } j \text{ (by (3))} \iff x_{sj} \geq x'_{sj}, \text{ all } j. \text{ Hence } W_x = W_{x'} \implies x_{si} = x'_{si}, \text{ all } i, \text{ and } W_x \geq W_{x'} \implies x_{si} > x'_{si}, \text{ all } i. \text{ So } x_{si} \text{ is a strictly increasing function of } W_x.

(ii) If (i) holds, 

\[
- \frac{d \log u_i'(x)}{dx} = \frac{1}{A_i + Bx} = \frac{1}{B} \frac{d \log (A_i + Bx)}{dx}
\]

so 

\[
u_i'(x) = C_i (A_i + Bx)^{-1/B}, \text{ for some constant } C_i.
\]

Hence (3') reduces to 

\[
\lambda_i C_i (A_i + Bx_{si})^{-1/B} = \lambda_j C_j (A_j + Bx_{sj})^{-1/B}
\]

or

\[
\frac{P_i}{P_j} (A_i + Bx_{si}) = A_j + Bx_{sj}
\]

where \( P_j = (A_j C_j)^{-1} \). (5) implies upon summing over \( j \)

\[
\sum_j \frac{P_i}{P_j} (A_i + Bx_{si}) = \sum_j \frac{1}{P_j} A_j + B \sum_j x_{sj} = \sum_j A_j + B W_x
\]

so \( x_{si} \) is linear in \( W_x \).

An equilibrium in the asset market is price vector \( q \) and portfolio allocation \( \{z_{ki}\}_{k=1}^q; i = 1, \ldots, T \), such that

1. \( \{z_{ki}\}_{k=1}^q \) maximizes \( u_i(z_{oi}) + \sum_k \pi_k u_i(z_{ki}) \)

subject to 

\[
x_{oi} + \sum_k q_{ki} z_{ki} \leq w_{oi} + \sum_k q_{ki} \bar{z}_{ki};
\]

\[
x_{si} \leq \sum_k q_{ki} \bar{z}_{ki}, \quad (\text{with } r_{s1} = 1, r_{s2} = 2)
\]

\[
x_{oi} \geq 0, x_{si} \geq 0, \text{ all } i.
\]
and (b) \( \sum_i z_{ki} = \sum_i \bar{z}_{ki} \), \( k = 1, 2 \).

Any such equilibrium,
\[ x_{oi} = \sum_k r_{sk} \bar{z}_{ki} = \bar{z}_{oi} + r_{oi} \bar{z}_{oi} \]

and \( W_2 = \sum_i \bar{z}_{oi} + r_{oi} \sum_i \bar{z}_{oi} \), so \( r_{oi} \) is linear in \( W_2 \); implying \( x_{oi} \) is linear in \( W_2 \), which implies (3'). To check (3'), note that the maximization problem (a) reduces to the unconstrained maximization
\[ u_i (w_{oi} + \sum_k r_{sk} \bar{z}_{ki} - \sum_k q_{sk} \bar{z}_{ki}) + \sum_k \pi_k u_i (\sum_k r_{sk} \bar{z}_{ki}) \]

generating the first-order condition
\[ \theta_k u_i'(x_{oi}) = \sum_k \pi_k u_i'(x_{oi}) r_{sk} \]

\[ = \sum_k \frac{\pi_k}{\lambda_i} \frac{\theta_k}{\pi_k} r_{sk} \text{ using (3)} \]

\[ = \frac{1}{\lambda_i} \sum_k \pi_k r_{sk} \]

so \( \lambda_i u_i'(x_{oi}) = \frac{1}{\theta_k} \sum_k \pi_k r_{sk} \), independent of (1)

so (2) holds.