Ec 703 Microeconomic Theory

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# Ec703 SOLUTIONS TO MIDTERM EXAMINATION, 2008

*1. Consider an economy with I households and J firms. Each firm has a strictly convex and compact production set. Each household has (i) a given ownership share $\theta_{ij}$ in firm j, (ii) strictly monotone preferences, (iii) an excess demand function $Z^i(p)$ defined for all $p >> 0$, incorporating incomes resulting from i's share in different firms' profits. $Z^i(.)$ is continuous, homogeneous of degree zero, bounded from below and unbounded above (i.e., with respect to the maximum excess demand across all commodities, as some price tends to zero).*

*Provide detailed arguments to show that the economy has: (i) a well-defined excess demand function, and (ii) at least one competitive equilibrium. You can invoke theorems proven in class.*

(i) Firm $j$'s profit maximization problem is Max $p.y$ subject to $y \in Y^j$, where $Y^j$ is strictly convex and compact. This maximization problem has a unique solution $y^j(p)$, defining a supply function for firm $j$. The Theorem of the Maximum applies to the maximization problem, as the objective function is continuous in $p$, and the feasible set is independent of $p$. So the set of solutions to the maximization problem forms a upper hemicontinuous correspondence in $p$. Since the solution is unique, it follows that $y^j(p)$ is continuous.

Define the excess demand function for the economy as a whole: $E(p) \equiv \sum_i Z^i(p) - \sum_j y^j(p)$, which is clearly continuous. We check the other conditions on $E(p)$ that guarantee existence of a competitive equilibrium:

(ii) *Walras Law:* $p.E(p) = \sum_i p.Z^i(p) - \sum_j py^j(p) = 0$ because $p.Z^i(p) = \sum_j \theta_{ij} y^j(p)$ and summing over households we get $\sum_i p.Z^i(p) = \sum_j py^j(p)$.

(iii) *Homogeneity of degree zero:* This is straightforward, as $Z^i(p)$ and $y^j(p)$ are homogenous of degree zero.

(iv) *Bounded below:* This follows since $Z^i(p)$ is bounded below, and so is $y^j(p)$, as firm $j$'s production set is bounded.

(v) *Unbounded above:* If $p_n \to p$ where some prices are zero at $p$, then $\text{Max}_l Z^i_l(p_n) \to \infty$, while $y^j(p_n)$ is bounded. So $\text{Max}_l E(p_n) \to \infty$.

It follows that $E(p)$ is an excess demand function defined over all strictly positive price vectors, satisfying assumptions (i)-(v), so a competitive equilibrium exists.

*2. Answer either (a) or (b) of the following two questions.*

*(a) Consider an exchange economy with $L$ commodities and $I$ consumers each of whom have a continuous, strictly convex and strictly monotone preferences. Consider any interior Pareto optimal allocation $x \equiv (x_1, x_2, \ldots, x_I) >> 0$. Suppose a social planner were to assign endowments $w_i = x_i$ for each $i$. Show that starting with these endowments, a competitive equilibrium cannot involve any trade. Comment on the significance of this result.*

*(b) Consider an exchange economy with $L$ commodities where for every household $i$ there is a good $l(i)$ on which household $i$ spends* **all** *its income. Under what additional conditions will competitive equilibrium in this economy be unique?*

(a) Suppose there exists a CE where at least one agent $i$ trades, and obtains a consumption $\tilde{x}_i \neq x_i$ . Then this agent must be strictly better off at $\tilde{x}_i$ than at $x_i$, owing to strict convexity of preferences. And every agent $j$ must be at least as well off as at $x_j$. This contradicts Pareto optimality of $x$.

Since the endowments are interior, the conditions on preferences implies at least one CE exists. By the above argument, it must involve no trade. Hence this provides an alternate, simple proof of the Second Welfare Theorem.

(b) Household $i$'s excess demand for commodity $l$ is $Z_l^i(p) = \frac{1}{p_l} p.\omega_i - \omega_{il}$ if $l = l(i)$, and $-\omega_{il}$ otherwise. So if $l \neq m$:

$$\frac{\partial Z_l^i(p)}{\partial p_m} = \omega_{im} \geq 0$$

if $l = l(i)$ and 0 otherwise. If it is the case that for every commodity $l$ there exists at least one household $i$ such that $l = l(i)$ and $\omega_{im} > 0$ for all $m \neq l$, then the aggregate excess demand function will exhibit the *gross substitute* property: $\frac{\partial Z_l(p)}{\partial p_m} \geq \frac{\partial Z^i_{l(i)}(p)}{\partial p_m} = \omega_{im} > 0$, whenever $l \neq m$. This implies uniqueness of competitive equilibrium.

*3. An exchange economy has two dates $t = 0, 1$ and two states of nature $s = 1, 2$ which will be revealed at date 1. Use $s = 0$ to denote the date-event pair corresponding to date 0. There is one physical commodity, and two consumers $i = 1, 2$ whose endowments $\omega_{is}$ are as follows: $\omega_{10} = 2, \omega_{11} = 4, \omega_{12} = 3, \omega_{20} = 4, \omega_{21} = 2, \omega_{22} = 3$. Both share the von-Neumann-Moregenstern utility $\log c_0 + \log c_1$, where $c_t$ denotes date $t$ consumption. Consumer 1 believes $s = 1$ with probability $\frac{3}{4}$,* *while consumer 2 believes $s = 1$ with probability $\frac{1}{4}$. At date $0$, consumers trade in the commodity, besides two assets $k = 1, 2$ whose date-$1$ returns $r_{sk}$ are given by $r_{11} = 1, r_{12} = 2, r_{21} = 0, r_{22} = 1$. At date $1$, spot commodity markets open.*

*(a) Derive the entire set of ex ante Pareto optimal allocations in this economy. Are these allocations ex post Pareto optimal as well?*

An allocation in this economy is represented by consumer 1's consumption allocation $x_{1s}, s = 0, 1, 2$, since this determines consumer 2's consumption allocation $x_{2s} = 6 - x_{1s}$, as the aggregate endowment of the economy is 6 for every $s$. An *ex ante* Pareto optimal allocation maximizes

$$\log x_{10} + \frac{3}{4}\log x_{11} + \frac{1}{4}\log x_{12} + \lambda[\log(6 - x_{10}) + \frac{1}{4}\log(6 - x_{11}) + \frac{3}{4}\log(6 - x_{12})]$$

where $\lambda > 0$ is the relative welfare weight on consumer 2. We obtain the following allocation corresponding to $\lambda$:

$$x_{10} = \frac{6}{1+\lambda}, x_{11} = \frac{18}{3+\lambda}, x_{12} = \frac{6}{1+3\lambda}.$$

Varying $\lambda$, we obtain the class of *ex ante* Pareto optimal allocations fr this economy.

These allocation are all *ex post* optimal as well, since there is a single physical commodity.

*(b) Describe carefully the optimization problem defining the optimal asset demands of the two consumers at date $0$ (You need not derive the asset demand functions: show the objective function and the budget constraints.)*

Date 0 markets will involve spot-trading in the commodity (price $p_0$), and trading in the two assets (prices denoted $q_1, q_2$). At date 1, the state of the world is revealed, and as there is a single physical good, there is no scope for trades between the two consumers. We can seleect the commodity as numeraire at date 0, so $p_0 = 1$ without loss of generality.

Consumer 1's budget constraint at date 0 is then

$$x_{10} + q_1 z_{11} + q_2 z_{12} \leq 2 \tag{1}$$

and at date 1 is

$$x_{11} = 4 + z_{11}, x_{12} = 3 + 2z_{11} + z_{21} \tag{2}$$

. So consumer 1 maximizes expected utility $\log x_{10} + \frac{3}{4}\log x_{11} + \frac{1}{4}\log x_{12}$ subject to constraints (1) and (2).

Consumer 2's budget constraint at date 0 is

$$x_{20} + q_1 z_{21} + q_2 z_{22} \leq 4 \tag{3}$$

and at date 1 is

$$x_{21} = 2 + z_{21}, x_{22} = 3 + 2z_{21} + z_{22} \tag{4}$$

. So consumer 1 maximizes expected utility $\log x_{20} + \frac{1}{4}\log x_{21} + \frac{3}{4}\log x_{22}$ subject to constraints (3) and (4).

*(c) What can you say about existence and Pareto optimality of Radner equilibria in this economy?*

Since the returns of the two assets are linearly independent (rank of the return matrix is 2), the set of Radner equilibria of this economy is the same as the set of equilibria of an Arrow securities economy, which in turn is the same as the set of Arrow-Debreu equilibria. Since the Arrow-Debreu economy has two consumers with interior endowments, strictly convex and monotone preferences (with respect to contingent commodities), it has at least one Arrow-Debreu equilibrium. Moreover, the two welfare theorems apply: all equilibria are *ex ante* Pareto optimal, and all optimal allocations can be achieved as (Arrow-Debreu, hence Radner) equilibria with redistribution of endowments.