Ec 703 Microeconomic Theory

Spring 2005 Boston University

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Ec703 SOLUTIONS TO MIDTERM EXAMINATION, March 2, 2006

1. [**2+3+10=15 marks**] An exchange economy has two dates $t = 0, 1$, one physical good ($L = 1$), I households, and S states of nature. Households hold endowments of goods and consume only at $t = 1$. In state s (known at $t = 1$ but not at $t = 0$), i consumes $x_{is} \geq 0$, has endowment $\omega_{is} > 0$, and a von-Neumann Morgenstern utility function $u_{is}(x_{is})$ which is continuous, strictly increasing and strictly concave. At $t = 0$, household i believes state s will arise with probability $\pi_{is}(> 0$ for all $i, s)$. There are K assets in this economy, with asset k generating return r_{sk} of the commodity in state s, where $r_{sk} > 0$ for all s. Household i has an initial endowment $\bar{z}_{ik} > 0$ of asset k at $t = 0$. Households trade in assets at $t = 0$, but are subject to a short sales constraint: $z_{ik} \geq -B$, for all i, k, where z_{ki} is i's holding of asset k and B is a positive number.

(i) Define the expected utility of household i at $t = 0$ as a function of its asset portfolio.

(ii) Define an equilibrium in the asset market at $t = 0$.

(iii) Show that such an equilibrium must exist. (You can invoke Propositions that have been proven in class or in problem sets, there is no need to prove them again. But make sure to check the validity of conditions required by any Proposition you invoke).

(i) $W_i(z_{i1}, z_{i2}, \ldots, z_{iK}) = \sum_s \pi_{is} u_{is}(\omega_{is} + \sum_k r_{sk} z_{ik})$ (ii) An equilibrium in the asset market at $t = 0$ is an asset allocation $\{z^*_{i1}, z^*_{i2}, \ldots, z^*_{iK}\}_{i=1,\ldots,I}$ and asset prices $q^*_k, k = 1, \ldots, K$ such that (a) $(z^*_{i1}, z^*_{i2}, \ldots, z^*_{iK})$ maximizes W_i subject to budget constraint $\sum_k q^*_k z_{ik} \leq \sum_k q^*_k \bar{z}_{ik}$ and short sales constraint $z_{ik} \geq -B$, and (b) asset markets clear: $\sum_{i=1}^I [z^*_{ik} - \bar{z}_{ik}] = 0$ for all $k = 1, \ldots, K$. (iii) This is a Walrasian equilibrium of the exchange economy where each agent has positive initial endowment of each asset, and a utility function W_i which is continuous, strictly concave and strictly monotone (because each state is assigned positive probability, and each asset generates a positive return). A feasible asset portfolio is bounded below because of the short sales constraint. Hence the feasible set for each household (intersection of the budget constraint and the short sales constraint) is compact at any strictly positive price vector q. So each household will have a unique optimal asset portfolio (uniqueness follows from strict concavity of W_i) at any $q >> 0$, which will be well-defined and continuous at any $q >> 0$, homogeneous of degree zero, satisfying Walras' Law (owing to strictly monotone preferences), bounded below (by $-B$) and unbounded above (with respect to any sequence of strictly positive price vectors converging to a limit where one or more assets have a zero price). Hence all five conditions required by excess demand functions for existence of a Walrasian equilibrium will be satisfied.

2. [**10 marks**] Consider an exchange economy with three goods, and one type of consumer with Cobb-Douglas preferences and strictly positive endowment of each good. Show by direct computation that the index of every (strictly positive) price vector in this economy must be $+1$.

The excess demand function for goods 1 and 2 is (with p_3 set equal to 1, and with $p_1, p_2 > 0$):

$$z_1 = \frac{\alpha_1(\omega_1 p_1 + \omega_2 p_2 + \omega_3)}{p_1} - \omega_1, z_2 = \frac{\alpha_2(\omega_1 p_1 + \omega_2 p_2 + \omega_3)}{p_2} - \omega_2.$$

Hence the determinant of the Jacobian of the excess demand function for goods 1 and 2 at any strictly positive price vector $(p_1, p_2, 1)$ is

$$[\alpha_1.\alpha_2]\{[-\frac{p_2\omega_2 + \omega_3}{p_1^2}][-\frac{p_1\omega_1 + \omega_3}{p_2^2}] - [\frac{\omega_2}{p_1}.\frac{\omega_1}{p_2}]\}$$

which equals

$$[\alpha_1.\alpha_2]\{\frac{(\omega_3)^2 + \omega_3\omega_1 p_1 + \omega_3\omega_2 p_2}{\alpha_1\alpha_2(p_1p_2)^2}\} > 0.$$

So the index of any price vector is $(-1)^{3-1}[+1] = +1$.

3. [**5*5 = 25 marks**] Consider an exchange economy with L goods, I households, S states of the world $s = 1, \ldots S$ at $t = 1$, and trading in K financial assets at $t = 0$, where asset

k pays off $r_{sk} > 0$ units of good 1 in state s. These assets trade at $t = 0$, while spot commodity markets open at $t = 1$ after s is revealed. All households share the same belief at $t = 0$ that state s will arise with probability $\pi_s > 0$. Household i has a state-independent von-Neumann Morgenstern utility function $u_i(x_{i1}, x_{i2}, \ldots, x_{iL})$ which is strictly increasing, strictly concave, twice continuously differentiable, with the marginal utility of any good l equal to ∞ if good l consumption equals zero.

In contrast to a Radner equilibrium where price expectations at $t = 0$ are actually realized, consider instead the Hicksian notion of a *temporary equilibrium* where household i has an arbitrary set of price expectations (at $t = 0$) denoted by $\bar{p}_{ils}$ for the spot price of good l that will prevail in state s. These price expectations are exogenously given and may turn out to not be realized.

(a) Formulate the expected utility at $t = 0$ of household i as a function of its asset portfolio.

(b) Provide a complete definition of a temporary equilibrium allocation and prices (for assets at $t = 0$ and date 1 spot commodity prices). Explain how the definition differs from a Radner equilibrium.

(c) Define an *ex ante* Pareto optimal allocation in this economy, and obtain first-order conditions that characterize such an allocation.

(d) Define an *ex post* Pareto optimal allocation in this economy, and obtain first-order conditions that characterize such an allocation. Explain how these differ from those that characterize an *ex ante* Pareto optimal allocation.

(e) What can you say about the *ex ante* or *ex post* Pareto optimality of a temporary equilibrium allocation?

(a)

$$W_i(\{z_{ik}\}) = \sum_s \pi_s V_i(\bar{p}_{is}, \bar{p}_{is}\omega_{is} + \sum_k r_{sk} z_{ik})$$

where $V_i(p, Y)$ denotes the indirect utility function corresponding to direct utility function u_i, with the spot price of good 1 set equal to one in every state. (b) A temporary equilibrium is an asset allocation $\{z^*_{ik}\}$, asset price vector $\{q^*_k\}$, date 1 consumption allocation $\{x^*_{ils}\}$ and spot commodity price vector $\{p^*_{ils}\}$ such that: (i) for each i: $\{z^*_{ik}\}$ maximizes W_i subject to $\sum_k q^*_k z_{ik} \leq 0$; (ii) asset markets clear: $\sum_k z^*_{ik} = 0$ for each k; (iii) for each i and each state s, the consumption bundle $\{x^*_{ils}\}$ maximizes u_i subject to the budget constraint $\sum_l p^*_{ls}.[x_{ils} - \omega_{ils}] \leq \sum_k r_{sk} z^*_{ik}$; and (iv) spot commodity markets clear in every state s : $\sum_{i=1}^{I}[x^*_{ils} - \omega_{ils}] = 0$ for each l.

This definition differs from a Radner equilibrium insofar as households have exogenous expectations at $t = 0$ concerning spot prices at $t = 1$ which may deviate from the equilibrium spot prices, so the price expectations may not be fulfilled.

(c) Let λ_i denote an arbitrary positive welfare weight for household i. Then an *ex ante* P.O. consumption allocation $\{x_{ils}\}_{i,l,s}$ maximizes $\sum_i \sum_s \pi_s u_i(x_{i1}, x_{i2}, \ldots, x_{iL})$ subject to the resource constraint $\sum_i [x_{ils} - \omega_{ils}] = 0$ for each l, s. Note that this definition has nothing to do with prices or price expectations. This is a concave optimization problem, and under the assumptions made all solutions are interior, characterized by the first-order conditions: there are positive multipliers $\delta_{ls}, l = 1, \ldots, L$ and $s = 1, \ldots, S$ such that for any i, l, s:

$$\lambda_i \pi_s u'_{ils} = \delta_{ls} \tag{1}$$

where u'_{ils} denotes marginal utility of i for good l in state s. These imply the following generalization of the Arrow-Borch equations to the multi-good case: the ratio of marginal utilities

$$\frac{u'_{ils}}{u'_{jms'}} = \frac{\delta_{ls}}{\delta_{ms'}} \frac{\pi_{s'}}{\pi_s} \frac{\lambda_j}{\lambda_i} \tag{2}$$

for any pair of goods l, m, any pair of households i, j, and any pair of states s, s'.

(d) An *ex post* P.O. consumption allocation $\{x_{ils}\}_{i,l}$ in state s maximizes $\sum_i \lambda_{is} u_i(x_{i1s}, x_{i2s}, \ldots, x_{iLs})$ for some set of welfare weights λ_{is}, subject to the resource constraint $\sum_i [x_{ils} - \omega_{ils}] = 0$ for each l. Note that the welfare weights need not be the same across different states. The corresponding first-order conditions state the existence of positive multipliers $\delta_{ls}, l = 1, \ldots, L$ such that for any i, l, s:

$$\lambda_{is} u'_{ils} = \delta_{ls}. \tag{3}$$

These imply equal marginal rates of substitution (with respect to any pair of goods l, m) across all households within any state:

$$\frac{u'_{ils}}{u'_{ims}} = \frac{\delta_{ls}}{\delta_{ms}} \tag{4}$$

a condition which is also true for any *ex ante* P.O. allocation. However, the latter additionally satisfies equality of marginal rates of substitution for any given good *across states* s, s' between households:

$$\frac{u'_{ils}}{u'_{ils'}} = \frac{\delta_{ls}}{\delta_{ls'}} \frac{\pi_{s'}}{\pi_s} \tag{5}$$

a property which may not hold for an *ex post* P.O. allocation.

(e) A temporary equilibrium allocation will involve all households equating their marginal rates of substitution within any state to a common spot commodity price vector, so will be *ex post* P.O. It need not be *ex ante* P.O. since they may have different price expectations and thus need not have equal marginal rates of substitution across states for any given good.