SOLUTIONS TO Ec703 FINAL EXAMINATION, May 11, 2005

1. [5+10 = 15 marks] A labor market has \( K \) firms and \( N \) workers. Each firm hires at most one worker. All workers have identical but unknown productivity \( \mu \). Firm \( i \) privately receives a signal \( x_i = \mu + \epsilon_i \) about \( \mu \), where \( \epsilon_i \) is drawn i.i.d. from a given distribution \( F \) with zero mean. Half the workers have reservation wage 0, the remaining half have reservation wage \( w > 0 \). The wage rate on this labor market is set by a Walrasian auctioneer.

(a) Define a rational expectation equilibrium (REE) for this labor market.

(b) For what parameter values does there exist a REE which reveals the mean signal \( \bar{x} \equiv \frac{\sum_i x_i}{K} \) to all firms? Under these conditions derive the REE wage function.

(a) A REE is a wage function \( w(x_1, \ldots, x_K) \) such that (if we define the demand for labor for firm \( i \) to be \( D_i(x_i, w(x_1, \ldots, x_K)) = 1 \) if and only if \( E(\mu|x_i, w(x_1, \ldots, x_K)) \geq w(x_1, \ldots, x_K) \) and 0 otherwise, and the supply of labor for worker \( j \) with reservation wage \( w_j \) to be 1 if and only if \( w(x_1, \ldots, x_K) \geq w_j \) and 0 otherwise), for any \( x_1, \ldots, x_K \):

\[
\sum_{i=1}^{K} D_i(x_i, w(x_1, \ldots, x_K)) \leq \sum_{j} S_j(w(x_1, \ldots, x_K)),
\]

with equality holding if \( w(x_1, \ldots, x_K) > 0 \).

(b) The supply function is \( \frac{N}{2} \) at any wage between 0 and \( w \), and \( N \) at any wage above \( w \). Consider the full information economy where each firm knows the realization of \( \bar{x} \): then the demand for labor is \( K \) at any wage between 0 and \( \bar{x} \), and 0 at any wage above \( \bar{x} \). If \( K > N \)
then the full information Walrasian wage will be \( w(x_1, \ldots, x_n) = \bar{x}, \) i.e., will be one-to-one in \( \bar{x} \). In this case this is also a REE which reveals \( \bar{x} \) to all firms when they observe only their own private signals.

If \( K \) lies between \( N \) and \( \frac{N}{2} \) then the full information Walrasian wage will equal \( \bar{w} \) whenever \( \bar{x} > \bar{w} \), so will not be a one-to-one function of \( \bar{x} \). But if the distribution of the signal \( x_i \) is bounded above by \( \bar{w} \), then the Walrasian wage will again equal \( \bar{x} \) in all states.

Finally if \( K < \frac{N}{2} \) then the full information wage is always 0 and then a revealing REE cannot exist.

So a necessary and sufficient condition for existence of a revealing REE is that either (i) \( K > N \), or (ii) \( K \) lies between \( N \) and \( \frac{N}{2} \) and the range of the signal \( x_i \) is bounded above by \( \bar{w} \). In these situations \( w(x_1, \ldots, x_K) = \bar{x} \).

\[ 5+7+8=20 \text{ marks} \]

2. A labor market has workers whose productivity \( \theta \) is distributed over a set of \( n \) possible values \( \{\theta_1, \ldots, \theta_n\} \) with frequencies \( \alpha_1, \ldots, \alpha_n \), where \( \theta_i < \theta_{i+1} \) and \( \alpha_i > 0, \sum_i \alpha_i = 1 \). Each worker is privately informed about her own productivity. Each worker decides whether or not to submit to a (costless) test which if taken publicly reveals her productivity without error. All employers (there are at least two) have a CRS technology so can hire as many workers as they like. They observe whether any given worker submitted to the test, and if so the result of the test. Then they make wage offers simultaneously to each worker. All workers have a zero reservation wage.

(i) Define a Perfect Bayesian Equilibrium (PBE) in pure strategies for this market game.

(ii) Construct a PBE.

(iii) Show that all PBEs will result in the same allocation.

(i) A PBE is: (a) a strategy of each worker of type \( \theta_i \) which is whether or not to submit to the test \( (I_i = 1 \) or 0\); (b) (common) beliefs of employers \( \mu_j(I, \theta_i) \) which is a posterior belief that the worker is of type \( \theta_j \) after observing either \( (I = 1, \theta_i) \) for a worker of true type
θ_i, or I = 0; and (c) a wage offer w_k made thereafter by firm k. It satisfies the requirement that:

(B) the posterior belief is derived from the workers strategy upon applying Bayes rule, which implies that \( \mu_j(1, \theta_i) = 1 \) if and only if \( i = j \), and \( \mu_j(0) = \frac{\alpha_j}{\sum_{i: I_i=0} \alpha_i} \) for any \( j \) such that \( I_j = 0 \) (assuming such a \( j \) exists), and 0 otherwise. If every type \( i \) selects \( I_i = 1 \) then the beliefs \( \mu_j(0) \) consequent on \( I = 0 \) are arbitrary.

(S) the strategies are sequentially rational given the beliefs, which means for every firm \( k \) offers the following wage function: \( w(1, \theta_i) = \theta_i \), and \( w(0) = E[\theta | I_i = 0] \), where this expectation is formed on the basis of the posterior beliefs, and for any worker of type \( \theta_i: I_i = 1 \) if \( \theta_i > w(0) \), \( I_i = 0 \) if \( \theta_i < w(0) \), and \( I_i \) either 0 or 1 if \( \theta_i = w(0) \).

(ii) There is a PBE in which every type of worker decides to submit to the test, combined with pessimistic beliefs \( (\mu_1(0) = 1) \) held by firms consequent upon observing a worker that does not take the test. Then \( w(0) = \theta_1 \) so every worker of type \( \theta_2 \) or higher strictly prefers to take the test, while workers of type \( \theta_1 \) are indifferent.

(iii): Consider any PBE where some type \( \theta_i \) decides not to submit to the test: by (S) we have \( \theta_i \leq w(0) = E[\theta | I_i = 0] \) for any such type. It follows that only one such type must not take the test (otherwise, take the highest type that does not take the test, this condition must be violated for that type). Moreover since for any other type \( \theta_j \) we have \( I = 1 \) we must have \( \theta_j \geq w(0) = E[\theta | I_i = 0] \), so the only type that does not take the test must be type \( \theta_1 \). Then all types must reveal themselves: all types \( \theta_2 \) and above take the test, and type \( \theta_1 \) reveals herself by not taking the test.

3. [4+3+8=15 marks] A social planner is contemplating nationalization of a given sector of the economy with a labor market subject to adverse selection. Workers have productivity \( \theta \) drawn from a continuous distribution over the interval \([\underline{\theta}, \bar{\theta}]\) with a density function \( f(\theta) \) and distribution function \( F(\theta) \). Each worker knows his own productivity, but private employers and the social planner cannot identify the productivity of any given worker, although they know the distribution \( F \). A worker of productivity \( \theta \) has a reservation wage \( r(\theta) \) which is
strictly increasing. The utility of this worker is then 
\[ u(\theta) \equiv w(\theta)I(\theta) + [b(\theta) + r(\theta)][1 - I(\theta)], \]
where \( I(\theta) \in \{0, 1\} \) is the decision whether or not he is employed, and \( w(\theta) \) and \( b(\theta) \) are the payments made to him in the event that he is or is not employed respectively.

The social planner has a welfare objective given by a welfare function
\[ \int_{\theta}^{\bar{\theta}} \alpha(\theta)u(\theta)dF(\theta), \]
where \( \alpha(\theta) \) is a positive welfare weight on a type-\( \theta \) worker. The planner seeks to design a mechanism for the nationalized firm to maximize this welfare function, subject to the constraint that the firm break even, and that no worker can be compelled to participate in the mechanism (i.e., each worker retains the option of not working and at the same time receiving or making no transfer to the government).

(a) State the Revelation Principle as it applies to this setting. Use it to identify the class of mechanisms for the nationalized firm that the planner can confine attention to.

(b) Use your answer in (a) to represent the planner’s problem as an optimization problem.

(c) Show that the planner’s problem reduces to selecting a single pair of numbers: a wage \( w \) and an unemployment benefit \( b \), and then letting each worker decide whether or not to work.

(d) Prove that the highest wage competitive equilibrium allocation in the private labor market solves the planner’s problem for some set of welfare weights.

(a) The Revelation Principle states that the planner can confine attention to truth-telling equilibria of revelation mechanisms defined by functions \( \{I(\theta), w(\theta), b(\theta)\} \). The class of feasible mechanisms satisfies the truth-telling constraint:
\[ u(\theta) \equiv I(\theta)w(\theta) + [1 - I(\theta)][b(\theta) + r(\theta)] \geq I(\theta^*)w(\theta^*) + [1 - I(\theta^*)][b(\theta^*) + r(\theta)] \quad (IC) \]
for all \( \theta, \theta^* \in [\underline{\theta}, \bar{\theta}] \). It must also satisfy the constraint that participation of the worker is voluntary:
\[ u(\theta) \geq r(\theta) \quad (PC) \]
for all \( \theta \in [\underline{\theta}, \bar{\theta}] \).
(b) The planners problem is to choose the functions \( \{I(\theta), w(\theta), b(\theta)\} \) to maximize

\[
\int_{\theta}^{\hat{\theta}} \alpha(\theta)u(\theta)dF(\theta)
\]

subject to constraints (IC) and (PC).

(c) Take any pair of types \( \theta, \theta^* \) that both work, i.e., such that \( I(\theta) = I(\theta^*) = 1 \). Then (IC) implies that \( w(\theta) = w(\theta^*) \). Conversely take any pair of types that do not work: then they must receive the same benefit. Letting \( w \) and \( b \) denote the wage and benefit for those that do and do not work, (IC) then reduces to letting each worker choose whether or not to work (i.e., whether or not to report a type whose employment status differs from one’s own).

(d) This follows the same proof as in the class lecture, or Proposition 13.B.2 in the text.