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EC703

PROBLEM SET 6, WITH SOLUTIONS

1. **(Continuation of Problem 2 from Problem Set 5.)** Consider the exchange economy described in that problem: two physical goods $l = 1, 2$, two consumers that share the same von-Neumann Morgenstern utility function $\log c_1 + \log c_2$, where c_i denotes consumption of good i; two states of nature A and B. Both consumers have endowment w of the second good in either state. In state A the first consumer has endowment $\frac{w}{2}$ of the first good, while the second consumer has endowment $\frac{3w}{2}$ of this good. In state B their endowments of the first good get reversed. Both consumers assign equal probability to the two states.

(c) Suppose the consumers trade in Arrow securities (corresponding respectively to the two states) at $t = 0$ before the state of nature is revealed, and subsequently trade in commodities after the state is revealed. Given a set of state-contingent *ex post* commodity spot prices anticipated *ex ante* by each consumer (p_{1j}, p_{2j}), derive the optimal asset demand z^{iA}, z^{iB} of each consumer i as a function of date 0 security prices.

SOLUTION: Let z^j denote quantity of the Arrow security corresponding to state j. Then consumer 1's indirect utility in state j as a function of asset portfolio and commodity spot prices p_{1j}, p_{2j} is given by the maximum value of $\log c_{1j} + \log c_{2j}$ subject to the constraint $p_{1j}c_{1j} + p_{2j}c_{2j} = Y_j$, where $Y_A \equiv p_{1A}(\frac{w}{2} + z^A) + p_{2A}w$, and $Y_B \equiv p_{1B}(\frac{3w}{2} + z^B) + p_{2B}w$. This indirect utility function is seen to equal $2\log Y_j - \log(2p_{1j}) - \log(2p_{2j})$. Hence consumer 1's asset portfolio is selected to maximize $[\log Y_A + \log Y_B]$ subject to the constraint $qz^A + z^B = 0$, where q denotes the relative price of the security corresponding to state A. This reduces to the unconstrained maximization of $\log[p_{1A}(\frac{w}{2} + z^A) + p_{2A}w] + \log[p_{1B}(\frac{3w}{2} - qz^A) + p_{2B}w]$. Hence the demand for z^A by consumer 1 is given by

$$z^{1A} = \frac{w}{2}[\frac{3}{2q} - \frac{1}{2} + \frac{p_{2B}}{qp_{1B}} - \frac{p_{2A}}{p_{1A}}]$$

with the corresponding demand for the other asset satisfying $z^{1B} = -qz^{1A}$.

A similar calculation yields the optimal asset demand for consumer 2:

$$z^{2A} = \frac{w}{2}[\frac{1}{2q} - \frac{3}{2} + \frac{p_{2B}}{qp_{1B}} - \frac{p_{2A}}{p_{1A}}]$$

(d) Given a set of anticipated *ex post* commodity spot prices, calculate security prices that clear the securities market. Finally, derive a Radner equilibrium for this economy. Is the resulting allocation the unique Radner equilibrium allocation?

SOLUTION: Given anticipated spot prices p_{ij}, the equilibrium asset price is obtained by the condition that the security markets clear, i.e., $z^{1A} + z^{2A} = 0$:

$$\frac{3}{2q} - \frac{1}{2} + 2[\frac{p_{2B}}{qp_{1B}} - \frac{p_{2A}}{p_{1A}}] + \frac{1}{2q} - \frac{3}{2} = 0$$

which yields

$$q = [1 + \frac{p_{2B}}{p_{1B}}][1 + \frac{p_{2A}}{p_{1A}}]^{-1}.$$

A Radner equilibrium requires in addition that the anticipated spot prices do clear the commodity markets, given the asset portfolios selected by the consumers at the above asset price. Note that if we consider the following spot commodity prices $p_{ij} = 1$, all i, j, then the corresponding equilibrium asset price is $q = 1$, in which case $z^{1A} = \frac{w}{2} = z^{2B}, z^{2A} = -\frac{w}{2} = z^{1B}$. Then both consumers end up with identical endowments in both states as a result of their security holdings, and it is easily checked that the spot prices $p_{ij} = 1$ do subsequently clear the commodity markets. So this is a Radner equilibrium.

This must be the unique Radner equilibrium allocation, since we know that the set of Radner equilibrium allocations is identical to the set of competitive equilibrium allocations in the Arrow Debreu economy with a complete set of contingent commodity markets, and we know from the solution to part (b) of Problem 5.2 that the latter has a unique competitive equilibrium.

2. **NOTE: Parts (a)–(c) help you review key concepts of the theory. Parts (d) and (e) provide an alternative calculus-based proof of the theorem concerning**

equilibria of the Arrow securities economy that will be discussed in class on Thursday. If you are running out of time you can postpone submission of parts (d) and (e) to the following week.

An exchange economy has L physical goods $l = 1, \ldots, L$, S states of nature $s = 1, \ldots, S$, and I households $i = 1, \ldots, I$. Household i has endowment vector ω_{is} in state s, beliefs π_{is} over states of nature, and von-Neumann Morgenstern utility function $u_{is}(x_{is})$ which is differentiable and concave.

(a) Provide a clear and brief definition of an *ex ante* Pareto optimal allocation for this economy.

SOLUTION: An allocation is *ex ante* Pareto optimal if for some set of nonnegative welfare weights λ_i it maximizes $\Sigma_i \lambda_i \Sigma_s \pi_{is} u_{is}(x_{is})$ subject to the resource constraints $\Sigma_i x_{ils} \leq \Sigma_i \omega_{ils}$ for every pair l, s. Or equivalently it is not feasible to make some individual(s) better off *ex ante* without making someone else worse off.

(b) Derive first order conditions that characterize an *ex ante* Pareto optimal allocation.

SOLUTION: The first order conditions for the maximization problem in (a):

$$\lambda_i \pi_{is} \frac{\partial u_{is}(x_{is})}{\partial x_{ils}} = \delta_{ls} \tag{4a}$$

where δ_{ls} denotes the multiplier for the resource constraint for good l in state s.

(c) Now suppose there is a complete set of Arrow securities which are traded before the state of nature is revealed. Provide a clear and concise definition of a Radner equilibrium for this economy.

SOLUTION: It is a consumption allocation x_{ils}, an asset portfolio for every household z_{is}, and security prices q_s, spot commodity prices p_s such that: (a)every household i is maximizing $\Sigma_s \pi_{is} u_{is}(x_{is})$ subject to $x_{ils} \geq 0$, and the sequence of budget constraints $p_s(x_{is} - \omega_{is}) \leq p_{1s} z_{is}$ and $\Sigma_s q_s z_{is} \leq 0$; and (b) security and commodity markets clear: $\Sigma_i [x_{is} - \omega_{is}] \leq 0, \Sigma_i z_{is} \leq 0$.

(d) Formulate household i's problem of selecting an optimal asset portfolio, using the *ex post* indirect utility function $V_{is}(p_s, W_{is})$ that corresponds to the direct utility function u_{is}, where p_s denotes the spot commodity prices, and W_{is} the financial wealth of the household in state s resulting from the chosen portfolio in state s.

SOLUTION: Maximize $\Sigma_s \pi_{is} V_{is}(p_s, z_{is})$ subject to $\Sigma_s q_s z_{is} \leq 0$, upon normalizing $p_{1s} = 1$.

(e) Use the first order conditions from the portfolio choice problem of households to obtain a direct proof that the Radner equilibrium of the economy is *ex ante* Pareto optimal.

SOLUTION: The first order condition for the optimal security portfolio is

$$\pi_{is} \frac{\partial V_{is}}{\partial W_{is}} = \theta_i q_s \tag{4e}$$

On state s spot commodity markets, household i seelcts x_{is} to maximize $u_{is}(x_{is})$ subject to the budget constraint $p_s x_{is} \leq p_s \omega_{is} + W_{is}$, implying that $\frac{\partial u_{is}}{\partial x_{ils}} = \frac{\partial V_{is}}{\partial W_{is}} p_{ls}$. Hence (4e) can be rewritten as

$$\frac{\pi_{is}}{\theta_i} \frac{\partial u_{is}}{\partial x_{ils}} = p_{ls} q_s$$

which is exactly the set of first order conditions obtained in (b) above that characterize an *ex ante* Pareto optimum.