Ec 703 Microeconomic Theory (Spring 2004)
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SOLUTIONS TO Ec703 FINAL EXAMINATION, May 3, 2004

1. (15 marks) An economy has two states $s = H, L$ that are equally likely ex ante, two traders $i = 1, 2$ and two goods $x, y$. Both traders share common state-dependent preferences: $u_H(x, y) = \frac{1}{3} \log x + \frac{2}{3} \log y; u_L(x, y) = \frac{2}{3} \log x + \frac{1}{3} \log y$. Their endowments are as follows: Trader 2 has one unit endowment of each good in both states, and so does trader 1 in state $H$. In state $L$ trader 1’s endowment vector is $(7, 1)$. Trader 1 is fully informed (knows which state is realized), while trader 2 is uninformed.

Find the set of rational expectations equilibria for this economy (i.e., compute the equilibrium prices and trades if such equilibria exist; otherwise prove that no such equilibria exist).

First lets look for a fully revealing REE, in which the price of good $x$ satisfies $p_H \neq p_L$, so trader 2 learns the state from observing the price. Each trader’s demand for good $y$ in state $H$ is $\frac{2}{3}(1 + p_H)$, so market clearing requires $\frac{4}{3}(1 + p_H) = 2$, or $p_H = \frac{1}{2}$. In state $L$, trader 1’s demand for $y$ is $\frac{1}{3}(7p_L + 1)$, while trader 2’s demand is $\frac{1}{3}(p_L + 1)$. So then market clearing implies $\frac{8}{3}p_L + \frac{2}{3} = 2$, or $p_L = \frac{1}{2}$, which is equal to $p_H$. So a revealing REE does not exist.

Next suppose there is a nonrevealing REE, where $p_H = p_L = p$. In this REE trader 2 will not learn the state, so 2’s demand for $y$ will be $\frac{1}{2}(p + 1)$ in both states. Trader 1’s demand for $y$ will be $\frac{2}{3}(p + 1)$ in state $H$, and $\frac{1}{3}(7p + 1)$ in state $L$. Market clearing requires $p_H = \frac{5}{7}$ and $p_L = \frac{7}{17}$, so $p_H \neq p_L$. So there is no REE in this economy.
2. (5+10+5 = 20 marks) A firm sells a product which is either high (H) or low (L) quality. One unit of quality \(i\) product costs \(c_i\) to produce, and is valued at \(v_i\) by a consumer, where \(v_H > v_L > c_H > c_L\). The consumer can consume at most one unit of the product. The firm knows its quality, while the consumer does not, and believes the quality is high with probability \(\lambda \in (0, 1)\). The firm and consumer obtain a zero payoff if there is no trade. If there is trade of a quality \(i\) good at price \(p\) then the firm’s payoff is \(p - c_i\) and the consumer’s payoff is \(v_i - p\).

The government regulates the price of the product, and has set it at a value of \(p\) satisfying \(v_H > p > v_L\). The firm can however spend \(a\) on advertising its product, and this expenditure is observed by the consumer before deciding whether to purchase the product.

(a) Carefully define a Perfect Bayesian Equilibrium (in pure strategies) for this economy.

A PBE is a tuple of strategies \(a_H, a_L\) for firm types, a strategy \(\sigma(a) : \mathbb{R}_+ \rightarrow \{1, 0\}\) for the customer specifying whether or not to buy the product after observing \(a\), and beliefs of the customer \(\beta(a) : \mathbb{R}_+ \rightarrow [0, 1]\) that the firm has high quality. It satisfies the following conditions:

(i) sequential rationality of strategies given beliefs: firm type \(i\) selects \(a_i \geq 0\) to maximize \(\sigma(a)[p - c_i] - a\), and the customer selects \(\sigma(a)\) to maximize \(\sigma[\beta(a)v_H + (1 - \beta(a))v_L - p]\) over \(\{0, 1\}\).

(ii) consistency of beliefs with strategies: if \(a_H \neq a_L\) then \(\beta(a_H) = 1, \beta(a_L) = 0\), and if \(a_H = a_L = a^*\) then \(\beta(a^*) = \lambda\).

(b) Does there exist a separating PBE in which the firms signal their quality to the consumer? If yes, construct the equilibrium. If no, provide a proof.

If there is a separating PBE, then \(a_H \neq a_L, \beta(a_H) = 1, \beta(a_L) = 0\). Sequential rationality of the customer’s strategy and \(v_H > p > v_L\) implies that \(\sigma(a_H) = 1, \sigma(a_L) = 0\). It follows that the low quality firm must select \(a_L = 0\), since it does not sell its product (if \(a_L > 0\) its profits would be negative). So the low quality firm earns zero profit.
On the other hand, the high quality firm sells its product, so must earn nonnegative profit: \( p - c_H - a_H \geq 0 \). The low quality firm can deviate to \( a_H \), in which case it will succeed in selling its product (given the customer’s strategy), and will earn profit \( p - c_L - a_H > p - c_H - a_H \geq 0 \). So the low quality firm would have a profitable deviation. This shows there cannot be a separating PBE.

\((c)\) Does there exist a pooling PBE? If yes, construct such an equilibrium, otherwise prove it does not exist. Make explicit all assumptions underlying your reasoning.

Consider the case where \( \lambda v_H + (1 - \lambda)v_L \geq p \). Then consider any \( a^* \in [0, p - c_H] \). There is a pooled PBE with \( a_H = a_L = a^* \), \( \sigma(a^*) = 1 \), and \( \beta(a) = \lambda \) for all \( a \geq a^* \), \( \sigma(a) = 0, \beta(a) = 0 \) for all \( a < a^* \). In the opposite case where \( \lambda v_H + (1 - \lambda)v_L < p \) there cannot be any pooled PBE where there is trade, implying there cannot be any advertising either. So there is a unique pooled PBE outcome with \( a_H = a_L = 0, \beta(a) = \lambda, \sigma(a) = 0 \) for all \( a \).

3. \((5 + 10 = 15\text{ marks})\) A risk-neutral principal \( P \) hires a risk-averse agent \( A \) to carry out a task, which can result either in a gross profit of \( \pi_H \) or \( \pi_L \) for \( P \), where \( \pi_H > \pi_L > 0 \). The probability of the outcome \( \pi_H \) depends on the effort \( e \) of \( A \), which can be either high (\( e_H \)) or low (\( e_L \)). This is denoted by the function \( p(e) \), with \( p(e_H) > p(e_L) > 0 \). The agent’s disutility of effort is represented by the function \( g(e) \), where \( g(e_H) > g(e_L) > 0 \). The agent is infinitely risk-averse with respect to lotteries over wage income, i.e., evaluates any given lottery \( \{w_i; q_i\}_{i=1,...,m} \) (where wage income is \( w_i \) with probability \( q_i \), following choice of effort \( e \) ) by the function \( W(\{w_i; q_i\}; e) = \min_{e_i; q_i} \{ w_i \} - g(e) \). Finally, payments \( w \) that can be made to \( A \) are constrained to be nonnegative (\( w \geq 0 \)) and the reservation utility of \( A \) is 0.

\((a)\) Describe the first-best contract that \( P \) will offer \( A \) when effort \( e \) is observed by the principal and the wage paid to \( A \) can be conditioned on \( e \).

In the first-best, if \( P \) wants to implement \( e_H \), he will have to pay \( w = g(e_H) \) conditional on \( A \) selecting \( e_H \) and 0 otherwise. In this case \( P \) earns a payoff of \( p(e_H)\pi_H + [1 - p(e_H)]\pi_L - g(e_H) \). If \( P \) implements \( e_L \) he will pay \( w = g(e_L) \) and earn a payoff of \( p(e_L)\pi_H + [1 -
\[ p(e_L)\pi_L - g(e_L). \] The choice between implementing \( e_H \) and \( e_L \) will depend on whether \( p(e_H)\pi_H + [1 - p(e_H)]\pi_L - g(e_H) \) is bigger than \( p(e_L)\pi_H + [1 - p(e_L)]\pi_L - g(e_L). \)

(b) Now suppose \( e \) cannot be observed by \( P \). Derive the exact conditions under which the second-best contract differs from the first-best contract. Be explicit about any assumptions you need to make.

If the first-best contract involves \( e_L \) then it obviously remains feasible in the second best. So consider the case where the first-best involves \( e_H \). If \( p(e_H) = 1 \), then the first-best can be implemented with the following contract: the agent is paid \( w = g(e_H) \) only if the outcome \( \pi_H \) is realized, and 0 otherwise. This is incentive compatible and satisfies the participation constraint because the payoff of \( A \) under \( e_H \) is \( g(e_H) - g(e_H) = 0 \), whereas under \( e_L \) is \( \min\{g(e_H), 0\} - g(e_L) = -g(e_L) < 0 \).

On the other hand if \( p(e_H) < 1 \), then the first-best cannot be implemented. Then both outcomes \( \pi_L \) and \( \pi_H \) arise with positive probability under either effort level. If \( A \) gets paid \( w_i \) following outcome \( \pi_i, i = H, L \) then either effort level gives rise to a lottery which is evaluated by \( A \) in the same way \( (W = \min_i w_i) \), so \( A \) will have no incentive to select \( e_H \). Therefore the second-best contract in this case will involve \( e_L \), and \( P \) will pay \( w_i = g(e_L), i = H, L \). The expected profit of \( P \) will therefore be less than in the first-best if and only if the first-best involves \( e_H \) and \( p(e_H) < 1 \).