1. Consider an economy with \( I \) households and \( J \) firms. Each firm has a strictly convex and compact production set. Each household has (i) a given ownership share \( \theta_{ij} \) in firm \( j \), (ii) strictly monotone preferences, (iii) an excess demand function \( Z^i(p) \) defined for all \( p >> 0 \), incorporating incomes resulting from \( i \)'s share in different firms' profits. \( Z^i(.) \) is continuous, homogeneous of degree zero, bounded from below and unbounded above (i.e., with respect to the maximum excess demand across all commodities, as some price tends to zero). Show that the economy has at least one competitive equilibrium.

Firm \( j \)'s profit maximization problem is \( \text{Max } p.y \text{ subject to } y \in Y^j \), where \( Y^j \) is strictly convex and compact. This maximization problem has a unique solution \( y^j(p) \), defining a supply function for firm \( j \). The Theorem of the Maximum applies to the maximization problem, as the objective function is continuous in \( p \), and the feasible set is independent of \( p \). So the set of solutions to the maximization problem forms a upper hemicontinuous correspondence in \( p \). Since the solution is unique, it follows that \( y^j(p) \) is continuous.

Define the excess demand function for the economy as a whole:

\[ E(p) = \sum_i Z^i(p) - \sum_j y^j(p), \]

which is clearly continuous. We check the other conditions on \( E(p) \) that guarantee existence of a competitive equilibrium:

(ii) Walras Law: \( p.E(p) = \sum_i p.Z^i(p) - \sum_j p y^j(p) = 0 \) because \( p.Z^i(p) = \sum_j \theta_{ij} y^j(p) \) and summing over households we get \( \sum_i p.Z^i(p) = \sum_j p y^j(p) \).

(iii) Homogeneity of degree zero: This is straightforward, as \( Z^i(p) \) and \( y^j(p) \) are homogenous of degree zero.

(iv) Bounded below: This follows since \( Z^i(p) \) is bounded below, and so is \( y^j(p) \), as firm \( j \)'s production set is bounded.

(v) Unbounded above: If \( p_n \to p \) where some prices are zero at \( p \), then \( \text{Max}_i Z^i(p_n) \to \infty \), while \( y^j(p_n) \) is bounded. So \( \text{Max}_i E(p_n) \to \infty \).

It follows that \( E(p) \) is an excess demand function defined over all strictly positive price vectors, satisfying assumptions (i)-(v), so a competitive equilibrium exists.
2. Consider an exchange economy with $L$ commodities where for every household $i$ there is a good $l(i)$ on which household $i$ spends all its income. Under what additional conditions will competitive equilibrium in this economy be unique?

Household $i$’s excess demand for commodity $l$ is $Z_{il}(p) = \frac{1}{p_l}p_i \omega_i - \omega_{il}$ if $l = l(i)$, and $-\omega_{il}$ otherwise. So if $l \neq m$:

$$\frac{\partial Z_{il}(p)}{\partial p_m} = \omega_{im} \geq 0$$

if $l = l(i)$ and 0 otherwise. If it is the case that for every commodity $l$ there exists at least one household $i$ such that $l = l(i)$ and $\omega_{im} > 0$ for all $m \neq l$, then the aggregate excess demand function will exhibit the gross substitute property:

$$\frac{\partial Z_{il}(p)}{\partial p_m} \geq \frac{\partial Z_{il}(p)}{\partial p_m} = \omega_{im} > 0,$$

whenever $l \neq m$. This implies uniqueness of competitive equilibrium.

3. An exchange economy has two dates $t = 0, 1$ and two states of nature $s = 1, 2$ which will be revealed at date 1. Use $s = 0$ to denote the date-event pair corresponding to date 0. There is one physical commodity, and two consumers $i = 1, 2$ whose endowments $\omega_is$ are as follows: $\omega_{10} = 2, \omega_{11} = 4, \omega_{12} = 3, \omega_{20} = 4, \omega_{21} = 2, \omega_{22} = 3$. Both share the von-Neumann-Morgenstern utility $\log c_0 + \log c_1$, where $c_t$ denotes date $t$ consumption. Consumer 1 believes $s = 1$ with probability $\frac{3}{4}$, while consumer 2 believes $s = 1$ with probability $\frac{1}{4}$. At date 0, consumers trade in the commodity, besides two assets $k = 1, 2$ whose date-1 returns $r_{sk}$ are given by $r_{11} = 1, r_{12} = 2, r_{21} = 0, r_{22} = 1$.

At date 1, spot commodity markets open.

(a) Derive the entire set of ex ante Pareto optimal allocations in this economy. Are these allocations ex post Pareto optimal as well?

An allocation in this economy is represented by consumer 1’s consumption allocation $x_{1s}, s = 0, 1, 2$, since this determines consumer 2’s consumption allocation $x_{2s} = 6 - x_{1s}$, as the aggregate endowment of the economy is 6 for every $s$. An ex ante Pareto optimal allocation maximizes

$$\log x_{10} + \frac{3}{4} \log x_{11} + \frac{1}{4} \log x_{12} + \lambda \log(6 - x_{10}) + \frac{1}{4} \log(6 - x_{11}) + \frac{3}{4} \log(6 - x_{12})$$

where $\lambda > 0$ is the relative welfare weight on consumer 2. We obtain the following allocation corresponding to $\lambda$:

$$x_{10} = \frac{6}{1 + \lambda}, x_{11} = \frac{18}{3 + \lambda}, x_{12} = \frac{6}{1 + 3\lambda}.$$

Varying $\lambda$, we obtain the class of ex ante Pareto optimal allocations for this economy.

These allocation are all ex post optimal as well, since there is a single physical commodity.
(b) Describe carefully the optimization problem defining the optimal asset demands of the two consumers at date 0 (You need not derive the asset demand functions: show the objective function and the budget constraints.)

Date 0 markets will involve spot-trading in the commodity (price $p_0$), and trading in the two assets (prices denoted $q_1, q_2$). At date 1, the state of the world is revealed, and as there is a single physical good, there is no scope for trades between the two consumers. We can select the commodity as numeraire at date 0, so $p_0 = 1$ without loss of generality.

Consumer 1’s budget constraint at date 0 is then

$$x_{10} + q_1 z_{11} + q_2 z_{12} \leq 2 \quad (1)$$

and at date 1 is

$$x_{11} = 4 + z_{11}, x_{12} = 3 + 2z_{11} + z_{21} \quad (2)$$

So consumer 1 maximizes expected utility $\log x_{10} + \frac{3}{4}\log x_{11} + \frac{1}{4}\log x_{12}$ subject to constraints (1) and (2).

Consumer 2’s budget constraint at date 0 is

$$x_{20} + q_1 z_{21} + q_2 z_{22} \leq 4 \quad (3)$$

and at date 1 is

$$x_{21} = 2 + z_{21}, x_{22} = 3 + 2z_{21} + z_{22} \quad (4)$$

So consumer 1 maximizes expected utility $\log x_{20} + \frac{1}{4}\log x_{21} + \frac{3}{4}\log x_{22}$ subject to constraints (3) and (4).

(c) What can you say about existence and Pareto optimality of Radner equilibria in this economy?

Since the returns of the two assets are linearly independent (rank of the return matrix is 2), the set of Radner equilibria of this economy is the same as the set of equilibria of an Arrow securities economy, which in turn is the same as the set of Arrow-Debreu equilibria. Since the Arrow-Debreu economy has two consumers with interior endowments, strictly convex and monotone preferences (with respect to contingent commodities), it has at least one Arrow-Debreu equilibrium. Moreover, the two welfare theorems apply: all equilibria are ex ante Pareto optimal, and all optimal allocations can be achieved as (Arrow-Debreu, hence Radner) equilibria with redistribution of endowments.