1. Consider an exchange economy with $L$ commodities and $I$ households, where household $i$ has consumption set $R^L_+$, a nonnegative endowment vector, and a utility function $U_i(x_{i1}, x_{i2}, \ldots) = \min\{\frac{x_{i1}}{\lambda_{i1}}, \frac{x_{i2}}{\lambda_{i2}}, \ldots\}$, with $\lambda_{il} > 0$ for all $i, l$. Provide a complete proof that this economy has a competitive equilibrium.

2. (a) Suppose that the aggregate excess demand function of households is well-defined, and does not satisfy the Weak Axiom of Revealed Preference. Describe conditions on the technology of firms that ensures that competitive equilibrium allocations are not unique.

   (b) Consider an exchange economy with one physical good, $S$ states of nature, and $I$ households, where all households share the same beliefs, and the same state independent von-Neumann Morgenstern utility function $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 0$ and different from 1. The aggregate endowment of the economy (i.e., the endowment of the single good aggregated across all households) depends on the state of nature. Show that in any ex ante Pareto optimal allocation, the consumption of any individual $i$ in state $s$ is proportional to per capita consumption $\bar{c}_s$ in the economy, i.e., takes the form $c_{is} = \lambda_i \bar{c}_s$, $s = 1, \ldots, S$ for some set of multipliers $\lambda_i$ that are independent of $s$. 

Answer any three questions. You have three hours. Answers should be complete, explaining all steps in the reasoning, while omitting irrelevant details.
3. Consider an exchange economy with one consumption good, $S$ states of nature, and $I$ households, where household $i$ has endowment $\omega_{is} > 0$ in state $s$, beliefs $\pi_{is}$ over states of nature, and von-Neumann Morgenstern utility function $u_{is}$ which is strictly increasing, strictly concave, twice differentiable, and satisfies $u'_{is}(0) = +\infty$. If in addition the elasticity of marginal utility of consumption $u'_{is}$ in every state $s$ is less than 1, show that this economy has a unique Arrow Debreu equilibrium.

4. An exchange economy has $L$ physical goods $l = 1, \ldots, L$, $S$ states of nature $s = 1, \ldots, S$, and $I$ households $i = 1, \ldots, I$. Household $i$ has endowment vector $\omega_{is}$ in state $s$, beliefs $\pi_{is}$ over states of nature, and von-Neumann Morgenstern utility function $u_{is}(x_{is})$ which is strictly increasing, differentiable and concave.

(a) Define an $ex$ $ante$ Pareto optimal allocation for this economy.

(b) Derive conditions that characterize an $ex$ $ante$ Pareto optimal allocation.

(c) Now suppose there is a complete set of Arrow securities which are traded before the state of nature is revealed. Define a Radner equilibrium for this economy.

(d) Formulate household $i$’s problem of selecting an optimal asset portfolio, using the $ex$ $post$ indirect utility function $V_{is}(p_s, W_{is})$ that corresponds to the direct utility function $u_{is}$, where $p_s$ denotes the spot commodity prices, and $W_{is}$ the financial wealth of the household in state $s$ resulting from the chosen portfolio in state $s$.

(e) Use the first order conditions from the portfolio choice problem of households to obtain a direct proof that the Radner equilibrium of the economy is $ex$ $ante$ Pareto optimal. (Hint: show that they imply the conditions characterizing Pareto optimal allocations that you derived in (b) above.)