Time Series
Why Time is Important

- Explicit in many environmental models
  - Can generate complex/chaotic feedbacks
  - External/environmental factors change over time

- Measurements often made repeatedly over time
  - Data usually correlated in time
  - Response to treatments

- Importance of separating process and measurement error
  - Measurement error does not propagate
Characteristics of Time Series Data

- Single/small number of long time series
  - Often concerned with identifying trends, periodicity, autocorrelation, cross-correlation, etc.
- Longitudinal / repeated measures
  - Many short time series
  - Intervention analysis
  - Mark-recapture (boolean data)
Dynamic process models

\[ x_t = f(x_{t-1}, x_{t-2}, \ldots, \theta) \]

- Recursive: state at current time point a function of the previous state
- Any model that depends only on the most recent state \((x_{t-1})\) is called a Markov model
- Higher order models (additional lags) introduce memory to the system
Random Walk

\[ X_t = X_{t-1} + \epsilon_t \]

- Mean = \( X_0 \)
- Var = \( t \sigma^2 \)
Random Walk

\( X_t = X_{t-1} + \epsilon_t \)

- Mean = 0
- Var = t \( \sigma^2 \)

Approaches

→ Random effects?
→ Autocorrelation?
→ State space?
Bayesian State Space Model

\[
X_t = f(X_{t-1}) + \epsilon_t \quad \text{Process Model}
\]

\[
Y_t = g(X_t) + \omega_t \quad \text{Data Model}
\]

- \( X \) = latent time series
- \( Y \) = observed data
- \( \epsilon \) = process error
- \( \omega \) = observation error
Random Walk State Space Model

Y's are conditionally independent given the X's.
Random Walk State Space Model

\[ X_t \sim N(X_{t-1}, \sigma^2) \]
\[ Y_t \sim N(X_t, \tau^2) \]
\[ \sigma^2 \sim IG(s1, s2) \]
\[ \tau^2 \sim IG(t1, t2) \]
\[ X_0 \sim N(X_{ic}, V_X) \]

Process Model
Data Model
Process Error prior
Observation Error prior
Initial Condition prior

• What are the parameters?
• What is the joint (full) posterior?
• What are the conditional distributions for each parameter?
Random Walk State Space Model

\[ X_t \sim N(X_{t-1}, \sigma^2) \]
\[ Y_t \sim N(X_t, \tau^2) \]
\[ \sigma^2 \sim IG(s1, s2) \]
\[ \tau^2 \sim IG(t1, t2) \]
\[ X_0 \sim N(X_{ic}, V_X) \]

- What are the parameters?
  - \(X's, \sigma^2, \tau^2\)
Random Walk State Space Model

• What is the joint (full) posterior?

\[
\begin{align*}
X_t &\sim N(X_{t-1}, \sigma^2) \\
Y_t &\sim N(X_t, \tau^2) \\
\sigma^2 &\sim IG(s1, s2) \\
\tau^2 &\sim IG(t1, t2) \\
X_0 &\sim N(X_{ic}, V_X)
\end{align*}
\]
Random Walk State Space Model

- What is the joint (full) posterior?

\[
p(\tilde{X}, \sigma^2, \tau^2 | \tilde{Y}, \ldots) \sim \\
\prod_{t=1}^{n} N(Y_t | X_t, \tau^2) \times \\
\prod_{t=1}^{n} N(X_t | X_{t-1}, \sigma^2) \times \\
IG(\tau^2 | t_1, t_2) \times IG(\sigma^2 | s_1, s_2) \times N(X_0 | X_{ic}, V_X)
\]
Random Walk State Space Model

- What are the conditional distributions?

\[
\begin{align*}
    X_t & \sim N(X_{t-1}, \sigma^2) \\
    Y_t & \sim N(X_t, \tau^2) \\
    \sigma^2 & \sim IG(s_1, s_2) \\
    \tau^2 & \sim IG(t_1, t_2) \\
    X_0 & \sim N(X_{ic}, V_X)
\end{align*}
\]
Random Walk State Space Model

• What are the conditional distributions?

\[ \sigma^2 \sim IG(s1, s2) \times \prod_{t=1}^{n} N(X_t | X_{t-1}, \sigma^2) \]

\[ \tau^2 \sim IG(\tau^2 | t1, t2) \times \prod_{t=1}^{n} N(Y_t | X_t, \tau^2) \]

\[ X_t \sim N(X_{t-1}, \sigma^2) \]

\[ Y_t \sim N(X_t, \tau^2) \]

\[ \sigma^2 \sim IG(s1, s2) \]

\[ \tau^2 \sim IG(t1, t2) \]

\[ X_0 \sim N(X_{ic}, V_x) \]
Random Walk State Space Model

- What are the conditional distributions?

\[
X_t \sim N(X_{t-1}, \sigma^2) \times N(X_t, \sigma^2) \times N(Y_t, \tau^2)
\]

- Three special cases
  - First
  - Last
  - Missing Y

\[
X_t \sim N(X_{t-1}, \sigma^2) \\
Y_t \sim N(X_t, \tau^2) \\
\sigma^2 \sim IG(s1, s2) \\
\tau^2 \sim IG(t1, t2) \\
X_0 \sim N(X_{ic}, V_X)
\]
Random Walk State Space Model

- First

\[
X_0 \sim N(X_0 \mid X_{ic}, V_X) \times N(X_1 \mid X_0, \sigma^2)
\]

- Last

\[
X_n \sim N(X_n \mid X_{n-1}, \sigma^2) \times N(Y_n \mid X_n, \tau^2)
\]

- Missing Y

\[
X_t \sim N(X_t \mid X_{t-1}, \sigma^2) \times N(X_{t+1} \mid X_t, \sigma^2)
\]

\[
X_t \sim N(X_{t-1}, \sigma^2)
\]
\[
Y_t \sim N(X_t, \tau^2)
\]
\[
\sigma^2 \sim IG(s1, s2)
\]
\[
\tau^2 \sim IG(t1, t2)
\]
\[
X_0 \sim N(X_{ic}, V_X)
\]
How do we sample?

- All amenable to Gibbs sampling (NxB, NxIG)
- X's need to be updated sequentially because each X is conditioned on the one BEFORE it and the one AFTER it

\[
X_t^{(g+1)} \sim N \left( X_t | X_{t-1}^{(g+1)}, \sigma^2 \right) \times \\
N \left( X_{t+1}^{(g)} | X_t, \sigma^2 \right) \times \\
N \left( Y_t | X_t, \tau^2 \right)
\]
Bayesian State Space Model

\[ X_t = f(X_{t-1}) + \varepsilon_t \]  \hspace{1cm} \text{Process Model}

\[ Y_t = g(X_t) + \omega_t \]  \hspace{1cm} \text{Data Model}

- \( Y \) = observed data
- \( X \) = \textit{latent} time series
- \( \varepsilon \) = process error
- \( \omega \) = observation error
Population growth

\[ \frac{dN}{dt} = rN \]

Exponential Growth

\[ N_t = N_0 e^{rt} \]

Let \( X = \log(N) \)

\[ X_t = X_{t-1} + r + \epsilon_t \quad \text{Discrete time recursion} \]

\[ Y_t = X_t + \omega_t \quad \text{Observation error model} \]

- \( r = \) intrinsic growth rate
- \( Y = \log(\text{observation}) \)
- Can interpret \( \epsilon \) as a random effect on \( r \)
Exponential Growth State Space

Data Model

Process Model

Parameter Model

Y's are conditionally independent given the X's
Example

- \( r = 0.1 \)
- Process s.d. varied from 0.01 to 1.0
- Run 1 = dominated by process model
- Run 2 = process model and process error similar
- Run 3 = dominated by process error
  - Env or wrong model
\[ \text{sigma}_2 = 0.3 \]

\[ \text{tau}_2 = 0.05 \]
Missing Data

![Graph showing missing data observations](image)

- **true**
- **obs**
- **fit**

**Missing Observations**
Generality of the State Space Model

- Neither X nor Y need be Normal
- X and Y don't need to be the same type of data
- X and Y don't need to have the same time scale
- Easily handles missing data (gaps) and irregularly spaced data
- Easily handles multiple data sources (Y's), which don't need to be the same type or synchronous
- Easily handles time-integrated observations