Bayes' Theorem

Rev. Thomas Bayes
1702-1761
Conditional Probability

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\Pr(A \mid B) = \frac{\Pr(A,B)}{\Pr(B)}
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**BAYES RULE**
False Positives

• If a patient has a disease the test returns a positive 99% of the time
• If a patient does not have the disease, the test returns positive 5% of the time
• 0.1% of the population has the disease
• What is the probability that someone who tested positive has the disease?
Suppose $A = \text{has disease}$  
$B = \text{tested positive}$

\[
P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\text{not } A)P(\text{not } A)}
\]

\[
P(A|B) = \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.05 \cdot 0.999}
\]

\[
P(A|B) \approx \frac{0.00099}{0.00099 + 0.04995} \approx 0.019
\]
What is the probability that the sun exploded??
Bayes' Theorem

\[ P(\theta | y) = \frac{P(y | \theta) P(\theta)}{P(y)} \]

\[ = \frac{\int_{-\infty}^{\infty} P(y | \theta) P(\theta) d\theta}{P(y)} \]
Bayes' Billiard Table

\[ P(\theta) = \text{Unif} \]

\[ P(\theta|y) = ?? \]
\[ P(\theta|y) \propto P(y|\theta) P(\theta) \]

Unif(0,1)

What is \( P(y | \theta) \)?

\[ L = P(y|\theta) = \text{Binom}(y|n, \theta) \]
\[
P(\theta|y) = \frac{\int_0^1 \text{Binom}(y|n, \theta) \text{Unif}(\theta|0,1)}{1} \]

\[
P(\theta|y) = \frac{\binom{n}{y} \theta^y (1-\theta)^{n-y} \cdot 1}{\int_0^1 \binom{n}{y} \theta^y (1-\theta)^{n-y} \cdot 1}
\]
\[ P(\theta|y) = \frac{\theta^y (1-\theta)^{n-y}}{\int_0^1 \theta^y (1-\theta)^{n-y}} \]

\[ Beta(x|\alpha, \beta) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{\int_0^1 x^{\alpha-1} (1-x)^{\beta-1}} \]

\[ P(\theta|y) = Beta(x|y+1, n-y+1) \]
• Posterior is a PDF
• $\theta$ is a random variable
• Interested in full distribution
Data updates the prior

\[ n=0 \]
Priors

• Makes it possible to calculate a posterior density of the model parameter rather than the likelihood of the data

• Provides a way of incorporating information that is external to the data set(s) at hand

• Inherently sequential

    Previous Posterior = New Prior
Where do Priors come from?

- Uninformative / vague
  - Chosen to have minimal information content, allows the likelihood to dominate the analysis

- Previous analyses
  - Must be equivalent
  - Variance inflation

- “The literature”
  - Meta-analysis

- Expert knowledge
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Prior specification must be “blind” to the data in the analysis!!

No “double dipping” -- leads to falsely overconfident results
How do I choose a prior PDF?

- Analogous to how we choose the data model
  - Range restrictions, shape, etc.
- Conjugacy
  - A prior is conjugate to the likelihood if the posterior PDF is in the same family as the prior
  - Allow for closed-form analytical solutions to either full posterior or (in multiparameter models) for the conditional distribution of that parameter.
  - Modern computational methods no longer require conjugacy
Example: Tree mortality rate

• Data: observed $n=4$ trees, $y=1$ died this year

$$L = P(y|\theta) = \text{Binom}(y|\theta, n)$$

• Prior: last year observed $n_0=2$ trees, $y_0=1$ died

$$Prior = P(\theta) = \text{Beta}(\theta|y_0, n_0 - y_0)$$
\[ P(\theta | y) \propto \text{Binom}(y | \theta, n) \text{Beta}(\theta | y_0, n_0 - y_0) \]

\[ P(\theta | y) \propto \binom{n}{y} \theta^y (1-\theta)^{n-y} \times \frac{\theta^{y_0-1} (1-\theta)^{n_0-y_0-1}}{\text{B}(y_0, n_0 - y_0)} \]

\[ P(\theta | y) \propto \theta^{y+y_0-1} (1-\theta)^{n-y+n_0-y_0-1} \]

\[ P(\theta | y) = \text{Beta}(\theta | y+y_0, n+n_0 - y - y_0) \]

Beta-Binomial Model
\[ P(\theta|y) = \text{Beta}(\theta|y+y_0, n+n_0-y-y_0) = \text{Beta}(\theta|2,4) \]
How much impact does the prior have on the analysis?