Online Appendices to Capital Structure, Product Market Dynamics, and the Boundaries of the Firm

Dirk Hackbarth[*]   Richmond Mathews[†]   David Robinson[‡]

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[*]Boston University School of Management, 595 Commonwealth Avenue, Boston, MA 02215, USA. Email: dhackbar@bu.edu. Tel: (617) 358-4206.

[†]Robert H. Smith School of Business, University of Maryland, 4426 Van Munching Hall, College Park, MD 20742, USA. Email: rmathews@rhsmith.umd.edu. Tel: (301) 405-4113.

[‡]Fuqua School of Business, Duke University, 100 Fuqua Drive, Durham, NC 27708, USA. Email: davidr@duke.edu. Tel: (919) 660-8023.

Online Appendix A. Derivations of Lemma 1 and Proposition 1

Within the present model, the exponential law holds for all $t \geq 0$:

$$P(Y_t = 0) = e^{-\rho t} \,. \tag{A.1}$$

This implies that at any time t the expected time to obsolescence of the option is given by:

$$E[\mathcal{T}_Y] = \int_t^\infty \rho\,(s-t)\,e^{-\rho\,(s-t)}\,ds = \frac{1}{\rho}\,. \tag{A.2}$$

Consistent with economic intuition, equation (A.2) indicates that the expected time to obsolescence is inversely related to the risk of obsolescence.

Recall that for $t < \mathcal{T}_G$ the small firm invests to maximize the (equity) value of the levered assets obtained from exercise as long as exercise takes place prior to obsolescence (i.e., for $\mathcal{T}_G < \mathcal{T}_Y$):

$$V_S(X) = \sup_{\mathcal{T}_G} E\left[1_{\mathcal{T}_G < \mathcal{T}_Y}\, e^{-r\mathcal{T}_G} \left(V_S^+(X_{\mathcal{T}_G}) - \kappa \right) \right]\,. \tag{A.3}$$

On the other hand, if $\mathcal{T}_G > \mathcal{T}_Y$, then the small firm vanishes: $V_S(X) = 0$.

For $t > \mathcal{T}_G$, the small firm operates assets in place and hence its owners receive capital gains of $E\left[dV_S^+(X)\right]$ and cash flows $(1 - \tau)\,\pi X + \tau C_S^+$ over each time interval dt. The required rate of return for investing in the small firm is the risk-free rate r. Thus, the Bellman equation in the continuation region (i.e., for $t < \mathcal{T}_D$) is:

$$rV_S^+(X)\,dt = E\left[dV_S^+(X)\right] + \left[(1-\tau)\,\pi X + \tau C_S^+\right]dt\,. \tag{A.4}$$

Applying Ito's lemma to expand the right-hand side of the Bellman equation, it is straightforward that the value of the matured, small firm before default (i.e., for $X > \underline{X}_S^+$) satisfies:

$$rV_S^+(X) = (1-\tau)\,\pi X + \tau C_S^+ + \mu X \frac{\partial V_S^+(X)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_S^+(X)}{\partial X^2}\,. \tag{A.5}$$

The ordinary differential equation has a solution of the form:

$$V_S^+(X) = \pi\Lambda X + \frac{\tau C_S^+}{r} + A_1 X^{\vartheta'} + A_2 X^{\xi'}\,, \tag{A.6}$$

where $\vartheta' < 0$ is given in (4) and $\xi' > 1$ is given by: $\xi' = (\frac{1}{2} - \mu/\sigma^2) + \sqrt{(\frac{1}{2} - \mu/\sigma^2)^2 + 2r/\sigma^2}$. The constants A_1 and A_2 are determined by the value-matching conditions of the levered firm:

$$V_S^+(\underline{X}_S^+) = \pi\,(1-\alpha)\Lambda\underline{X}_S^+\,, \tag{A.7}$$

$$\lim_{X\to\infty} V_S^+(X) = \pi\Lambda X + \frac{\tau C_S^+}{r}\,. \tag{A.8}$$

The first condition captures the value of the firm net of bankruptcy costs that will be transferred to the new owners in case of default. The second condition states that firm value is bounded above by the default-risk-free value of assets and tax shields, which implies that $A_2 = 0$. Solving the first equation for the remaining unknown constant A_1 yields firm value after investment, given in the first equation of Lemma 1.

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Similar arguments as in equations (A.3), (A.5), and (A.6) yield equity value after investment, given in the second equation of Lemma 1, if we solve the value-matching conditions for equity:

$$E_S^+(\underline{X}_S^+) = 0 \,, \tag{A.9}$$

$$\lim_{X \to \infty} E_S^+(X) = \pi \Lambda X - \frac{(1-\tau)C_S^+}{r} \,. \tag{A.10}$$

The first condition ensures that equity is worthless in case of default, while the second equation corresponds to a no-bubble condition implying again that $A_2 = 0$. In addition to these value-matching conditions, equity value satisfies an optimality condition. That is, the default threshold that maximizes equity value solves the smooth-pasting condition:

$$\left. \frac{\partial E_S^+(X)}{\partial X} \right|_{X=\underline{X}_S^+} = 0 \,, \tag{A.11}$$

which implies the closed-form solution for $\underline{X}_S^+$ given in Lemma 1. Substituting the result for $\underline{X}_S^+$ into $V_S^+(X)$, the first-order condition of firm value after investment with respect to C_S^+ is given by:

$$\frac{\partial V_S^+(X)}{\partial C_S^+} = 0 \,, \tag{A.12}$$

which can be solved analytically to produce the closed-form solution for C_S^+ given in Lemma 1. This solution is indeed optimal given that it is straightforward to verify that the second-order condition for this optimization problem is negative.

Plugging the expressions for C_S^+ and $\underline{X}_S^+$ into equation (2) and simplifying yields:

$$V_S^+(X) = \pi \Lambda X \left(1 + \frac{\tau}{1-\tau} \left(1 - \vartheta'(1 - \alpha + \alpha/\tau) \right)^{1/\vartheta'} \right) \,, \tag{A.13}$$

which we can substitute into the solution for equation (A.3). In the continuation region of cash flow levels below which investment is optimal (i.e., $X < \overline{X}_S$), standard arguments imply that equity value satisfies the Bellman equation:

$$rV_S(X)\,dt = \mathrm{E}[dV_S(X)] \,. \tag{A.14}$$

Applying Ito's lemma to expand the right-hand side of the Bellman equation yields for $X < \overline{X}_S$:

$$rV_S(X) = \mu X \frac{\partial V_S(X)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_S(X)}{\partial X^2} + \rho\,[0 - V_S(X)] \,, \tag{A.15}$$

which has a solution of the form:

$$V_S(X) = A_3 X^\vartheta + A_4 X^\xi \,, \tag{A.16}$$

where $\vartheta < 0$ and $\xi > 1$ are given in (32) and (15). The unknown constants A_3 and A_4 are determined by the value-matching conditions for the small firm's equity value:

$$\lim_{X \to 0} V_S(X) = 0 \,, \tag{A.17}$$

$$V_S(\overline{X}_S) = V_S^+(\overline{X}_S) - \kappa \,. \tag{A.18}$$

The first condition stipulates that the option ought to be worthless when cash flows become arbitrarily small, implying that $A_3 = 0$. The second is a no-arbitrage condition as it says that, at the time of exercise, equity

value before exercise, V_S, equals firm value after exercise, V_S^+, net of the exercise cost, κ. Solving the second equation for the remaining unknown constant, A_4, and substituting the optimal investment threshold $\overline{X}_S$ yields equity value before investment, given in the first equation of Proposition 1. In addition to the value-matching conditions, equity value satisfies an optimality condition. That is, the optimal exercise threshold solves the smooth-pasting condition:

$$\left. \frac{\partial V_S(X)}{\partial X} \right|_{X=\overline{X}_S} = \left. \frac{\partial V_S^+(X)}{\partial X} \right|_{X=\overline{X}_S} , \tag{A.19}$$

which implies the closed-form solution for $\overline{X}_S$ given in the last equation of Proposition 1. ∎

Online Appendix B. Derivations of Lemma 2, Lemma 3 and Proposition 2

Similar arguments as the ones used for deriving Lemma 1 can be used to derive the closed-form solutions given in Lemma 2 and Lemma 3. In particular, observe that if we replace the variable π, the size of the growth opportunity, in all expressions given in Lemma 1 by the variable $(1 - \gamma)$, the reduced size of assets in place resulting from cannibalization due to option exercise, then we obtain all the expressions given in Lemma 2. Similarly, notice that we can replace the size of the growth opportunity, π, in all expressions given in Lemma 1 by the reduced size of assets in place resulting from the preemption cost, $(1 - \delta)$, to produce closed-form solutions given in Lemma 3.

In the continuation region of cash flow levels below which investment is optimal (i.e., $X < \overline{X}_S$) for the small firm and above which default is optimal (i.e., $X > \underline{X}_L$) for the large firm, standard arguments imply that for all $t < \mathscr{T}_Y$ firm value satisfies:

$$r V_L(X) dt = \mathrm{E}[dV_L(X)] + [(1 - \tau)X + \tau C_L] dt . \tag{B.1}$$

Ito's lemma then says that the ordinary differential equation before investment or obsolescence is:

$$r V_L(X) = (1 - \tau)X + \tau C_L + \mu X \frac{\partial V_L(X)}{\partial X} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 V_L(X)}{\partial X^2} + \rho \left[V_L^\circ(X) - V_L(X)\right] . \tag{B.2}$$

It is easy to guess and verify that this ordinary differential equation has a solution of the form:

$$V_L(X) = \Lambda X + \frac{\tau C_L}{r} + A_5 X^\vartheta + A_6 X^\xi - \frac{\rho}{r + \rho - \mu} \delta \Lambda X \tag{B.3}$$

$$- \frac{\rho}{r + \rho - \vartheta' \mu - \frac{1}{2} \vartheta' (\vartheta' - 1) \sigma^2} \left(\frac{\tau C_L}{r} + \alpha \Lambda \underline{X}_L^\circ \right) \left(\frac{X}{\underline{X}_L^\circ} \right)^{\vartheta'} .$$

The first four terms on the right-hand side of (B.3) correspond to the ones in (A.6). In addition, the fifth and the sixth term capture obsolescence risk in that they reflect the preemption cost and the change in net tax shields, respectively, which enter equation (B.2) via $V_L^\circ(X)$.

The unknown constants A_5 and A_6 are determined by the value-matching conditions:

$$V_L(\underline{X}_L) = (1 - \alpha) \Lambda X_L - \frac{\rho}{r + \rho - \mu} \delta \Lambda \left[\underline{X}_L - \overline{X}_S \left(\frac{X_L}{\overline{X}_S} \right)^\xi \right] - \gamma \Lambda \overline{X}_S \left(\frac{X_L}{\overline{X}_S} \right)^\xi , \tag{B.4}$$

$$V_L(\overline{X}_S) = V_L^+(\overline{X}_S) . \tag{B.5}$$

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The first condition captures the value of the firm net of bankruptcy costs that will be transferred to the new owners in case of default. In particular, it not only reflects the bankruptcy costs but also the cannibalization cost and the preemption cost (i.e., the second and the third terms in equation (B.4)), which remain associated with the large firm's assets through and beyond the restructuring process. The second condition states that firm value is bounded above by its post-investment value given in Lemma 2, which excludes the growth option's payoff. Solving these two equations for the two unknown constants yields firm value before investment or obsolescence, given in the first equation of Proposition 2.

Similar arguments as in equations (B.1), (B.2), and (B.3) yield equity value before investment or obsolescence, given in Proposition 2, if we instead solve the value-matching conditions for equity:

$$E_L(\underline{X}_L) = 0 \,, \tag{B.6}$$

$$E_L(\overline{X}_S) = E_L^+(\overline{X}_S) \,. \tag{B.7}$$

The first condition ensures that equity is worthless in case of default, while the second condition ensures that equity value is bounded above by its post-investment value given in Lemma 2. In addition to the value-matching conditions, equity value also needs to satisfy a smooth-pasting condition:

$$\left.\frac{\partial E_L(X)}{\partial X}\right|_{X=\underline{X}_L} = 0 \,. \tag{B.8}$$

As equation (B.8) is non-linear in $\underline{X}_L$, we obtain a quasi-closed form solution in form of a fairly complex equation (available upon request), which renders numerical solution necessary. Therefore, the optimal coupon choice of the firm numerically maximizes $V_L(X)$ with respect to C_L at time zero:

$$\max_{C_L} V_L(X; C_L, \underline{X}_L, \underline{X}_L^+, \underline{X}_L^\circ, \cdot) \,, \tag{B.9}$$

which takes into account C_L's effect on the optimally chosen default boundaries $\underline{X}_L$, $\underline{X}_L^+$, and $\underline{X}_L^\circ$ for the various regions. ∎

Online Appendix C. Derivations of Lemma 4, Lemma 5 and Proposition 3

Similar arguments as the ones used for deriving Lemma 1 can be used to derive the closed-form solutions given in Lemma 4. That is, if we replace the factor π, the size of the growth opportunity, in all expressions given in Lemma 1 by the factor $(1 + \pi - \gamma)$, the increased size of assets in place resulting from option exercise but net of cannibalization due to option exercise, then we obtain all the expressions given in Lemma 4. It turns out that the equations in Lemma 5 are actually identical to the ones in Lemma 3 up to some subscripts, namely, L and I, which accommodates different coupon choices and correspondingly different default thresholds. Apart from these differences, the large firm's value functions after obsolescence and the integrated firm's value functions after obsolescence ought to be the same since they did not invest and operate the same assets in place.

Similar arguments as in equations (B.1), (B.2), and (B.3) yield integrated firm value before investment or obsolescence, given in Proposition 3, if we solve the modified value-matching conditions:

$$V_I(\underline{X}_I) = (1 - \alpha) \Lambda X_I - \frac{\rho}{r + \rho - \mu} \delta \Lambda \underline{X}_I \,, \tag{C.1}$$

$$V_I(\overline{X}_I) = V_I^+(\overline{X}_I) \,. \tag{C.2}$$

Equation (C.1) captures the value of the firm net of bankruptcy costs that will be transferred to the new owners in case of default. In particular, it only reflects the bankruptcy costs and the preemption cost induced by product introduction by a competitor, which remain associated with the large firm's assets through and beyond the restructuring process, because the growth option cannot be brought through the restructuring process. An alternative assumption, which we have explored in unreported numerical solutions, is to partially transfer the option to the new owners, which corresponds to simply including a fractional value of the last two terms in (B.4) in (C.1). However, all the results are qualitatively very similar to the ones reported in the paper and hence suppressed for brevity. The intuition is that transferring some of the option to the debtholders increases their recoveries, which are multiplied by a fairly small number (default probability) for computing time zero firm value. The small increase in firm value relative to Non-Integration lowers the cost of debt (credit spread) for a given coupon, which, in equilibrium, increases the optimal coupon and hence magnifies the overhang problem (i.e., a decrease in firm value relative to Non-Integration). Equation (C.2) states that firm value is bounded above by its post-investment value given in Lemma 4 net of the capital outlay required for option exercise. Solving these two equations for the two unknown constants yields firm value before investment or obsolescence, given in the first equation of Proposition 3.

Similar arguments as in equations (B.1), (B.2), and (B.3) yield equity value before investment or obsolescence, given in Proposition 3, if we instead solve the value-matching conditions for equity:

$$E_I(\underline{X}_I) = 0 \,, \tag{C.3}$$

$$E_I(\overline{X}_I) = E_I^+(\overline{X}_I) - \kappa \,. \tag{C.4}$$

The first condition ensures that equity is worthless in case of default, while the second condition ensures that equity value pastes correctly to its post-investment value given in Lemma 4 net of the equity-financed exercise cost, κ. In addition to these two value-matching conditions, optimality requires that equity value satisfies smooth-pasting conditions for default and investment:

$$\left. \frac{\partial E_I(X)}{\partial X} \right|_{X=\underline{X}_I} = 0 \,, \tag{C.5}$$

$$\left. \frac{\partial E_I(X)}{\partial X} \right|_{X=\underline{X}_I} = \left. \frac{\partial E_I^+(X)}{\partial X} \right|_{X=\underline{X}_I} \,. \tag{C.6}$$

Equations (C.5) and (C.6) are non-linear in $\underline{X}_I$ and $\overline{X}_I$, respectively, so that we obtain only quasi-closed form solutions in form of two tedious equations (available upon request). Non-linearity renders numerical solution necessary. Therefore, the optimal coupon choice of the firm numerically maximizes $V_I(X)$ with respect to C_I at time zero:

$$\max_{C_I} V_I(X; C_I, \underline{X}_I, \underline{X}_I^+, \underline{X}_I^\circ, \overline{X}_I, \cdot) \,, \tag{C.7}$$

which is clearly similar to the problem in equation (B.9). Yet, an important difference is that the optimal choice of C_I incorporates not only its effect on the optimally chosen default boundaries $\underline{X}_I$, $\underline{X}_I^+$, and $\underline{X}_I^\circ$ for the various regions, but also its effect on the jointly optimal investment boundary $\overline{X}_I$, which uniquely delivers endogenous overhang effects of the Integrated form. The numerical procedure for the constrained optimization is suppressed here but available upon request. ■

Online Appendix D. Alternative Financing Arrangements

In this section we analyze the effect of an alternative financing arrangement in the Integrated case, in which the integrated firm is able to follow a similar capital structure policy as the small, non-integrated firm at the time of option exercise.[18]

Specifically, at the time of option exercise in the Integrated case we grant the integrated firm the option to recapitalize with respect to the new collateral pool from the option (i.e., π), but not with respect to the existing collateral pool that we normalized to one (i.e., assets in place). In addition to the time zero debt with coupon C_I, we assume the integrated firm issues a second, time $\mathcal{T}_G$ debt tranche with coupon payments C_I^+ specified as in (6). Specifying this amount of debt (i.e., the amount that would be chosen by an all-equity stand-alone firm for this set of assets at the time of exercise) gives the integrated firm a limited measure of flexibility that most closely matches the flexibility advantage enjoyed by the small firm in the Non-Integrated case. Note that under this alternative specification, the large, separated firm does not have an option to recapitalize after its time zero debt choice, thus keeping capital structure flexibility relatively constant across organizational designs with respect to the pool of assets in place.[19] We also need to specify debt priority in bankruptcy, which primarily affects our result through the floatation value of the new debt. We assume that the two classes of debt receive equal treatment in bankruptcy.[20] We continue to assume that the exercise time is chosen by the firm's equityholders to maximize their own value plus the value of the new debt. These assumptions can be modified or relaxed, but they are made to be as consistent as possible with the treatment of the separated, small firm in Lemma 1 and Proposition 1.

The increase in the integrated firm's financial flexibility in this extension causes the wedge in net tax benefits across forms seen in Panel (d) of Figure 1 to shrink, so that net tax benefit differences arising from financial flexibility are no longer a key driver of organizational design. This extension also attenuates debt overhang concerns because the integrated firm chooses less debt at time zero and can use the proceeds from the later debt issue to fund a fraction of the exercise cost. Hence the integrated firm gets closer to maximizing the pure growth option value in its initial financing choice. However, note that despite the integrated firm's greater flexibility, there are still two important factors that may prevent it from completely replicating the performance of the Non-Integrated form. First, since the integrated firm still optimally takes on some debt at time zero, there will remain a (potentially significant) measure of debt overhang. Second, because of the overhang concern it will optimally take on less leverage with respect to the assets in place than the large non-integrated firm, which does not have to worry about overhang at all. It is therefore important to re-examine some of the key equilibrium maps from Section 4. to see whether the behavior of the critical cutoff value for γ^* would be materially altered under this setup.

Figure 9 provides four equilibrium maps, which trace again the critical value γ^* as a function of various model parameters. In particular, we re-examine for this model extension the optimal organizational form as a function of γ and ρ (panel (a)), γ and σ (panel (b)), γ and α (panel (c)), and γ and τ (panel (d)), holding all

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[18]We have also considered having all capital structure decisions made at the time of investment (i.e., all firms are debt-free prior to exercise). However, the integrated firm would then have an incentive to speed up its exercise decision so it could more quickly issue debt against the assets in place. This would not be possible for the large non-integrated firm, and would therefore be an additional source of non-comparability across forms.

[19]We could also allow all firms to recapitalize at the time of option exercise with respect to their full collateral pool. However, this would be significantly less tractable and is not expected to affect the qualitative results.

[20]That is, the relative weights implied by the two debt coupons apportion the firm's recovery value according to an equal priority (*pari passu*) rule. For an analysis of optimal priority structure, see, e.g., Hackbarth and Mauer (2012).

other parameters constant at their base levels. While the cutoff for the cannibalization parameter to provoke integration is lower, the comparative statics of γ^* are qualitatively unchanged relative to the base case model. In other words, the predictions of γ^*'s directional behavior from the main model are unaffected by giving the Integrated form more financial flexibility. The fact that the level of γ^* at the base parameter values is now lower makes intuitive sense because granting the Integrated form an additional refinancing option (weakly) increases its value. Hence the Integrated form's joint profit maximization incentives are less important, leading to the lower critical cannibalization threshold γ^*. Finally, we have verified that the equilibrium maps for the other model parameters are directionally consistent with the ones discussed in Section 4.. However, for brevity we do not show all of the results here (the equilibrium maps for the remaining parameters are available upon request).
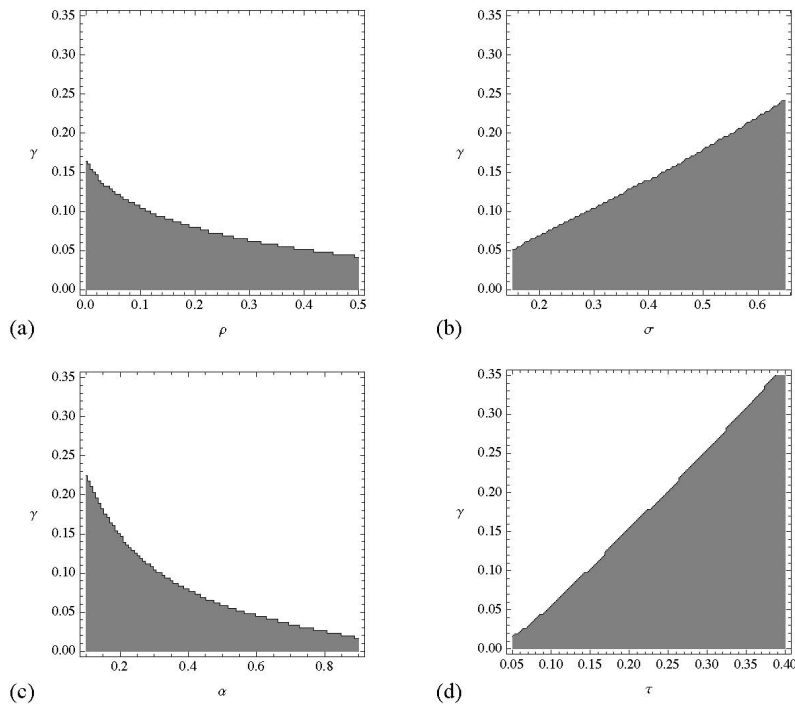

Figure 9. Equilibrium maps showing the optimal organizational form as a function of two parameters holding all others constant for the model extension on Alternative Financing Arrangements. Non-Integration is optimal in shaded regions. Panel (a) varies cannibalization cost, γ, and obsolescence risk, ρ. Panel (b) varies γ and cash flow uncertainty, σ. Panel (c) varies γ and bankruptcy costs, α. Panel (d) varies γ and the corporate tax rate, τ. All parameter values are set as in the baseline environment.