Online Appendix to Financing Asset Sales and Business Cycles

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March 11, 2016

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Appendix A. Derivations

Appendix A.1. The stochastic discount factor, risk-free rates, and market prices of risk

We assume that the representative agent has the continuous-time analog of Epstein-Zin-Weil preferences of stochastic differential utility type (e.g., Duffie and Epstein 1992a, Duffie and Epstein 1992b). The utility index $U_t$ over a consumption process $C_s$ solves

$$U_t = E^P \left[ \int_t^\infty \rho \frac{C_s^{1-\delta} - ((1 - \gamma) U_s)^{1-\delta}}{((1 - \gamma) U_s)^{1-\delta} - 1} ds \mid \mathcal{F}_t \right],$$

(A.1)

in which $\rho$ is the rate of time preference, $\gamma$ the coefficient of relative risk aversion for a timeless gamble, and $\Psi := \frac{1}{\delta}$ the elasticity of intertemporal substitution for deterministic consumption paths.

Incorporating the separability of time and state preferences and assuming that $\Psi > 1$, i.e., that the representative agent has a preference for early resolution of uncertainty and require expected returns that increase in the uncertainty about future consumption, are necessary to capture the impact of aggregate risk on corporate security values.

According to Bhamra, Kuehn, and Strebulaev (2010) and Chen (2010), solving the Bellman equation associated with the consumption problem of the representative agent implies that the stochastic discount factor $m_t$ follows the dynamics

$$\frac{dm_t}{m_t} = -r_i dt - \eta_i dW_t^i + (e^{\kappa_i} - 1) dM_t,$$

(A.2)

in which $M_t$ is the compensated process associated with the Markov chain, $r_i$ are the regime-dependent risk-free interest rates, and $\eta_i$ denote the risk prices for systematic Brownian shocks affecting aggregate output. The market prices of consumption risk $\eta_i$ increase with the agents’ risk aversion and consumption volatility. The parameters $\kappa_i$ denote the relative jump sizes of the discount factor when the Markov chain leaves state $i$, i.e., they are the market prices of the discount factor jump risk.
Risk-free rates, and the market prices of consumption and jump risk are defined as

\[
\begin{align*}
    r_i &= \bar{r}_i + \lambda_i \left[ \frac{\gamma - \delta}{\gamma - 1} \left( w^{\frac{\gamma - 1}{\gamma - 2}} - 1 \right) - (w^{-1} - 1) \right], \\
    \eta_i &= \gamma \sigma_i^C, \\
    \kappa_i &= (\delta - \gamma) \log \left( \frac{h_j}{h_i} \right),
\end{align*}
\]  

(A.3)

(A.4)

(A.5)

with \( i, j = G, B, i \neq j \). The parameters \( h_G, h_B \) solve the following non-linear system of equations (e.g. Bhamra, Kuehn, and Strebulaev 2010):

\[
0 = \rho \left( \frac{1 - \gamma}{1 - \delta} \right) h_i^{\delta - \gamma} + \left( (1 - \gamma) \theta_i - \frac{1}{2} \gamma (1 - \gamma) \left( \sigma_i^C \right)^2 - \rho \frac{1 - \gamma}{1 - \delta} \right) h_i^{1 - \gamma} + \lambda_i \left( h_j^{1 - \gamma} - h_i^{1 - \gamma} \right) 
\]  

(A.6)

The risk-free rates \( r_i \) contain the interest rate if the economy stayed in state \( i \) forever, \( \bar{r}_i \), plus a second term that incorporates a possible switch in the state. The no-jump part of the interest rates, \( \bar{r}_i \), are

\[
\bar{r}_i = \rho + \delta \theta_i - \frac{1}{2} \gamma (1 + \delta) \left( \sigma_i^C \right)^2,
\]  

(A.7)

and

\[
w := e^{\kappa_B} = e^{-\kappa_G}
\]  

(A.8)

measures the size of the jump in the real-state price density when the economy shifts from bad states to good states (see for example Proposition 1 in Bhamra, Kuehn, and Strebulaev 2010).

**Appendix A.2. Derivation of the values of corporate securities after investment**

**The valuation of corporate debt.** Our valuation of corporate debt of a firm that consists of only invested assets in a two state setting follows (Hackbarth, Miao, and Morellec 2006), and (Arnold, Wagner, and Westermann 2013). We illustrate the case in which the default boundary in good states is lower than the one in bad states, i.e., \( \hat{D}_G < \hat{D}_B \). If a firm defaults, debtholders receive a fraction \( \Lambda_i \alpha_i \) of the unleveraged after tax asset value \( (1 - \tau)Xy_i \). A debt investor requires an instantaneous return equal to the risk-free rate \( r_i \). The instantaneous debt return corresponds to the realized rate of return plus the coupon proceeds from debt. Hence, an application of Ito’s lemma with possible state switches shows that debt satisfies the following system of ODEs.
For $0 \leq X \leq \hat{D}_G$:
\[
\begin{align*}
\hat{d}_G(X) &= \alpha_G \Lambda_G (1 - \tau) X y_G \quad \text{and} \\
\hat{d}_B(X) &= \alpha_B \Lambda_B (1 - \tau) X y_B.
\end{align*}
\] (A.9)

For $\hat{D}_G < X \leq \hat{D}_B$:
\[
\begin{align*}
r_G \hat{d}_G(X) &= c + \mu_G X \hat{d}_G''(X) + \frac{1}{2} \sigma_G^2 X^2 \hat{d}_G''(X) + \lambda_G \left( \alpha_B \Lambda_B (1 - \tau) X y_B - \hat{d}_G(X) \right) \\
\hat{d}_B(X) &= \alpha_B \Lambda_B (1 - \tau) X y_B.
\end{align*}
\] (A.10)

For $X > \hat{D}_B$:
\[
\begin{align*}
r_G \hat{d}_G(X) &= c + \mu_G X \hat{d}_G''(X) + \frac{1}{2} \sigma_G^2 X^2 \hat{d}_G''(X) + \lambda_G \left( \hat{d}_B(X) - \hat{d}_G(X) \right) \\
r_B \hat{d}_B(X) &= c + \mu_B X \hat{d}_B''(X) + \frac{1}{2} \sigma_B^2 X^2 \hat{d}_B''(X) + \lambda_B \left( \hat{d}_G(X) - \hat{d}_B(X) \right).
\end{align*}
\] (A.11)

The boundary conditions read
\[
\begin{align*}
\lim_{X \to \infty} \frac{\hat{d}_i(X)}{X} &< \infty, \quad i = G, B, \quad \text{(A.12)} \\
\lim_{X \searrow \hat{D}_B} \hat{d}_G(X) &= \lim_{X \nearrow \hat{D}_B} \hat{d}_G(X), \quad \text{(A.13)} \\
\lim_{X \searrow \hat{D}_B} \hat{d}_G'(X) &= \lim_{X \nearrow \hat{D}_B} \hat{d}_G'(X), \quad \text{(A.14)} \\
\lim_{X \searrow \hat{D}_G} \hat{d}_G(X) &= \alpha_G \Lambda_G (1 - \tau) D_G y_G, \quad \text{(A.15)}
\end{align*}
\]

and
\[
\lim_{X \searrow \hat{D}_G} \hat{d}_B(X) = \alpha_B \Lambda_B (1 - \tau) D_B y_B. \quad \text{(A.16)}
\]

Condition (A.12) captures the no-bubbles condition. The other boundary conditions are the value-matching conditions (A.13), (A.15), and (A.16), and the smooth-pasting condition at the higher default threshold $\hat{D}_B$ for the debt function in the good state $\hat{d}_G(\cdot)$, Eq. (A.14). The functional form
of the solution is

\[
\hat{d}_i(X) = \begin{cases} 
\alpha_i \Lambda_i (1 - \tau) X y_i & X \leq \hat{D}_i, \quad i = G, B \\
\hat{C}_1 X^{\hat{G}} + \hat{C}_2 X^{\hat{G}} + C_3 X + C_4 & \hat{D}_G < X \leq \hat{D}_B, \quad i = G \\
\hat{A}_i X^{\gamma_1} + \hat{A}_i X^{\gamma_2} + A_{i5} & X > \hat{D}_B, \quad i = G, B,
\end{cases}
\]  

(A.17)

in which \(\hat{A}_{G1}, \hat{A}_{G2}, \hat{A}_{B1}, \hat{A}_{B2}, A_{G5}, A_{B5}, \hat{C}_1, \hat{C}_2, C_3, C_4, \gamma_1, \gamma_2, \beta_1^G, \) and \(\beta_2^G\) are real-valued parameters to be determined.

First, we consider the region \(X > \hat{D}_B\). We start with the standard approach by plugging the functional form \(\hat{d}_i(X) = \hat{A}_i X^{\gamma_1} + \hat{A}_i X^{\gamma_2} + A_{i5}\) into both equations of (A.11). Comparing coefficients and solving the resulting two-dimensional system of equations for \(A_{i5}\), we obtain

\[
A_{i5} = \frac{c (r_j + \lambda_i + \lambda_j)}{r_i r_j + r_j \lambda_i + r_i \lambda_j} = \frac{c}{r_i^2},
\]  

(A.18)

and find that \(\hat{A}_{Gk}\) is a multiple of \(\hat{A}_{Bk}\), \(k = 1, 2,\) with the factor \(l_k := \frac{1}{\lambda_G} (r_G + \hat{\lambda}_G - \hat{\mu}_G \gamma_k - \frac{1}{2} \sigma_G^2 \gamma_k (\gamma_k - 1))\), i.e., \(\hat{A}_{Bk} = l_k \hat{A}_{Gk}\). Using these results when comparing coefficients again, it can be shown that \(\gamma_1\) and \(\gamma_2\) are the negative roots of the quadratic equation

\[
(\hat{\mu}_B \gamma + \frac{1}{2} \sigma_B^2 \gamma (\gamma - 1) - \hat{\lambda}_B - \hat{r}_B) (\hat{\mu}_G \gamma + \frac{1}{2} \sigma_G^2 \gamma (\gamma - 1) - \hat{\lambda}_G - r_G) = \hat{\lambda}_B \hat{\lambda}_G.
\]  

(A.19)

To satisfy the no-bubbles condition for debt in Eq. (A.12), we take the negative roots.

Next, we consider the region \(\hat{D}_G \leq X \leq \hat{D}_B\). Plugging the functional form \(d_G(X) = \hat{C}_1 X^{\hat{G}} + \hat{C}_2 X^{\hat{G}} + C_3 X + C_4\) into the first equation of (A.10), we find by comparison of coefficients that

\[
\beta_{1,2}^G = \frac{1}{2} - \frac{\hat{\mu}_G}{\sigma_G^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\hat{\mu}_G}{\sigma_G^2}\right)^2 + \frac{2(\hat{r}_G + \hat{\lambda}_G)}{\sigma_G^2}}
\]

\[
C_3 = \frac{\hat{\lambda}_G \alpha_B \Lambda_B (1 - \tau) y_B}{r_G + \hat{\lambda}_G - \hat{\mu}_G}
\]

\[
C_4 = \frac{c}{r_G + \hat{\lambda}_G}.
\]  

(A.20)

We then plug the functional form (A.17) into conditions (A.13)–(A.16), and obtain a four-dimensional
linear system in the remaining four unknown parameters $\hat{A}_{G1}, \hat{A}_{G2}, \hat{C}_1,$ and $\hat{C}_2$:

$$
\hat{A}_{G1}\hat{D}^{\gamma_1}_B + \hat{A}_{G2}\hat{D}^{\gamma_2}_B + A_{G5} = \hat{C}_1\hat{D}^{G\gamma_1}_B + \hat{C}_2\hat{D}^{G\gamma_2}_B + C_3\hat{D}_B + C_4
$$

$$
\hat{A}_{G1}\gamma_1\hat{D}^{\gamma_1}_B + \hat{A}_{G2}\gamma_2\hat{D}^{\gamma_2}_B = \hat{C}_1\beta_1\hat{D}^{G\beta_1}_B + \hat{C}_2\beta_2\hat{D}^{G\beta_2}_B + C_3\hat{D}_B
$$

$$
\alpha_G\Lambda_G(1 - \tau)\hat{D}_{GyG} = \hat{C}_1\hat{D}^{G\beta_1}_B + \hat{C}_2\hat{D}^{G\beta_2}_B + C_3\hat{D}_B + C_4
$$

$$
l_1\hat{A}_{G1}\hat{D}^{\gamma_1}_B + l_2\hat{A}_{G2}\hat{D}^{\gamma_2}_B + A_{B5} = \alpha_B\Lambda_B(1 - \tau)\hat{D}_{B/yB}.
$$

We define the matrices

$$
\hat{M} := \begin{bmatrix}
\hat{D}^{\gamma_1}_B & \hat{D}^{\gamma_2}_B & -\hat{D}^{G\gamma_1}_B & -\hat{D}^{G\gamma_2}_B \\
\gamma_1\hat{D}^{\gamma_1}_B & \gamma_2\hat{D}^{\gamma_2}_B & -\beta_1\hat{D}^{G\beta_1}_B & -\beta_2\hat{D}^{G\beta_2}_B \\
0 & 0 & \hat{D}^{G\beta_1}_B & \hat{D}^{G\beta_2}_B \\
l_1\hat{D}^{\gamma_1}_B & l_2\hat{D}^{\gamma_2}_B & 0 & 0
\end{bmatrix}
$$

and

$$
\hat{b} := \begin{bmatrix}
C_3\hat{D}_B + C_4 - A_{G5} \\
C_3\hat{D}_B \\
\alpha_G\Lambda_G(1 - \tau)\hat{D}_{GyG} - C_3\hat{D}_B - C_4 \\
\alpha_B\Lambda_B(1 - \tau)\hat{D}_{B/yB} - A_{B5}
\end{bmatrix},
$$

such that $\hat{M} \begin{bmatrix} \hat{A}_{G1} \hat{A}_{G2} \hat{C}_1 \hat{C}_2 \end{bmatrix}^T = \hat{b}$. The solution for the unknown parameters is

$$
\begin{bmatrix} \hat{A}_{G1} \hat{A}_{G2} \hat{C}_1 \hat{C}_2 \end{bmatrix}^T = \hat{M}^{-1}\hat{b}.
$$

The value of the tax shield is calculated by using the formula for the value of debt, in which $c$ is replaced by $\tau c$, and $\alpha$ is equal to zero. Similarly, we obtain the value of bankruptcy costs by simply replacing $c$ with zero, and $\alpha$ with $1 - \alpha$.

**Default policy.** The value of equity equals the firm value minus the value of debt. The firm value is given by the value of assets in place plus the value of the growth option and the tax shield minus default costs. After debt has been issued, firms choose the ex post default policy that maximizes the value of equity. Formally, the default policy is determined by equating the first derivative of
the equity value to zero at the corresponding default boundary level:

\[
\begin{align*}
\hat{e}_G'(\hat{D}_G^*) &= 0 \\
\hat{e}_B'(\hat{D}_B^*) &= 0.
\end{align*}
\] (A.25)

We solve this problem numerically.

For a firm that receives scaled earnings after investment, the value of corporate securities is solved similarly by replacing \( X \) with the scaled level of earnings. For example, if the firm exercises the option in the good state, and finances the exercise cost by issuing equity, the scaled earnings correspond to \((s_G + 1)X\). The default boundaries \( \hat{D}_G^* \) and \( \hat{D}_B^* \) are then expressed in terms of the scaled earnings levels.

**Appendix A.3. Derivation of the value of the growth option**

*The case with \( X_G < X_B \):*

We present the derivation of the value of the growth option for a firm that finances the option exercise by issuing equity in good states and selling assets in bad states. The value of the growth option for a firm with an alternative financing strategy can be derived similarly. For each state \( i \), the option is exercised immediately whenever \( X \geq X_i \) (option exercise region); otherwise, it is optimal to wait (option continuation region). This structure results in the following system of ODEs for the option’s value function.

For \( 0 \leq X < X_G \):

\[
\begin{align*}
 r_G G_G(X) &= \tilde{\mu}_G X G'_G(X) + \frac{1}{2} \tilde{\sigma}^2_G X^2 G''_G(X) + \tilde{\lambda}_G (G_B(X) - G_G(X)) \\
 r_B G_B(X) &= \tilde{\mu}_B X G'_B(X) + \frac{1}{2} \tilde{\sigma}^2_B X^2 G''_B(X) + \tilde{\lambda}_B (G_G(X) - G_B(X)).
\end{align*}
\] (A.26)

For \( X_G \leq X < X_B \):

\[
\begin{align*}
 G_G(X) &= (1 - \tau)s_G X y_G - K_G(1 + \Upsilon_G) \\
 r_B G_B(X) &= \tilde{\mu}_B X G'_B(X) + \frac{1}{2} \tilde{\sigma}^2_B X^2 G''_B(X) + \tilde{\lambda}_B ((1 - \tau)s_G X y_G - K_B(1 + \Upsilon_B) - G_B(X)).
\end{align*}
\] (A.27)
For \( X \geq X_B \):

\[
\begin{align*}
G_G(X) &= (1 - \tau)s_GXy_G - K_G(1 + \Upsilon_G) \\
G_B(X) &= (1 - \tau)s_BXy_B - K_B/\Lambda_B.
\end{align*}
\]  

(A.28)

When \( X \) is in the option continuation region, which corresponds to system \((A.26)\) and the second equation of \((A.27)\), the required rate of return \( r_i \) (left-hand side) must be equal to the realized rate of return (right-hand side). We calculate the realized rate of return by using Ito’s lemma for state switches. In this region, the last term captures the possible jump in the value of the growth option due to a state switch. It can be expressed as the instantaneous probability of a shift in the state, \( \tilde{\lambda}_G \) or \( \tilde{\lambda}_B \), times the corresponding change in the value of the option. The first equation of \((A.27)\) and the system \((A.28)\) describe the payoff of the option at exercise. The process is in the option exercise region in these cases. The boundary conditions are

\[
\lim_{X \searrow 0} G_i(X) = 0, \quad i = G, B,
\]  

(A.29)

\[
\lim_{X \searrow X_G} G_B(X) = \lim_{X \nearrow X_G} G_B(X),
\]  

(A.30)

\[
\lim_{X \searrow X_G} G_B'(X) = \lim_{X \nearrow X_G} G_B'(X),
\]  

(A.31)

\[
\lim_{X \nearrow X_B} G_B(X) = (1 - \tau)s_BXy_B - K_B/\Lambda_B,
\]  

(A.32)

and

\[
\lim_{X \nearrow X_G} G_G(X) = (1 - \tau)s_GXy_G - K_G(1 + \Upsilon_G).
\]  

(A.33)

Condition \((A.29)\) ensures that the option value goes to zero as earnings approach zero. Conditions \((A.30)\) and \((A.31)\) are the value-matching and smooth-pasting conditions of the value function in bad states at the exercise boundary in good states. The other conditions \((A.32)-(A.33)\) are the value-matching conditions at the exercise boundaries in a good state and a bad state, respectively.
The functional form of the solution is

\[ G_i(X) = \begin{cases} 
\bar{A}_3 X^{\gamma_3} + \bar{A}_4 X^{\gamma_4} & 0 \leq X < X_G, \\
\bar{C}_1 X^{\beta_1^B} + \bar{C}_2 X^{\beta_2^B} + \bar{C}_3 X + \bar{C}_4 & X_G \leq X < X_B, \\
(1 - \tau) s_B X y_B - K_B / \Lambda_B & X \geq X_B \\
(1 - \tau) s_G X y_G - K_G(1 + \Upsilon_G) & X \geq X_G
\end{cases} \tag{A.34} \]

in which \( \bar{A}_G, \bar{A}_B, \bar{C}_1, \bar{C}_2, \bar{C}_3, \gamma_3, \gamma_4, \beta_1^B, \) and \( \beta_2^B \) are real-valued parameters that need to be determined.

First, we consider the region \( 0 \leq X < X_G \), and plug the functional form \( G_i(X) = \bar{A}_3 X^{\gamma_3} + \bar{A}_4 X^{\gamma_4} \) into both equations of \( (A.26) \). A comparison of coefficients implies that \( \bar{A}_G \) is a multiple of \( \bar{A}_B \), \( k = 3, 4 \), with the multiple factor \( \tilde{l}_k := \frac{1}{\lambda_G} (r_G + \tilde{\lambda}_G - \tilde{\mu}_G \gamma_k - \frac{1}{2} \tilde{\sigma}_G^2 \gamma_k(\gamma_k - 1)) \), i.e., \( \bar{A}_B = \tilde{l}_k \bar{A}_G \).

Using this result when comparing coefficients, we find that \( \gamma_3 \) and \( \gamma_4 \) are the positive roots of the quadratic equation

\[ (\tilde{\mu}_B \gamma + \frac{1}{2} \tilde{\sigma}_B^2 \gamma(\gamma - 1) - \tilde{\lambda}_B - r_B)(\tilde{\mu}_G \gamma + \frac{1}{2} \tilde{\sigma}_G^2 \gamma(\gamma - 1) - \tilde{\lambda}_G - r_G) = \tilde{\lambda}_B \tilde{\lambda}_G. \tag{A.35} \]

Boundary condition \( (A.29) \) implies to take the positive roots.

Next, we consider the region \( X_G \leq X < X_B \). Plugging the functional form \( G_B(X) = \bar{C}_1 X^{\beta_1} + \bar{C}_2 X^{\beta_2} + \bar{C}_3 X + \bar{C}_4 \) into the second equation of \( (A.27) \), we find by comparison of coefficients that

\[ \beta_{1,2}^B = \frac{1}{2} - \frac{\tilde{\mu}_B}{\tilde{\sigma}_B^2} \pm \frac{1}{2} \frac{\tilde{\sigma}_B^2}{\tilde{\sigma}_B^2} \pm \frac{2(r_B + \tilde{\lambda}_B)}{\tilde{\sigma}_B^2} \]

\[ \bar{C}_3 = \frac{\tilde{\lambda}_B (1 - \tau) s_G y_G}{r_B - \tilde{\mu}_B + \tilde{\lambda}_B}, \tag{A.36} \]

\[ \bar{C}_4 = \frac{-\tilde{\lambda}_B K_B / \Lambda_B}{r_B + \tilde{\lambda}_B}. \]

The remaining unknown parameters are \( \bar{A}_G, \bar{A}_B, \bar{C}_1, \) and \( \bar{C}_2 \). Plugging the functional form \( (A.34) \)}
into conditions (A.30)–(A.33) yields

\[
\begin{align*}
\bar{C}_1 X^B_G + \bar{C}_2 X^B_G + \bar{C}_3 X_G + \bar{C}_4 &= \bar{l}_3 \bar{A}_G X^\gamma_G + \bar{l}_4 \bar{A}_G X_{G}^\gamma_G, \\
\bar{C}_1 \beta^B_1 X^B_G + \bar{C}_2 \beta^B_2 X^B_G + \bar{C}_3 X_G &= \bar{l}_3 \bar{A}_G \gamma^G_3 X^\gamma_G + \bar{l}_4 \gamma^G_4 \bar{A}_G X_{G}^\gamma_G, \\
\bar{C}_1 \beta^B_B + \bar{C}_2 \beta^B_B + \bar{C}_3 X_B + \bar{C}_4 &= (1 - \tau s_{BYB} X_B - K_B / \Lambda_B, \\
\end{align*}
\]

(A.37)

\[
\begin{align*}
\bar{l}_3 \beta^G_3 X_{G}^\gamma + \bar{l}_4 \beta^G_4 X_{G}^\gamma &= (1 - \tau s_{GYG} X_G - K_G (1 + \Upsilon_G).
\end{align*}
\]

(A.40)

This four-dimensional system is linear in its four unknowns $\bar{A}_G$, $\bar{A}_G$, $\bar{C}_1$ and $\bar{C}_2$. We define the matrices

\[
\begin{align*}
\bar{M} := 
\begin{bmatrix}
\bar{l}_3 X^\gamma_G & \bar{l}_4 X^\gamma_G & -X^B_G & -X^B_G \\
\bar{l}_3 \gamma^G_3 X_{G}^\gamma & \bar{l}_4 \gamma^G_4 X_{G}^\gamma & -\beta^B_1 X^B_G & -\beta^B_2 X^B_G \\
0 & 0 & X^B_G & X^B_G \\
X^\gamma_G & X_{G}^\gamma & 0 & 0
\end{bmatrix}, \\
\end{align*}
\]

(A.41)

and

\[
\begin{align*}
\bar{b} := 
\begin{bmatrix}
\bar{C}_3 X_G + \bar{C}_4 \\
\bar{C}_3 X_G \\
-\bar{C}_3 X_B - \bar{C}_4 + (1 - \tau s_{BYB} X_B - K_B / \Lambda_B, \\
(1 - \tau s_{GYG} X_G - K_G (1 + \Upsilon_G)
\end{bmatrix}, \\
\end{align*}
\]

(A.42)

such that $\bar{M} \begin{bmatrix} \bar{A}_G & \bar{A}_G & \bar{C}_1 & \bar{C}_2 \end{bmatrix} = \bar{b}$. The solution to the remaining four unknowns is

\[
\begin{align*}
\begin{bmatrix}
\bar{A}_G & \bar{A}_G & \bar{C}_1 & \bar{C}_2
\end{bmatrix} = \bar{M}^{-1} \bar{b}.
\end{align*}
\]

(A.43)

The unleveraged value of the growth option. The unleveraged value of the growth option can be
calculated by additionally imposing the smooth-pasting boundary conditions at option exercise:

$$\lim_{X \nearrow X_B^{\text{unlev}}} G_B^{\text{unlev}}(X) = (1 - \tau)s_{BYB} \quad (A.44)$$

and

$$\lim_{X \nearrow X_G^{\text{unlev}}} G_G^{\text{unlev}}(X) = (1 - \tau)s_{GYG}. \quad (A.45)$$

The solution method is analog to the one for the leveraged growth option value up to and including Eq. (A.36). The system of equations (A.37)–(A.40) needs to be augmented by the two equations corresponding to the additional smooth-pasting boundary conditions:

$$\bar{C}_1^{\text{unlev}} \beta_1^{\beta_B - 1} \left(B_B^{\text{unlev}} \right)^{\beta_B - 1} + \bar{C}_2^{\text{unlev}} \beta_2^{\beta_B - 1} \left(B_B^{\text{unlev}} \right)^{\beta_B - 1} + \bar{C}_3 = (1 - \tau)s_{BYB} \quad (A.46)$$

and

$$\bar{A}_3^{\text{unlev}} \gamma_3 \left(G_G^{\text{unlev}} \right)^{\gamma_3 - 1} + \bar{A}_4^{\text{unlev}} \gamma_4 \left(G_G^{\text{unlev}} \right)^{\gamma_4 - 1} = (1 - \tau)s_{GYG}. \quad (A.47)$$

The full system is six-dimensional with the six unknowns $A_{G3}^{\text{unlev}}, A_{G4}^{\text{unlev}}, \bar{C}_1^{\text{unlev}}, \bar{C}_2^{\text{unlev}} X_G^{\text{unlev}},$ and $X_B^{\text{unlev}},$ linear in the first four unknowns and nonlinear in the last two unknowns. We solve it numerically.

The case with $X_G \geq X_B$:

The solution of the case $X_G \geq X_B$ can be obtained immediately by renaming states in the solution of the presented case for $X_G < X_B$.

Appendix A.4. Firms with invested assets and an expansion option

We first present a proof for the value of corporate debt in the case in which $D_G < D_B, \bar{D}_G < \bar{D}_B,$ and $X_B > X_G$. The argument of the proof is adapted from (Arnold, Wagner, and Westermann 2013).
Proof of Proposition 2. A debt investor requires an instantaneous return equal to the risk-free rate $r_i$. The application of Ito’s lemma with state switches shows that debt must satisfy the following system of ODEs.

For $0 \leq X \leq D_G$:

$$
\begin{align*}
  d_G(X) &= \alpha_G \Lambda_G ( (1 - \tau) X y_G + G_G^{unlev}(X) ) \\
  d_B(X) &= \alpha_B \Lambda_B ( (1 - \tau) X y_B + G_B^{unlev}(X) ) .
\end{align*}
$$

For $D_G < X \leq D_B$:

$$
\begin{align*}
  r_G d_G(X) &= c + \tilde{\mu}_G X d'_G(X) + \frac{1}{2} \tilde{\sigma}_G^2 X^2 d''_G(X) \\
  &+ \tilde{\lambda}_G ( \alpha_B \Upsilon_B ( (1 - \tau) X y_B + G_B^{unlev}(X) ) - d_G(X) ) \\
  d_B(X) &= \alpha_B \Lambda_B ( (1 - \tau) X y_B + G_B^{unlev}(X) ) .
\end{align*}
$$

For $D_B < X < X_G$:

$$
\begin{align*}
  r_G d_G(X) &= c + \tilde{\mu}_G X d'_G(X) + \frac{1}{2} \tilde{\sigma}_G^2 X^2 d''_G(X) \\
  r_B d_B(X) &= c + \tilde{\mu}_B X d'_B(X) + \frac{1}{2} \tilde{\sigma}_B^2 X^2 d''_B(X) + \tilde{\lambda}_B ( d_G(X) - d_B(X) ) \\
  d_B(X) &= \alpha_B \Lambda_B ( (1 - \tau) X y_B + G_B^{unlev}(X) ) .
\end{align*}
$$

For $X_G \leq X < X_B$:

$$
\begin{align*}
  d_G(X) &= \tilde{d}_G ( (s_G + 1) X ) \\
  r_B d_B(X) &= c + \tilde{\mu}_B X d'_B(X) + \frac{1}{2} \tilde{\sigma}_B^2 X^2 d''_B(X) + \tilde{\lambda}_B \left( d_G((s_G + 1) X) - d_B(X) \right) .
\end{align*}
$$

For $X \geq X_B$:

$$
\begin{align*}
  d_G(X) &= \tilde{d}_G ( (s_G + 1) X ) \\
  d_B(X) &= \tilde{d}_B \left( s_B + 1 - \frac{K_B / \Lambda_B}{(1 - \tau) X y_B} \right) X .
\end{align*}
$$

In system (A.48), the firm is in the default region in both good and bad states. In this region, debtholders receive $\alpha_i \Lambda_i ( (1 - \tau) X y_i + G_i^{unlev}(X) )$ at default. The firm is in the continuation region in good states, and in the default region in bad states in system (A.49). For the continuation region in good states, the left-hand side of the first equation is the instantaneous rate of return required by investors for holding corporate debt. The right-hand side is the realized rate of return, computed by Ito’s lemma as the expected change in the value of debt plus the coupon payment.
c. The last term expresses the possible jump in the value of debt in case of a state switch that 
triggers immediate default. Eqs. (A.50) describe the case in which the firm is in the continuation 
region in both good and bad states. The next system, (A.51), treats the case in which the firm is 
in the exercise region in good states and in the continuation region in bad states. After exercising 
the option, the firm owns total assets in place with value \((1 - \tau)Xy_i + (1 - \tau)s_iXy_i\), reflecting the 
notion that the exercise cost of the growth option can be financed by issuing equity in good states. 
The value of debt must then be equal to the value of debt of a firm with only invested assets, i.e., 

\[ d_G(X) = \hat{d}_G((s_G + 1)X), \] 

which is the first equation in (A.51). We obtain the second equation in 
this case by using the same approach as in (A.50). The last term captures the notion that a switch 
from bad states to good states triggers immediate exercise of the expansion option with equity 
financing. Finally, (A.52) describes the case in which the firm is in the exercise region in both good 
and bad states. In good states, the earnings of the firm are scaled by \(s_G + 1\). In bad states, the 
exercise cost \(K_B\) is financed by selling \(\frac{K_B/\Lambda_B}{(1 - \tau)Xy_B}\) of the assets in place, such that the earnings of 
the firm are scaled by \((s_B + 1 - \frac{K_B/\Lambda_B}{(1 - \tau)Xy_B})X_B\).

The system is subject to the following boundary conditions.

\[
\lim_{X \downarrow D_B} d_G(X) = \lim_{X \uparrow D_B} d_G(X), \tag{A.53}
\]

\[
\lim_{X \downarrow D_B} d'_G(X) = \lim_{X \uparrow D_B} d'_G(X), \tag{A.54}
\]

\[
\lim_{X \downarrow D_G} d_G(X) = \alpha_G \Lambda_G \left((1 - \tau)D_Gy_G + G_{unlev}^G(D_G)\right), \tag{A.55}
\]

\[
\lim_{X \downarrow D_B} d_B(X) = \alpha_B \Lambda_B \left((1 - \tau)D_By_B + G_{unlev}^B(D_B)\right), \tag{A.56}
\]

\[
\lim_{X \downarrow X_G} d_B(X) = \lim_{X \uparrow X_G} d_B(X), \tag{A.57}
\]

\[
\lim_{X \downarrow X_G} d'_B(X) = \lim_{X \uparrow X_G} d'_B(X), \tag{A.58}
\]

\[
\lim_{X \uparrow X_G} d_G(X) = \hat{d}_G((s_G + 1)X_G), \tag{A.59}
\]

and

\[
\lim_{X \uparrow X_B} d_B(X_B) = \hat{d}_B \left((s_B + 1 - \frac{K_B/\Lambda_B}{(1 - \tau)Xy_B})X_B\right). \tag{A.60}
\]

Eqs. (A.53) and (A.54) are the value-matching and smooth-pasting conditions for the debt value
in the good state at the default boundary of the bad state. Eqs. (A.57) and (A.58) reflect the corresponding conditions for the debt value in the bad state at the option exercise boundary of the good state. Eqs. (A.55) and (A.56) show the value-matching conditions at the default thresholds, and Eqs. (A.59) and (A.60) are the value-matching conditions at the option exercise boundaries. The default thresholds and option exercise boundaries are chosen by equityholders. Hence, we do not need the corresponding smooth-pasting conditions for debt.

To solve this system, we start with the functional form of the solution, in which

$$A_{G1}, A_{G2}, A_{B1}, A_{B2}, C_1, C_2, C_3, C_4, C_5, C_6, B_1, B_2, B_4, \beta_1^G, \beta_2^G, \beta_1^B, \beta_2^B, \gamma_1, \gamma_2, \gamma_3, \text{ and } \gamma_4$$

are real-valued parameters to be determined.

We first consider the region $D_B < X \leq X_G$. Plugging the functional form $d_i(X) = A_{i1}X^{\gamma_1} + A_{i2}X^{\gamma_2} + A_{i3}X^{\gamma_3} + A_{i4}X^{\gamma_4} + A_{i5}$ into both equations of (A.50) and comparing coefficients, we obtain

$$A_{i5} = \frac{c (r_j + \tilde{\lambda}_i + \tilde{\lambda}_j)}{r_i r_j + r_j \tilde{\lambda}_i + r_i \tilde{\lambda}_j} = \frac{c}{r_i}.$$  \hspace{1cm} (A.61)

As in Appendix A.2, $A_{Gk}$ is a multiple of $A_{Bk}, k = 1, \ldots, 4$, with the multiple factor $l_k := \frac{1}{\lambda_G} (r_G + \tilde{\lambda}_G - \tilde{\mu}_G \gamma_k - \frac{1}{2} \sigma_G^2 \gamma_k (\gamma_k - 1))$, i.e., $A_{Bk} = l_k A_{Gk}$. Using this relation and comparing coefficients, it can be shown that $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$ correspond to the roots of the quadratic equation

$$(\tilde{\mu}_B \gamma + \frac{1}{2} \sigma_B^2 \gamma (\gamma - 1) - \tilde{\lambda}_B - r_B) (\tilde{\mu}_G \gamma + \frac{1}{2} \sigma_G^2 \gamma (\gamma - 1) - \tilde{\lambda}_G - r_G) = \tilde{\lambda}_B \tilde{\lambda}_G.$$ \hspace{1cm} (A.62)

According to Guo (2001), this quadratic equation always has two negative and two positive distinct real roots. The value of debt in both states is subject to boundary conditions from below (default) and above (exercise of expansion option). To satisfy these boundary conditions, we use four terms with the corresponding factors $A_{ik}$ as well as the exponents $\gamma_k$, which requires the usage of all four roots of Eq. (A.62). We do not incorporate the no-bubbles condition again because it is already implemented in the value function $\hat{d}_i$ of a firm with only invested assets. The unknown parameters for this region are $A_{Gk}, k = 1, \ldots, 4$.

Next, we examine the region $D_G \leq X \leq D_B$. Plugging the functional form $d_G(X) = C_1 X^{\beta_1^G} + C_2 X^{\beta_2^G} + C_3 X + C_4 + C_5 X^{\gamma_3} + C_6 X^{\gamma_4}$ into the second equation of (A.49), we find by comparison
of coefficients that

\[
\beta_{1,2} = \frac{1}{2} \frac{\bar{\mu}_G}{\hat{\sigma}_G^2} \pm \sqrt{\left(\frac{1}{2} \frac{\bar{\mu}_G}{\hat{\sigma}_G^2}\right)^2 + 2 (r_G + \tilde{\lambda}_G)} ,
\]

(A.63)

\[
C_3 = \tilde{\lambda}_G \frac{\alpha_B \Lambda_B (1 - \tau) y_B}{r_G + \bar{\lambda}_G} ,
\]

(A.64)

\[
C_4 = \frac{c}{r_G + \bar{\lambda}_G} ,
\]

(A.65)

\[
C_5 = \alpha_B \Lambda_B \bar{l}_3^{unlev} A_{G3} ,
\]

(A.66)

and

\[
C_6 = \alpha_B \Lambda_B \bar{l}_4^{unlev} A_{G4} .
\]

(A.67)

The unknown parameters left in this region are \(C_1\) and \(C_2\).

Finally, we consider the region \(X_G < X \leq X_B\). Plugging the functional form \(B_1 X^{\beta_1} + B_2 X^{\beta_2} + Z(X) + \tilde{\lambda}_B \frac{c}{r_B + \lambda_B} \) into the second equation of (A.51) and comparing coefficients, we find that

\[
Z(X) = \tilde{\lambda}_B B_5 X^{\gamma_1} + \tilde{\lambda}_B B_6 X^{\gamma_2} .
\]

(A.68)

(A.69)

The parameters \(B_5\) and \(B_6\) are given by

\[
B_5 = \frac{(s_B + 1)^{\gamma_1} \hat{A}_{G1}}{r_B - \bar{\mu}_B \gamma_1 - \frac{1}{2} \hat{\sigma}_B^2 \gamma_1 (\gamma_1 - 1) + \bar{\lambda}_B} ,
\]

(A.70)

and

\[
B_6 = \frac{(s_B + 1)^{\gamma_2} \hat{A}_{G2}}{r_B - \bar{\mu}_B \gamma_2 - \frac{1}{2} \hat{\sigma}_B^2 \gamma_2 (\gamma_2 - 1) + \bar{\lambda}_B} .
\]

(A.71)

The unknown parameters in this region are \(B_1\) and \(B_2\).

To obtain the unknown parameters \(A_{G1}, A_{G2}, A_{G3}, A_{G4}, C_1, C_2, B_1,\) and \(B_2\), we plug the func-
tional form into the system of boundary conditions (A.53)–(A.60):

\[ \sum_{k=1}^{4} A_{Gk} D_{B}^{\gamma_k} + A_{G5} = C_{1} D_{B}^{\beta_1 G} + C_{2} D_{B}^{\beta_2 G} + C_{3} X + C_{4} + C_{5} X^{\gamma_3} + C_{6} X^{\gamma_4} \]

\[ \sum_{k=1}^{4} A_{Gk} \gamma_k D_{B}^{\gamma_k} = C_{1} \beta_1 G D_{B}^{\beta_1 G} + C_{2} \beta_2 G D_{B}^{\beta_2 G} + C_{3} X + C_{5} \gamma_3 X^{\gamma_3} + C_{6} \gamma_4 X^{\gamma_4} \]

\[ \alpha_{G} \Lambda_{G} \left( (1 + \tau) D_{G} y_{G} + \Gamma^{\text{unlev}}_{G}(D_{G}) \right) = C_{1} D_{G}^{\beta_1 G} + C_{2} D_{G}^{\beta_2 G} + C_{3} D_{G} + C_{4} + C_{5} D_{G}^{\gamma_2} + C_{6} D_{G}^{\gamma_3} \]

\[ \sum_{k=1}^{4} l_{k} A_{Gk} D_{B}^{\gamma_k} + A_{B5} = \alpha_{B} \Lambda_{B} \left( (1 + \tau) D_{B} y_{B} + \Gamma^{\text{unlev}}_{B}(D_{B}) \right) \]

\[ \sum_{k=1}^{4} l_{k} A_{Gk} X_{G}^{\gamma_k} + A_{B5} = B_{1} X_{G}^{\beta_1 B} + B_{2} X_{G}^{\beta_2 B} + Z(X_{G}) + B_{4} \]

\[ \sum_{k=1}^{4} l_{k} A_{Gk} \gamma_k X_{G}^{\gamma_k} = B_{1} \beta_1 B X^{\beta_1 B}_{G} + B_{2} \beta_2 B X^{\beta_2 B}_{G} + X_{G} Z'(X_{G}) \]

\[ \sum_{k=1}^{4} A_{Gk} X_{G}^{\gamma_k} + A_{G5} = \hat{d}_{G} ((s_{G} + 1) X_{G}) \]

\[ B_{1} X_{B}^{\beta_1 B} + B_{2} X_{B}^{\beta_2 B} + Z(X_{B}) + B_{4} = \hat{d}_{B} \left( s_{B} + 1 - \frac{K_{B}/\Lambda_{B}}{(1 - \tau) X_{B}} \right) X_{B} \]

Using matrix notation, we can write

\[
M := \begin{bmatrix}
D_{B}^{\gamma_1} & D_{B}^{\gamma_2} & D_{B}^{\gamma_3} & D_{B}^{\gamma_4} & -D_{B}^{\beta_1 G} & -D_{B}^{\beta_2 G} & 0 & 0 \\
\gamma_1 D_{B}^{\gamma_1} & \gamma_2 D_{B}^{\gamma_2} & \gamma_3 D_{B}^{\gamma_3} & \gamma_4 D_{B}^{\gamma_4} & -\beta_1 G D_{B}^{\beta_1 G} & -\beta_2 G D_{B}^{\beta_2 G} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{G}^{\beta_1 G} & D_{G}^{\beta_2 G} & 0 & 0 \\
l_{1} D_{B}^{\gamma_1} & l_{2} D_{B}^{\gamma_2} & l_{3} D_{B}^{\gamma_3} & l_{4} D_{B}^{\gamma_4} & 0 & 0 & 0 & 0 \\
l_{1} X_{G}^{\gamma_1} & l_{2} X_{G}^{\gamma_2} & l_{3} X_{G}^{\gamma_3} & l_{4} X_{G}^{\gamma_4} & 0 & 0 & -X_{G}^{\beta_1 G} & -X_{G}^{\beta_2 G} \\
l_{1} \gamma_1 X_{G}^{\gamma_1} & l_{2} \gamma_2 X_{G}^{\gamma_2} & l_{3} \gamma_3 X_{G}^{\gamma_3} & l_{4} \gamma_4 X_{G}^{\gamma_4} & 0 & 0 & -\beta_1 G X_{G}^{\beta_1 G} & -\beta_2 G X_{G}^{\beta_2 G} \\
X_{G}^{\gamma_1} & X_{G}^{\gamma_2} & X_{G}^{\gamma_3} & X_{G}^{\gamma_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_{B}^{\beta_1 B} & X_{B}^{\beta_2 B}
\end{bmatrix}
\]

(A.73)
\[
\begin{bmatrix}
-A_G + C_3D_B + C_4 + C_5D_G^\gamma_1 + C_6D_B^\gamma_2 \\
C_3D_B + \gamma_1C_5D_B^\gamma_1 + \gamma_2C_6D_B^\gamma_2 \\
-C_3D_G - C_4 - C_5D_G^\gamma_3 - C_6D_G^\gamma_4 + \alpha_G\Lambda_G \left((1 - \tau)D_Gy_G + G^\text{unlev}(D_G)\right) \\
-A_B + \alpha_B\Lambda_B \left((1 - \tau)D_By_B + G^\text{unlev}(D_B)\right) \\
-A_B + Z(X_G) + B_4 \\
X_GZ'(X_G) \\
-A_G + \bar{d}_G((s_G + 1)X_G) \\
-Z(X_B) + B_4 + \bar{d}_B \left((s_B + 1 - \frac{K_B/\Lambda_B}{(1-\tau)X_By_B})X_B\right)
\end{bmatrix}
\]

The solution to the remaining unknowns is

\[
\begin{bmatrix}
A_G1 & A_G2 & A_G3 & A_G4 & C_1 & C_2 & B_1 & B_2
\end{bmatrix}^T = M^{-1}b.
\] 

(A.75)

The case with \(D_G < D_B, \bar{D}_G < \bar{D}_B,\) and \(X_G > X_B:\)

Going through the same steps as in the previous case gives us

\[
M :=
\begin{bmatrix}
D_B^\gamma_1 & D_B^\gamma_2 & D_B^\gamma_3 & D_B^\gamma_4 & -D_B^{\beta_G} & -D_B^{\beta_G} & 0 & 0 \\
\gamma_1D_B^\gamma_1 & \gamma_2D_B^\gamma_2 & \gamma_3D_B^\gamma_3 & \gamma_4D_B^\gamma_4 & -\beta_1^G\alpha_B^{\beta_G} & -\beta_2^G\alpha_B^{\beta_G} & 0 & 0 \\
0 & 0 & 0 & 0 & D_B^{\beta_G} & D_B^{\beta_G} & 0 & 0 \\
l_1D_B^\gamma_1 & l_2D_B^\gamma_2 & l_3D_B^\gamma_3 & l_4D_B^\gamma_4 & 0 & 0 & 0 & 0 \\
X_B^\gamma_1 & X_B^\gamma_2 & X_B^\gamma_3 & X_B^\gamma_4 & 0 & 0 & -X_B^{\beta_G} & -X_B^{\beta_G} \\
\gamma_1X_B^\gamma_1 & \gamma_2X_B^\gamma_2 & \gamma_3X_B^\gamma_3 & \gamma_4X_B^\gamma_4 & 0 & 0 & -\beta_1^G\alpha_B^{\beta_G} & -\beta_2^G\alpha_B^{\beta_G} \\
l_1X_B^\gamma_1 & l_2X_B^\gamma_2 & l_3X_B^\gamma_3 & l_4X_B^\gamma_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_G^{\beta_G} & X_G^{\beta_G}
\end{bmatrix}
\] 

(A.76)
and

\[
\begin{bmatrix}
-A_{G5} + C_3D_B + C_4 + C_5D_{B1} + C_6D_{B2} \\
C_3D_B + \gamma_1C_5D_{B1} + \gamma_2C_6D_{B2} \\
-C_3D_G - C_4 - C_5D_{G1} - C_6D_{G2} + \alpha_G\Lambda_G ((1 - \tau)D_{GYG} + G_{unlev}^G(D_G)) \\
-A_{B5} + \alpha_B\Lambda_B ((1 - \tau)D_{BYB} + G_{unlev}^B(D_B)) \\
-A_{G5} + Z(X_B) + B_4 \\
X_B Z'(X_B) \\
-A_{B5} + \hat{d}_B ((s_B + 1 - \frac{K_B/\Lambda_B}{(1 - \tau)X_{yB}})X_B) \\
-Z(X_G) + B_4 + \hat{d}_G ((s_G + 1)X_G)
\end{bmatrix}.
\]  

(A.77)

The solution to the unknowns is again determined by

\[
\begin{bmatrix}
A_{G1} & A_{G2} & A_{G3} & A_{G4} & C_1 & C_2 & B_1 & B_2
\end{bmatrix}^T = M^{-1}b.
\]  

(A.78)

Appendix A.5. Bankruptcy costs

For the calculation of bankruptcy costs, the ODEs are given by the following system:

For \(0 \leq X \leq D_G\):

\[
\begin{align*}
b_G(X) &= (1 - \alpha_G\Lambda_G)(1 - \tau)X_{yG} + G_G(X) - \alpha_G\Lambda_GG_{unlev}^G(X) \\
b_B(X) &= (1 - \alpha_B\Lambda_B)(1 - \tau)X_{yB} + G_B(X) - \alpha_B\Lambda_BG_{unlev}^B(X).
\end{align*}
\]  

(A.79)

For \(D_G < X \leq D_B\):

\[
\begin{align*}
r_G b_G(X) &= \tilde{\mu}_G X b'_G(X) + \frac{1}{2}\tilde{\sigma}_G^2 X^2 b''_G(X) \\
&+ \tilde{\lambda}_G ((1 - \alpha_B\Lambda_B)(1 - \tau)X_{yB} + G_B(X) - \alpha_B\Lambda_BG_{unlev}^B(X) - b_G(X)) \\
b_B(X) &= (1 - \alpha_B\Lambda_B)(1 - \tau)X_{yB} + G_B(X) - \alpha_B\Lambda_BG_{unlev}^B(X).
\end{align*}
\]  

(A.80)
For $D_B < X < X_G$:

$$
\begin{align*}
\begin{cases}
    r_Gd_G(X) &= c + \mu_G X b_G'(X) + \frac{1}{2} \sigma_G^2 X^2 b_G''(X) + \tilde{\lambda}_G (b_B(X) - b_G(X)) \\
    r_Bd_B(X) &= c + \mu_B X d_B'(X) + \frac{1}{2} \sigma_B^2 X^2 d_B''(X) + \tilde{\lambda}_B (b_G(X) - b_B(X)).
\end{cases}
\end{align*}
$$

(A.81)

For $X_G \leq X < X_B$:

$$
\begin{align*}
\begin{cases}
    b_G(X) &= \hat{d}_G ((s_G + 1)X) \\
    r_B(X) &= c + \mu_B X b_B'(X) + \frac{1}{2} \sigma_B^2 X^2 b_B''(X) + \tilde{\lambda}_B \left( \hat{d}_G ((s_G + 1)X) - b_B(X) \right).
\end{cases}
\end{align*}
$$

(A.82)

For $X \geq X_B$:

$$
\begin{align*}
\begin{cases}
    b_G(X) &= \hat{b}_G ((s_G + 1)X) \\
    b_B(X) &= \hat{b}_B \left( (s_B + 1 - \frac{K_B}{\Lambda_B} (s_B + 1)X_B - \frac{K_B}{\Lambda_B} (s_B + 1)X_B) X \right).
\end{cases}
\end{align*}
$$

(A.83)

The boundary conditions are as follows:

$$
\begin{align*}
&\lim_{X \searrow D_B} b_G(X) = \lim_{X \nearrow D_B} b_G(X), \\
&\lim_{X \searrow D_B} b_G'(X) = \lim_{X \nearrow D_B} b_G'(X), \\
&\lim_{X \searrow D_G} b_G(X) = (1 - \alpha_G \Lambda_B)(1 - \tau) X_G + G_G(D_G) - \alpha_G \Lambda_G G^{unlev}_G(D_G), \\
&\lim_{X \searrow D_B} b_B(X) = (1 - \alpha_G \Lambda_G)(1 - \tau) X_B G_B(D_B) - \alpha_B G_B^{unlev}(D_B), \\
&\lim_{X \searrow X_G} b_B(X) = \lim_{X \nearrow X_G} b_B(X), \\
&\lim_{X \searrow X_G} b_B'(X) = \lim_{X \nearrow X_G} b_B'(X), \\
&\lim_{X \searrow X_G} b_G(X) = \hat{b}_G ((s_G + 1)X_G), \\
&\lim_{X \nearrow X_B} b_B(X) = \hat{b}_B \left( (s_B + 1 - \frac{K_B}{\Lambda_B} (s_B + 1)X_B - \frac{K_B}{\Lambda_B} (s_B + 1)X_B) X \right),
\end{align*}
$$

(A.84)\text{ to } (A.91)

Eqs. (A.84) and (A.85) are the value-matching and smooth-pasting conditions for bankruptcy costs.
in good states at the default boundary in bad states. Similarly, Eqs. (A.88) and (A.89) are the corresponding conditions for bankruptcy costs in bad states at the option exercise boundary in good states. Eqs. (A.86) and (A.87) are the value-matching conditions at the default thresholds. They incorporate the fact that upon default, the value of the leveraged growth option switches to the value of the unleveraged growth option. Eqs. (A.90) and (A.91) are the value-matching conditions at the option exercise boundaries.

To solve for the unknown parameters, we plug the functional form

\[
b_i(X) = \begin{cases} 
(1 - \alpha_i \Lambda_i)(1 - \tau)X y_i - \alpha_i \Lambda_i G_i^{unlev}(X) + G_i(X) & X \leq D_i, \quad i = G, B \\
C_i X^{\beta_i} + C_2 X^{\beta_2} + C_3 X^{\gamma_3} + C_4 X^{\gamma_4} + \hat{\lambda}_G \frac{c}{r_G + \hat{\mu}_G + \hat{\lambda}_G} \frac{X}{r_G} & D_G < X \leq D_B, \quad i = G \\
A_i X^{\gamma_1} + A_{i2} X^{\gamma_2} + A_{i3} X^{\gamma_3} + A_{i4} X^{\gamma_4} + \frac{c}{r_i} & D_B < X \leq X_G, \quad i = G, B \\
B_1 X^{\beta_1} + B_2 X^{\beta_2} + Z(X) + \hat{\lambda}_B \frac{c}{r_B + \lambda_B} \frac{X}{r_B + \lambda_B} & X_G < X \leq X_B, \quad i = B \\
\hat{b}_G ((s_G + 1)X) & X > X_G, \quad i = G \\
\hat{b}_B \left( (s_B + 1 - \frac{K_B/\Lambda_B}{(1 - \tau)X^{1/2}})X \right) & X > X_B, \quad i = B
\end{cases}
\]

(A.92)

into the system of boundary conditions (A.84)-(A.91). The solution to the unknown parameters is given by

\[
\begin{bmatrix} A_{G1} & A_{G2} & A_{G3} & A_{G4} & C_1 & C_2 & B_1 & B_2 \end{bmatrix}^T = M^{-1}b,
\]

(A.93)
in which
\[
M := \begin{bmatrix}
D_B^{71} & D_B^{72} & D_B^{73} & D_B^{74} & -D_B^{\beta G} & -D_B^{\beta G} & 0 & 0 \\
\gamma_1 D_B^{71} & \gamma_2 D_B^{72} & \gamma_3 D_B^{73} & \gamma_4 D_B^{74} & -\beta_1 D_B^{\beta G} & -\beta_2 D_B^{\beta G} & 0 & 0 \\
0 & 0 & 0 & 0 & D_G^{\beta G} & D_G^{\beta G} & 0 & 0 \\
l_1 D_B^{71} & l_2 D_B^{72} & l_3 D_B^{73} & l_4 D_B^{74} & 0 & 0 & 0 & 0 \\
l_1 X_G^{71} & l_2 X_G^{72} & l_3 X_G^{73} & l_4 X_G^{74} & 0 & 0 & -X_G^{\beta B} & -X_G^{\beta B} \\
l_1 \gamma_1 X_G^{71} & l_2 \gamma_2 X_G^{72} & l_3 \gamma_3 X_G^{73} & l_4 \gamma_4 X_G^{74} & 0 & 0 & -\beta_1 X_G^{\beta B} & -\beta_2 X_G^{\beta B} \\
X_G^{71} & X_G^{72} & X_G^{73} & X_G^{74} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_B^{\beta B} & X_B^{\beta B}
\end{bmatrix}, \tag{A.94}
\]

\[
b := \begin{bmatrix}
-A_{G5} + C_3 D_B + C_4 + C_5 D_B^{\gamma_1} + C_6 D_B^{\gamma_2} \\
C_3 D_B + \gamma_1 C_5 D_B^{\gamma_3} + \gamma_2 C_6 D_B^{\gamma_4} \\
- C_3 D_G - C_4 - C_5 D_G^{\gamma_3} - C_6 D_G^{\gamma_4} + (1 - \alpha G \Lambda G) ((1 - \tau) D_G y_G - \alpha_G \Lambda_G G_{G^{\text{unlev}}} (D_G)) + G_G (D_G) \\
- A_{B5} + (1 - \alpha_B \Lambda_B) ((1 - \tau) D_{B y_B} - \alpha_B \Lambda_B G_{B^{\text{unlev}}} (D_B)) + G_B (D_B) \\
-A_{B5} + Z (X_G) + B_4 \\
X_G Z' (X_G) \\
-A_{G5} + \hat{d}_G ((s_G + 1) X_G) \\
-Z (X_B) + B_4 + \hat{d}_B \left( (s_B + 1 - \frac{K_B / \Lambda_B}{(1 - \tau) X_{G y_B}}) X_B \right)
\end{bmatrix}, \tag{A.95}
\]

\[
C_5 = \frac{\bar{t}_3}{t_3} \left( \tilde{A}_{G3}^{\text{lev}} - \alpha_B \Lambda_B \tilde{A}_{G3}^{\text{unlev}} \right), \tag{A.96}
\]

and

\[
C_6 = \frac{\bar{t}_4}{t_4} \left( \tilde{A}_{G4}^{\text{lev}} - \alpha_B \Lambda_B \tilde{A}_{G4}^{\text{unlev}} \right). \tag{A.97}
\]

The case in which \( D_G < D_B, \tilde{D}_G < \tilde{D}_B, \) and \( X_G > X_B \) can be solved analogously.
References


