Quiz 6

Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page.

Question:	1	2	3	4	Total
Points:	10	10	5	0	25
Score:					

Name and section: ______________________________

1. Are the following state or path-dependent functions? Circle the correct answer.

   (a) (5 points) $f(a,b) = a\mathrm{e}^{ab+1}\,\mathrm{d}a + b\mathrm{e}^{ab+1}\,\mathrm{d}b$

   Solution:

   $$s(a,b) = a\mathrm{e}^{ab+1} \qquad t(a,b) = b\mathrm{e}^{ab+1}$$
   $$\frac{\partial s}{\partial b} \stackrel{?}{=} \frac{\partial t}{\partial a}$$
   $$a^2\mathrm{e}^{ab+1} \neq b^2\mathrm{e}^{ab+1}$$

   $\therefore$ path-dependent function

   (b) (5 points) $f(x,y) = \left(\mathrm{e}^{x-y} - \cos x \cos y\right)\mathrm{d}x - \left(\mathrm{e}^{x-y} - \sin x \sin y\right)\mathrm{d}y$

   Solution:

   $$s(x,y) = \mathrm{e}^{x-y} - \cos x \cos y \qquad t(x,y) = -\mathrm{e}^{x-y} + \sin x \sin y$$
   $$\frac{\partial s}{\partial y} = -\mathrm{e}^{x-y} + \cos x \sin y$$
   $$\frac{\partial t}{\partial x} = -\mathrm{e}^{x-y} + \cos x \sin y$$

   $\therefore$ state function

2. (10 points) For one of the above that is a state function, find the function F such that $\mathrm{d}F = f$ where $\mathrm{d}F$ is the total derivative of F.

   Solution:

   $$\int \mathrm{d}x \left(\mathrm{e}^{x-y} - \cos x \cos y\right) = \mathrm{e}^{x-y} - \sin x \cos y$$
   $$\int \mathrm{d}y \left(-\mathrm{e}^{x-y} + \sin x \sin y\right) = \mathrm{e}^{x-y} - \sin x \cos y$$
   $$\therefore$$
   $$F = \mathrm{e}^{x-y} - \sin x \cos y$$

3. (5 points) Suppose we have a refinery that must ship finished goods to some storage tanks. Suppose further that there are two pipelines, A and B, to do the shipping. The cost of shipping x units on A is ax^2, and the cost of shipping y units on B is by^2, where $a > 0$ and $b > 0$ are given. What is the minimum cost to ship Q units?

Solution:

Let the cost function be $f(x,y)$ and the constraint on the amount of product produced be $g(x,y)$:

$$\begin{aligned} f(x,y) &= ax^2 + by^2 \quad g(x,y) = x + y = Q \\ 2ax &= \lambda \quad 2by = \lambda \\ x &= \frac{\lambda}{2a} \quad y = \frac{\lambda}{2b} \\ \frac{\lambda}{2a} + \frac{\lambda}{2b} &= Q \\ \lambda &= Q\frac{2ab}{a+b} \\ x^* &= Q\frac{b}{a+b} \quad y^* = Q\frac{a}{a+b} \\ f_{\min}(x,y) &= Q^2\left(\frac{ab^2}{(a+b)^2} + \frac{a^2b}{(a+b)^2}\right) \\ &\boxed{f_{\min}(x,y) = Q^2\frac{ab}{a+b}} \end{aligned}$$

4. For fun if you finish early: Is the following a state or path dependent function?

$$g(x,y,z) = \mathrm{d}x\left(4y^2z^3\cos xy^2z^3\right) + \mathrm{d}y\left(8xyz^3\cos xy^2z^3\right) + \mathrm{d}z\left(12xy^2z^2\cos xy^2z^3\right)$$

Solution: For the function $f(x,y,z) = s(x,y,z)\,\mathrm{d}x + t(x,y,z)\,\mathrm{d}y + u(x,y,z)\,\mathrm{d}z$, the criteria for an exact differential are:

$$\frac{\partial s}{\partial y} = \frac{\partial t}{\partial x}\,; \quad \frac{\partial t}{\partial z} = \frac{\partial u}{\partial y}\,; \quad \frac{\partial s}{\partial z} = \frac{\partial y}{\partial x} \tag{1}$$

In this case, it might be easier to find $F(x,y,z)$ s.t. $\mathrm{d}F(x,y,z) = f(x,y,z)$. As can be seen,

$$F(x,y,z) = 4\sin(xy^2z^3). \tag{2}$$

Because the function $F(x,y,z)$ exists, this is by definition an exact differential and therefore a $\boxed{\text{state}}$ function.