Discussion 5

Name and section: ______________________________

1. Find the point (x^*, y^*, z^*) that is at the minimum of the function

$$f(x,y,z) = 2x^2 + 8y^2 + z^2 \tag{1}$$

subject to the constraint equation

$$g(x,y,z) = 6x + 4y + 4z - 72 = 0 \tag{2}$$

Solution:

$$\begin{aligned}
\mathcal{L} &= 2x^2 + 8y^2 + z^2 - \lambda\,(6x + 4y + 4z - 72 = 0) \\
\nabla\,\mathcal{L} &= (4x - 6\lambda,\ 16y - 4\lambda,\ 2z - 4\lambda) = 0 \\
6x = 9\lambda,\ 4y &= \lambda,\ 4z = 8\lambda \\
72 &= 18\lambda \\
\lambda &= 4 \\
x = 6,\ y &= 1,\ z = 8
\end{aligned}$$

2. A circle is centered about the axes and satisfies the equation

$$x^2 + y^2 = 4. \tag{3}$$

Find the point (x^*, y^*) on the circle that is closest to the point $(3, 2)$.

Solution: Because minimizing the distance or the square of the distance will give the same answer, this can be simplified some.

$$\begin{aligned}
f(x,y) &= (x-3)^2 + (y-2)^2 \\
g(x,y) &= x^2 + y^2 - 4 = 0 \\
\mathcal{L} &= (x-3)^2 + (y-2)^2 - \lambda\left(x^2 + y^2 - 4\right) \\
\nabla\,\mathcal{L} &= (2x - 6 - 2x\lambda,\ 2y - 4 - 2y\lambda) = 0 \\
x = \frac{3}{1-\lambda},\quad y &= \frac{2}{1-\lambda} \\
4 &= \frac{9+4}{(1-\lambda)^2} \\
\lambda &= 1 - \frac{\sqrt{13}}{2} \\
x = \frac{6}{\sqrt{13}},\quad y &= \frac{4}{\sqrt{13}}
\end{aligned}$$

3. You play a slot machine in Las Vegas. For every \$1 coin you insert, there are three outcomes:

   1. you lose your \$1; net profit of $-\$1$
   2. you win \$1; net profit of \$0
   3. you win \$5; net profit of \$4.

   Suppose you believe that your average expected profit over many trials is \$0. Find the maximum entropy distribution for the probabilities p_1, p_2, and p_3 of observing outcomes (1), (2), and (3) respectively.

   (Hint: What are the two constraints for the problem?)

Solution:

$$f(p_1, p_2, p_3) = -\sum_{i=1}^{3} p_i \ln p_i$$
$$g(p_1, p_2, p_3) = \sum_{i=1}^{3} p_i = 1$$
$$h(p_1, p_2, p_3) = \sum_{i=1}^{3} t_i p_i = 0$$

where $t_1 = -1$, $t_2 = 0$, and $t_3 = 4$.

$$\begin{aligned}
\mathcal{L} &= -\sum_{i=1}^{3} p_i \ln p_i \\
&\quad - \lambda_1 \left(\sum_{i=1}^{3} p_i - 1 \right) \\
&\quad - \lambda_2 \left(-1p_1 + 4p_3 \right) \\
\nabla_i \mathcal{L} &= -\ln p_i - 1 - \lambda_1 - \lambda_2 t_i = 0 \\
p_i &= \mathrm{e}^{-(1+\lambda_1+\lambda_2 t_i)} \\
0 &= \mathrm{e}^{-(1+\lambda_1)} \left(-\mathrm{e}^{\lambda_2} + 4\mathrm{e}^{-4\lambda_2} \right) \\
\lambda_2 &= \ln 4 - 4\lambda_2 \\
\lambda_2 &= \frac{\ln 4}{5} \\
1 &= \mathrm{e}^{-(1+\lambda_1)} \sum_{i=1}^{3} \mathrm{e}^{-\lambda_2 t_i} \\
\mathrm{e}^{-(1+\lambda_1)} &= \frac{1}{\sum_{i=1}^{3} \mathrm{e}^{-\lambda_2 t_i}} \\
p_i &= \frac{\mathrm{e}^{-\lambda_2 t_i}}{\sum_{i=1}^{3} \mathrm{e}^{-\lambda_2 t_i}} \\
p_1 \approx 0.498, &\quad p_2 \approx 0.377, \quad p_3 \approx 0.125
\end{aligned}$$