Complementarities in information acquisition with short-term trades

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Abstract

In a financial market where agents trade for short-term profit and where news can increase the uncertainty of the public belief, there are strategic complementarities in the acquisition of private information and if the cost of information is sufficient small, a continuum of equilibrium strategies. Imperfect observation of past prices reduces the continuum of Nash-equilibrium to a Strongly Rational-Expectations Equilibrium. In that equilibrium, there are two sharply different regimes for the evolution of the price, the volume of trade and the information acquisition.

Keywords: endogenous information, short-term gain, micro-structure, strategic complementarity, multiple equilibria, Strongly Rational-Expectations Equilibrium, trading frenzies.

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1 Introduction

Grossman and Stiglitz (1980) show how strategic substitutability arises in the acquisition of private information about the fundamental value of an asset in the sense that “the more individuals who are informed, the lower the expected utility of the informed to the uninformed”. They prove the property in a model with two essential assumptions: (i) agents hold their position until the revelation of the fundamental, (ii) the structure of information is Gaussian. In this paper, agents hold their position only for the short-term and the structure of information is not Gaussian. Strategic complementarity arises in information gathering, with multiple equilibria.

To describe the mechanism, assume that agents trade at most one unit of a claim to a fundamental, are risk-neutral, and hold their position for one period only. The payoff of information in period t depends on the expected absolute value of $p_{t+1} - p_t$. Information about the fundamental is valuable only if the price p_{t+1} moves significantly toward the fundamental, on average.

In any period, the price change from the previous period is determined by a Bayesian process similar to an “average” between the information of history and the trade in the current period where the latter’s weight is positively related to the uncertainty in the belief from history (*e.g.* the variance). In the model presented here with discrete values of the fundamental, some news that is generated by trade increase the uncertainty of the public belief.

When agents get more information about the fundamental in period t, the trade in that period may convey a stronger signal that increases the variance of the belief at the end of the period. In that case, the price in the next period p_{t+1} varies more toward the fundamental under the impact of trade in that period (because of the smaller weight of history). The effect increases the value of information in period t and generates a strategic complementarity in the acquisition of information.

The previous description does not apply either if agents trade for the long-run and hold their position until the revelation of the fundamental θ because the payoff depends on the expected value of $|\theta - p_t|$, or when the public belief is Gaussian because any news in period t reduces the uncertainty of the public belief at the end of the period and therefore the impact of the trade in the next period on the price $t + 1$.

The argument is proven in a model with no price rigidity where the information of the trade is used rationally to update the price[1]. The simplest model with this property is perhaps the model of Glosten and Milgrom (1985) which is expanded here to include information acquisition. The fundamental takes the value of 0 or 1 and news may increase the variance of the public belief[2]. Uncertainty is small when the probability of $\theta = 1$ is near 0 or 1, and highest in the middle-range. Two features are added: (i) agents who have private information hold the asset only for one period, (but the model could be extended to holdings for a few periods); (ii) some agents, called information agents, can obtain information about the fundamental, at some cost, before entering the market, and their decision depends on the information[3] publicly available at that time.

Two periods are the minimum with short-term trade and we begin with such a simple model in Section 2 in order to show that strategy complementarity may arise within a period. A strategy is a probability λ to get information (assumed to reveal the state, for simplicity), and is contingent on the information of an agent entering the market before his entry. Since the strategy depends only on the public information, it is common knowledge. The updating of the price by the market-maker depends on this strategy. We analyze how the payoff of getting information for an agent who enters the market in period 1 depends on the value of λ that would be taken by any other agent who could have entered the market instead and that is rationally computed by

[1] In Froot, Scharfstein and Stein (1992), trade orders are executed randomly in the present and in the next period because of some *ad hoc* friction. Information about the fundamental is useful in predicting the information of others who boost the demand and the price in the next period when, by assumption, the other half of the orders is executed. Agents can learn (at no cost) only one of the two independent components of the fundamental. There is strategic complementarity on the choice of the signal.

[2] For previous studies that depart from the Gaussian framework to generate time-variable uncertainty see, among others, Detemple (1991), David (1997), Veronesi (1998).

[3] Dow and Gorton (1994) analyze the efficiency of financial markets with short-term trading and exogenous information. In a model of the Glosten-Milgrom type, they make the key assumption that the probability of an informed agent increases exogenously as the maturity of the asset goes to zero. There is a fixed cost of trading. When the maturity is long, the probability that the price moves in the right direction in the next period (because of the occurrence of an informed trader) is small and because of the fixed cost, agents do not trade. Trade begins only when the maturity is sufficiently short. Vives (1995) analyzes the informational content of prices with short-term traders in the CARA-Gauss model when private information is accrued over time and when the fundamental is revealed at the end of the N-period game.

the market-maker. It is shown that if the "consensus is strong" before period 1, *i.e.*, if the public probability of $\theta = 1$ is near 1 or 0, an increase of λ increases the payoff of information.

In Section 3, the model is extended to an infinite number of periods where the fundamental is revealed in any period with a vanishingly small probabiility. An increase of information gathering in period $t+1$ has a positive impact on the magnitude of the variation of p_{t+1} in a rational-expectations equilibrium, and therefore a positive impact on the payoff of information in period t. Agents have rational expectations in any period t about the strategy λ_{t+1}.

The set of equilibrium strategies is rich but we consider only strategies where the functional relation between the last transaction price and the probability of investment is constant over time. As in the two-period model, if the public belief is sufficiently near one or zero, there is a strategic complementarity between the information acquisitions of different agents (Proposition 3). If the cost of information is sufficiently small, Proposition 4 shows that there is a continuum of equilibrium strategies where agents follow a trigger strategy: they invest in information ($\lambda_t = 1$) if and only if the last observed price p_{t-1} is in some interval (p^{**}, p^{*}); the boundaries p^{**} and p^{*} that define the trigger strategy can take any values within some intervals.

The continuum of equilibria under common knowledge opens the issue of robustness to a perturbation, and the problem of "equilibrium selection". The model is therefore extended in Section 5 with the very plausible assumption that agents, before they decide whether to get information about the fundamental, observe the last transaction price with a noise that can be vanishingly small. The model is similar to a "global game" (Carlsson and Van Damme, 1993), with two differences: market-makers have perfect information about the last transaction price, as suits their specialization, and more important, the iterated elimination of dominated strategies cannot be applied period by period separately as in standard models. Since the payoff of a strategy of short-term trade in any period t depends on the strategy of other agents in period $t+1$, the iteration has to be implemented backwards between an arbitrarily large number of periods.

Under a vanishingly small observation noise, there is a unique trigger strategy that survives the iterated elimination of dominated strategies and is therefore a Strongly Rational-Expectations Equilibrium (SREE). These results validate the relevance of trigger strategies for the equilibria of the model. They show that the existence of a continuum of equilibria depends on the common knowledge and is not robust to a perturbation. But one should emphasize that the essential property associated to multiple equilibria is a discontinuity in the behavior of agents. That property is strongly validated in the extension. In the unique equilibrium, when the price crosses a threshold value, the fraction of informed agents jumps up, and the average amplitude of price changes between periods changes abruptly[4].

Finally, the issue of strategic complementarity is directly relevant to "trade frenzies". In the present model, agents have a higher incentive to get information and trade when others do so, and the volume of trade is positively related to the amount of information that is conveyed by the market.

2 A simple two-period model

There is a financial asset a unit of which is a claim on the "fundamental" value θ. The value of θ is set by nature before the first period and equal to 1 with probability μ, and to 0 with probability $1 - \mu$, and it is constant over time. It is not directly observable and is revealed after the second period.

In the first period, the financial asset is traded in a setting that builds on the model of Glosten and Milgrom (1985): a new agent meets a risk-neutral profit maximizing market-maker and either trades one unit of the asset or does not trade. The new agent is of one of the following three types.

(i) With probability $\alpha \geq 0$, the agent has exogenous private information about the true state. To simplify, and without loss of generality[5], such an agent is perfectly informed about θ. In some special cases, which will be explicitly stated, α will be strictly positive.

[4]In Veronesi (1999), agents are risk-averse and there is a non-linear relation between the asset price and the public belief about the fundamental. Volatility depends on the asset price. Because of the risk-aversion, bad news when agents are fairly confident about a high fundamental have a strong impact because they reduce this confidence; good news have a weak impact because they also increase the uncertainty. In the present model, agents are risk-neutral and the price is always equal to the expected value of the fundamental.

[5]Imperfectly informed traders may be crowded out of the market into the bid-ask spread, and the analysis of the equilibrium may be more technical, without additional insight.

(ii) With probability $\overline{\beta}$, the agent is an *information agent* who can get, at a fixed[6] cost c, information about the true state θ before trading. As for the agents of the previous type, this information is assumed to be perfect: if the agent pays the cost c, he is said to *invest* and he gets to know θ. The investment decision is made at the beginning of the first period, knowing the public belief μ about θ. The strategy of an agent is defined by the probability to invest, $\beta/\overline{\beta}$ with $\beta \in [0, \overline{\beta}]$. The parameter β will also denote the strategy. An information trader who is informed knows the state and therefore trades according to that information, at any price: he buys (sells) when the state is good (bad). An information agent who is not informed will not trade, because of the spread between the ask and the bid, in equilibrium.

(iii) With probability $1 - \alpha - \overline{\beta}$, the agent trades for an exogenous "liquidity" or hedging purpose at any price. He sells, buys one unit of the asset or does not trade, each with probability 1/3. The issue of endogenous liquidity traders will be discussed briefly in Section 3.

The first period agent trades for the short-term: he cancels his position in the second period. His payoff is $\Big(E[p_2] - p_1\Big)x$ where $x = 1$ if he buys, $x = -1$ if he sells, p_1 and p_2 are the prices of the asset in the two periods, and the expectation $E[p_2]$ is conditional on the information of the agent. For simplicity, there is no discount. All trades take place with a market-maker who is perfectly competitive with other market-makers, holds his position until the revelation of the fundamental and maximizes his expected profit. Hence, $p = E[\theta]$, where the expectation depends only on the public information at the time of the trade (including the trade). The market-maker does not know the type of the agent but knows the probabilities of the different types.

In the second period, trade takes place in two steps: first, a new "young" agent comes to the market and is either informed or a noise trader; second, the "old" agent who traded in the first period cancels his position in the second period. The value of θ is revealed after the second period.

In order to focus on the essential mechanism, the young agent in period 2 is assumed to be either exogenously informed, with probability π_2, or to be a noise trader in which case he buy, sells or does not trade[7] with probability 1/3. If he is informed,

[6]Some heterogeneity in the cost would not change the results because the strategic complementarity will induce jumps in the gross value of the information investment (Section 4).

[7]The possibility of no trade is introduced to prepare for the next section.

he trades one unit to maximize his payoff that is equal to $\theta - p_2$.

The old agent in period 2 is identified as undoing his position whatever his type in the first period. (One may also assume that agents contract the holding for one period with market-makers). The trade of the old agent has therefore no impact on the price and the trade takes place at the price p_2. The sequence of events is represented by a time-line in Figure 1.

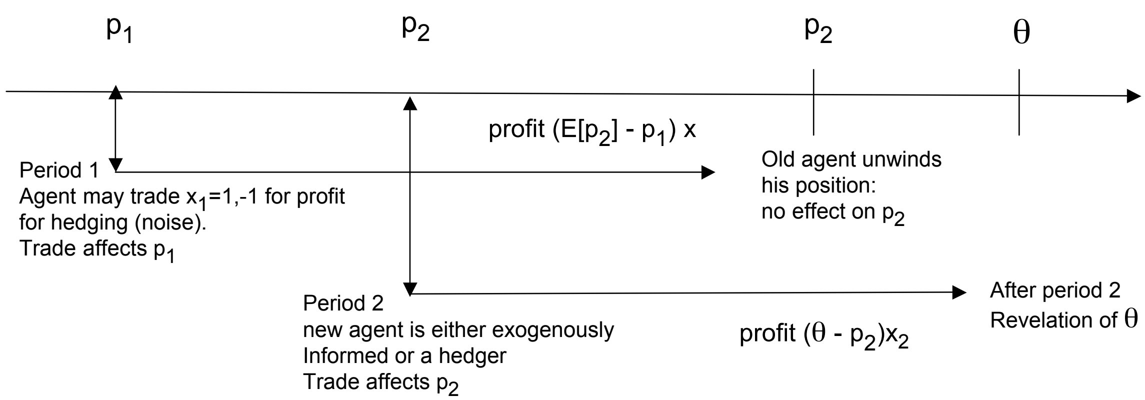

Figure 1: Time-line of trades

The evolution of the price

Let $x_1 \in \{-1, 0, 1\}$ describe the event when in period 1 an agent sells the asset, does not trade or buys the asset. By definition of the model, the probabilities of transactions in the good state ($\theta = 1$) and the bad state ($\theta = 0$) are

$$\begin{aligned} P(x_1 = -1|\theta = 1) &= P(x_1 = 1|\theta = 0) &&= \frac{1 - \alpha - \overline{\beta}}{3} &&= \pi^0, \\ P(x_1 = 1|\theta = 1) &= P(x_1 = -1|\theta = 0) &&= \frac{1 - \alpha - \overline{\beta}}{3} + \alpha + \beta_1 &&= \pi^0 + \pi_1, \\ P(x_1 = 0|\theta = 1) &= P(x_1 = 0|\theta = 0) &&= 1 - (2\pi^0 + \pi_1), \\ \text{with} \quad \pi_1 &= \alpha + \beta_1. \end{aligned} \tag{1}$$

The parameter $\pi_1 = \alpha + \beta_1$ measures the level of information (exogenous and endogenous) of an agent who comes to the market. It is also equal to the difference between the probabilities of a purchase and of a sale, conditional on the "good" fundamental $\theta = 1$. (The case $\theta = 0$ is symmetric). The strategy will depend only on the public information. Hence, the value of π_1 is common knowledge and is used by the market-maker in the updating of the public belief after the observation of the transaction x_1. Let $p^+(\mu, \pi_1)$ and $p^-(\mu, \pi_1)$ the values of the price in period 1 conditional on a buy ($x_1 = 1$), and a sale ($x_1 = -1$), given the beginning of period belief, μ. Using Bayes' rule and (1),

$$\begin{cases} p^+(\mu, \pi_1) = \dfrac{(\pi^0 + \pi_1)\mu}{(\pi^0 + \pi_1)\mu + \pi^0(1-\mu)} = \dfrac{(\pi^0 + \pi_1)\mu}{\pi^0 + \pi_1\mu}, \\ p^-(\mu, \pi_1) = \dfrac{\pi^0\mu}{\pi^0\mu + (\pi^0 + \pi_1)(1-\mu)} = \dfrac{\pi^0\mu}{\pi^0 + \pi_1(1-\mu)}. \end{cases} \tag{2}$$

The difference between the ask p^+, and the bid p^-, is the spread:

$$\Delta(\mu, \pi_1) = p^+(\mu, \pi_1) - p^-(\mu, \pi_1) = \frac{\mu(1-\mu)\pi_1(\pi_1 + 2\pi^0)}{\Big(\pi^0 + \pi_1\mu\Big)\Big(\pi^0 + \pi_1(1-\mu)\Big)}. \tag{3}$$

The probability of no trade is the same when $\theta = 1$ and $\theta = 0$. If there is no trade, there is no change in the public belief and by an abuse of notation, the price is set at $p = \mu$.

The value of information

Consider an agent with a private belief ν (which may be derived from the public and the private information), who trades in the first period and plans to unwind his position in the next period. The price in the next period p_2 is different from p_1 if there is a transaction in period 2, in which case it depends on π_2 according to an updating rule which has the same form as (2). In this simple model, we focus on the endogenous information in the first period and π_2 is fixed. Since the probabilities of a purchase and a sale in period 2, conditional on the good state $\theta = 1$, are $\pi^0 + \pi_2$ and π^0 (and vice-versa if $\theta = 0$), the value of holding one unit of the asset in period 1, $u(\nu)$ is the sum of the price p_1 and of the expected capital gain in each of the two states $\theta = 1$ and $\theta = 0$, multiplied by their probabilities ν and $1 - \nu$:

$$\begin{aligned} u(\nu) = p_1 + \nu\Big((\pi^0 + \pi_2)(p^+(p_1, \pi_2) - p_1) + \pi^0(p^-(p_1, \pi_2) - p_1)\Big) \\ +(1-\nu)\Big(\pi^0(p^+(p_1, \pi_2) - p_1) + (\pi^0 + \pi_2)(p^-(p_1, \pi_2) - p_1)\Big). \end{aligned}$$

Because the price p_1 satisfies the martingale property, we can replace in the previous equation u and μ by p_1. Taking the difference between the two equations,

$$u(\nu) - p_1 = \pi_2 \Delta(p_1, \pi_2)(\nu - p_1). \tag{4}$$

The absolute value of this expression represents the value of the optimal trade.

After the agent acquires information, his belief is either $\nu = 1$ in which case he buys at p_1^+, or $\nu = 0$ and he sells at p_1^-. The probability to learn that $\theta = 1$ is equal to μ, the public belief at the beginning of period 1. The value of acquiring the information is therefore

$$V(\pi; \mu)) = \pi_2 \Big(\mu(1 - p_1^+)\Delta(p_1^+, \pi_2) + (1 - \mu)p_1^- \Delta(p_1^-, \pi_2) \Big), \tag{5}$$

where p_1^+ and p_1^-, the ask and the bid in period 1, are given in (2). This function depends only on the probability that the trading agent is informed in the first period, $\pi = \alpha + \beta$, (where the time subscript for the first period will be omitted in the rest of this section), and on the public belief before that period. Recall the function is defined in an equilibrium where value of π is rationally computed by the market-maker who sets the bid and the ask.

Strategic complementarity in information

Let us consider the impact of the probability of an informed agent on the value of information. Suppose that the "consensus is strong" about the fundamental and that for example μ is near 1. (The argument will be similar if μ is near 0). If π increases, the ask p_1^+ increases and the bid p_1^- falls as the market-maker plays against an agent who is expected to be more informed. This effect was key in Grossman-Stiglitz (1980) and it reduces the value of information.

Now consider the impact on the variability of the price p_2 which is measured by the spreads in the right-hand side of (5). A higher ask p_1^+ in period 1, entails a higher confidence in the belief and therefore a lower impact of a transaction in the next period on p_2: the spread $\Delta(p_1^+, \pi_2)$ is reduced. But if a sale takes place in period 1, the lower confidence at the beginning of period 2 generates more variability of p_2 and a larger spread which increases the value of information. When μ is near 1 the last two effects are not symmetric: an increase of confidence when confidence is already high reduces the value of information by a small amount. But news that reduce the confidence have a larger impact on the value of information which dominates the previous effects.

This discussion is now formallized. Using (5), (3) and some manipulations, we find

$$V(\pi;\mu) = (\pi_2)^2(\pi_2+\pi^0)\mu^2(1-\mu)^2(\pi^0)^2\Bigg(\frac{\pi^0+\pi}{\Big(\pi^0+\pi_2 p^+\Big)\Big(\pi^0+\pi_2(1-p^+)\Big)\Big(\pi^0+\pi\mu\Big)^3} + \frac{\pi^0+\pi}{\Big(\pi^0+\pi_2 p^-\Big)\Big(\pi^0+\pi_2(1-p^-)\Big)\Big(\pi^0+\pi(1-\mu)\Big)^3}\Bigg). \quad (6)$$

If μ is near 1 or near 0, we can make the approximation (in both cases)

$$\frac{\partial V}{\partial \pi} \approx A\Bigg(1-2\Big(\frac{\pi^0}{\pi^0+\pi}\Big)^3\Bigg),$$

where $A > 0$ is a function of the parameters of the model that is independent of π. The right-hand side is positive if $\pi^0(2^{1/2}-1) < \pi$. Recall that we must have $\pi^0 + \pi \leq 1$. If $\pi^0 < 1/2$, the interval $(\pi^0(2^{1/2}-1), 1-\pi^0]$ is well defined. Since the previous expression is an approximation when μ is near 0 or 1, we have the following result.

Proposition 1

If the noise parameter π^0 is smaller than $1/2$, there exist $\bar{\mu}$ and $\bar{\pi}$ such that if the public belief μ at the beginning of the first period is smaller than $\bar{\mu}$ or greater than $1-\bar{\mu}$, and $\pi > \bar{\pi}$, then the value of information is an increasing function of the probability π that an agent is informed.

The result shows the strategic complementarity in getting information: assume that if an arbitrary information trader is called to the market, he invests in information with a higher probability and that probability is expected by the market-maker. Then the value of information for such an information trader is increased, under the conditions of Proposition 1.

The condition that noise trading should not be too large is intuitive: if noise trading is large, the trade of the informed agents have little impact on the price. As the variability of the price is smaller, there is also a smaller incentive for a short-term trader to get information.

3 Infinite horizon

We will see (Lemma 1 below) that an agent who trades for the short-term has a higher profit if agents in the next period get more information and generate more variability of the price toward the fundamental. More investment in a period may also affect the incentive for investment in the next period. In order to take into account these interactions, we need a multi-period model with endogenous information. If we would take a finite number of periods with revelation of the fundamental after some period, in that last period agents would trade effectively for the long-term and have a higher payoff of information. The investment rule would not be the same at the beginning of time and toward the last period. In order to avoid this non-stationarity, we extend the model of the previous section to an infinite number of periods with a small probability of revelation of the fundamental in each period.

As in the previous section, the fundamental θ is set randomly before the first period and is constant through time. In each period, θ is revealed with probability δ, conditional on no previous revelation. The value of δ will be small, in a sense that will be more precise later. Since the fundamental is revealed in finite time with probability one, the infinite number of periods does not introduce an additional effect (as in a model of money for example).

If θ is not revealed at the beginning of a period t (with probability $1 - \delta$), trading takes place as in the simple model: a new agent is of one of the three types described in the previous section, acquires information at a fixed cost c if he can and finds it profitable, and meets a risk-neutral profit maximizing market-maker to trade one unit of the asset or not to trade. The market-maker does not observe the type of the agent but has rational expectations and can compute the strategy of an information agent which is based on public information.

After the new agent meets the market-maker, if there is an old agent (who traded in the previous period), that agent unwinds his position and this action has no effect on the price of the financial asset.

The risk-neutral market-maker trades for the long-term until the eventual revelation of θ, or for the short-term (when he may unwind his position with another market-maker in the next period). Both assumptions are equivalent because of the law of iterated conditional expectations with rational agents[8].

[8] Let h_{t+1} the history at the end of period t, *i.e.*, the sequence of transactions including that of

Since p_{t-1} summarizes the public information at the beginning of period t, the strategies of the information agents will be assumed to be Markov strategies that are defined by measurable functions $B_t(p_{t-1})$ from $(0,1)$ to the closed interval $[0,\overline{\beta}]$. Without loss of generality, all information agents follow the same strategy which is common knowledge.

The evolution of the price and the value of information

The trade $x_t \in \{-1,0,1\}$ defines a public signal with the probabilities set in the equations (1) of the simple model. If θ is not revealed in period t and there is a transaction in that period, the ask and the bid are given by the same equations as in (2):

$$\begin{cases} p^+(p_{t-1},\pi_t) = \dfrac{(\pi^0+\pi_t)p_{t-1}}{(\pi^0+\pi_t)p_{t-1}+\pi^0(1-p_{t-1})}, \\ p^-(p_{t-1},\pi_t) = \dfrac{\pi^0 p_{t-1}}{\pi^0 p_{t-1}+(\pi^0+\pi_t)(1-p_{t-1})}. \end{cases} \tag{7}$$

The profit from a transaction $x_t \in \{-1,1\}$ by an agent with probability assessment ν of $\theta=1$ is found as in (4) with an additional term for the possible revelation of the fundamental. It is the product of x_t and of

$$\omega(\nu,p_t;\pi_{t+1}) - p_t = \Big[(1-\delta)\pi_{t+1}\Delta(p_t,\pi_{t+1})+\delta\Big](\nu-p_t). \tag{8}$$

Note the difference between the long-term and the short-term motive. If agents trade for the long-term, $\delta=1$. If δ is arbitrarily small, we have the short-term gain from trade which is proportional to the product of π_{t+1} and of the spread $\Delta(p_t,\pi_{t+1}) = p^+(p_t,\pi_{t+1}) - p^-(p_t,\pi_{t+1})$. This gain depends on the strategy in the next period $t+1$, which is common knowledge[9] in period t.

Let π^+_{t+1} and π^-_{t+1} be the values of the probability π_{t+1} of an informed agent in period $t+1$ after a price increase (with a purchase) and a price decrease (with a sale) in period t. The value of information at the beginning of period t is found as

period t. If the market maker trades for one period holding, $p_t = \delta E[\theta|h_{t+1}] + (1-\delta)E[p_{t+1}|h_{t+1}]$. $E[p_{t+1}|h_{t+1}] = E[\delta E[\theta|h_{t+2}] + (1-\delta)E[p_{t+2}|h_{t+2}]|h_{t+1}]$ which is equal to $\delta E[\theta|h_{t+1}] + (1-\delta)E[p_{t+2}|h_{t+1}]$. By iterations, $p_t = E[\theta|h_{t+1}]$ which applies when the market-maker trades for the long-term.

[9] If there is more than one equilibrium, we assume that agents coordinate on one strategy which is known. We ignore the issue of random coordination which does add new properties.

in equation (5) of the simple model and is equal to

$$V(p_{t-1},\pi_t,\pi^+_{t+1},\pi^-_{t+1}) = (1-\delta)\tilde{V}(p_{t-1},\pi_t,\pi^+_{t+1},\pi^-_{t+1}) + \delta L(p_{t-1},\pi_t), \quad (9)$$

$$\text{with}\quad \tilde{V}(p_{t-1},\pi_t,\pi^+_{t+1},\pi^-_{t+1}) = \begin{aligned} & p_{t-1}(1-p^+_t(p_{t-1},\pi_t))\pi^+_{t+1}\Delta(p^+_t,\pi^+_{t+1}) \\ & + (1-p_{t-1})p^-_t(p_{t-1},\pi_t)\pi^-_{t+1}\Delta(p^-_t,\pi^-_{t+1}), \end{aligned} \quad (10)$$

$$\text{and}\qquad L(p_{t-1},\pi_t) = \Big[p_{t-1}(1-p^+_t) + (1-p_{t-1})p^-_t\Big]. \quad (11)$$

The expression $\tilde{V}$ represents the gross payoff of information investment from short-term trading and is similar to the expression (5) in the simple model, while L represents the payoff from long-term trading (for an agent who waits for the revelation of the fundamental).

The model exhibits a symmetry between the high and the low values of the price p_{t-1} with respect to the middle price 1/2. This symmetry is expressed by the following property of the value function:

$$V(p_{t-1},\pi_t,\pi^+_{t+1},\pi^-_{t+1}) = V(1-p_{t-1},\pi_t,\pi^-_{t+1},\pi^+_{t+1}). \quad (12)$$

Definition of an equilibrium

An equilibrium strategy is defined by a sequence of measurable functions $\{B_t(p)\}_{t\geq 1}$ *from* $(0,1)$ *to* $[0,\overline{\beta}]$ *such that for any* $p\in(0,1)$,

- *if* $B_t(p)=0$, *then* $V(p,B_t(p),\pi^+_{t+1},\pi^-_{t+1})\leq c$,
- *if* $0<B_t(p)<1$, *then* $V(p,B_t(p),\pi^+_{t+1},\pi^-_{t+1})=c$,
- *if* $B_t(p)=\overline{\beta}$, *then* $V(p,B_t(p),\pi^+_{t+1},\pi^-_{t+1})\geq c$,

with $\pi^+_{t+1}=\alpha+B_{t+1}\Big(p^+(p,B_t(p))\Big)$, $\pi^-_{t+1}=\alpha+B_{t+1}\Big(p^-(p,B_t(p))\Big)$,

The impact of future investment on the value of current investment

A higher level of investment β_{t+1} in period $t+1$ (a higher probability $\beta_{t+1}/\overline{\beta}$), raises the probability $\pi_{t+1}=\alpha+\beta_{t+1}$ of an informed agent, and therefore the spread in that period. The larger spread increases the short-term payoff $\tilde{V}$ of information in period t, (equations (9) and (10)). The long-term payoff L is unaffected. More information investment in period $t+1$ raises the variability of the price in that period and therefore the value of information in the previous period t. The property is verified by simple algebra.

Lemma 1 (strategic complementarity from β_{t+1} to β_t)

The expected value of information investment in any period t, $V(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-)$, is an increasing function of $\pi_{t+1}^+ = \alpha + \beta_{t+1}^+$ and of $\pi_{t+1}^- = \alpha + \beta_{t+1}^-$.

The impact of the belief from history on the value of information

In the present setting, the uncertainty about the fundamental is measured by its variance $p_{t-1}(1 - p_{t-1})$. When p_{t-1} is greater than 1/2, this uncertainty is a decreasing function of p_{t-1}. There is a positive relation between uncertainty and the value of information. This intuitive property is formalized in the next result which is proven in the appendix. Throughout the paper, increasing (decreasing), will mean strictly increasing (decreasing).

Lemma 2

For any $(\pi_t, \pi_{t+1}^+, \pi_{t+1}^-) \in [\alpha, \alpha + \overline{\beta}]^3$, the value of information $V(p_{t-1}, \pi_t, \pi_{t+1}^+, \pi_{t+1}^-)$ defined in (9) is decreasing in p_{t-1} if $p_{t-1} > \hat{p}$ where $\hat{p}$ is defined by $Min_{\beta \in [0, \overline{\beta}]}(p_t^-(\hat{p}, \alpha + \beta)) = 1/2$, and increasing in p_{t-1} if $p_{t-1} < 1 - \hat{p}$.

We have seen that an equilibrium strategy B_t in period t depends on the strategy B_{t+1} in the next period (Lemma 1). In general the structure of equilibria will be complex but we will focus on stationary strategies where the function B_t does not depend on t: first, this class will be sufficiently rich; second, under imperfect information (Section 5), the unique equilibrium will be a stationary strategy.

4 Stationary equilibrium strategies

We have seen in Lemma 2 that the information payoff is decreasing in p if p is high, and increasing if p is low. It is therefore natural to consider strategies in which an information agent invests if and only if the price is in some interval (p^{**}, p^*).

Definition (trigger strategy)

A (stationary) trigger strategy is defined by an investment interval [10] (p^{**}, p^*) such that $B(p) = 1$ if $p \in (p^{**}, p^*)$, and $B(p) = 0$ if $p \notin (p^{**}, p^*)$.

[10] The interval (p^{**}, p^*) is open, but one could include boundaries without altering the equilibrium since the price is at one of the boundaries with zero probability.

Since there is a symmetry between the high and the low value of p (equation (12)), we focus on the determination of the upper-end of the investment interval, p^*. The value of information in period t depends on the level of information investment in period $t+1$ which depends on the price p_t. Assume that p_{t-1} is vanishingly close to p^*. If the information agent learns that $\theta = 1$, he buys at the ask p_t^+ which will be above p^* and, by definition of the stationary strategy, there will be no information investment in the next period. If he sells at the bid, the price p_t will be in the investment interval and $\beta_{t+1} = \bar{\beta}$. We are thus led to introduce *zero-one expectations* such that $\pi_{t+1}^+ = \alpha$, $\pi_{t+1}^- = \alpha + \overline{\beta}$. Under zero-one expectations, in the period that follows a transaction at the ask (at the bid), no information agent (any information agent) invests. Using (9), omitting the time subscript with $p_{t-1} \equiv p$, and recalling $\pi = \alpha + \beta$, the payoff of investment under zero-one expectations is equal to

$$\begin{aligned} W(p,\beta) &= V(p,\pi,\alpha,\alpha+\overline{\beta}) = (1-\delta)\tilde{W}(p,\beta) + \delta L(p,\pi), \\ \text{with} \quad \tilde{W}(p,\beta) &= \tilde{V}(p,\alpha+\beta,\alpha,\alpha+\overline{\beta}). \end{aligned} \tag{13}$$

The expression $\tilde{W}(p,\beta)$ defines the payoff of information with pure short-term trade and zero-one expectations as a function of the last transaction price p and the information investment β in the current period. We first analyze the properties of this function. We later show that the component from long-term trade, $\delta L(p,\pi)$, can be neglected is δ is sufficiently small. The next result shows that under some assumptions, the payoff from pure short-term trade generates strategic complementarities.

Proposition 2

For given α and $\overline{\beta}$ with $\alpha < \overline{\beta}/3$, there exists $\bar{p}$ such that for any $p \in (\bar{p}, 1)$, the value of information with pure short-term trading and zero-one expectations, $\tilde{W}(p,\beta)$, defined in (13), is decreasing in p and increasing in β.

The first part of the result is proven as Lemma 2. The second part which is proven in the Appendix, holds only if the exogenous level of information α is not too large relative to the range of values of the endogenous information, $\overline{\beta}$. Such an assumption is not surprising: a higher value of α generates a higher ask and a lower bid by the rational market-maker which reduces the payoff of information investment.

The properties of the function $\tilde{W}(p,\beta)$ in Proposition 2 are illustrated in Figure 2 where we can replace W by $\tilde{W}$. From Proposition 2 and since for given β, $\tilde{W}(p,\beta)$ is decreasing to 0 when p tends to 1, if c is not too high, the equation $\tilde{W}(p,0) = c$ has

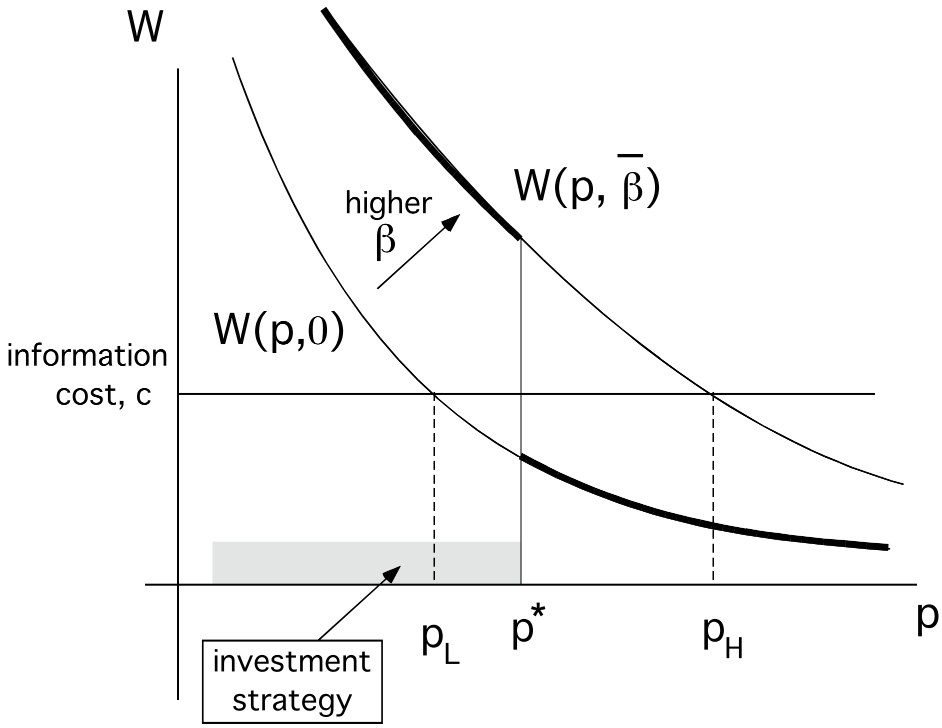

Figure 2: Continuum of constant equilibria

a unique solution $\tilde{p}_L$ such that $\tilde{p}_L > \bar{p}$ where $\bar{p}$ is defined in Proposition 2. In this case, there is another solution $\tilde{p}_H$ such that $\tilde{W}(\tilde{p}_H, \bar{\beta}) = c$ as represented in Figure 2 (where we can substitute $\tilde{p}_L$ for p_L and $\tilde{p}_H$ for p_H). If the probability of revelation δ is sufficiently small, then the function $\tilde{W}$ with pure short-term trading approximates arbitrarily closely the payoff of information W and we can have the Figure 2 for the function $W(p, \beta)$. This is the meaning of the next result.

Proposition 3

Assuming $\alpha < \bar{\beta}/3$, there exists $\bar{c}$ such that if $c < \bar{c}$, then there is $\bar{\delta}$ (which may be vanishingly small) such that if $\delta < \bar{\delta}$, the equation $W(\phi(\beta), \beta) = c$ has a unique solution $\phi(\beta) \in [1/2, 1]$ for $\beta \in [0, \bar{\beta}]$. Furthermore,

$$W(p, \beta) > c, \text{ if } p \in [1/2, \phi(\beta)), \quad W(p, \beta) < c \text{ if } p \in (\phi(\beta), 1),$$

$$\frac{\partial W(p, \beta)}{\partial \beta} > 0 \text{ if } p \in [p_L, p_H] \text{ with } W(p_L, 0) = W(p_H, \bar{\beta}) = c.$$

Choose a value $p^* \in (p_L, p_H)$ as represented in the figure, and a value $p^{**} \in (1 - p_H, 1 - p_L)$ (in the low range that is not represented). The next result shows that the trigger strategy (p^{**}, p^*) defines an equilibrium.

Proposition 4

Under the assumptions of Proposition 3, there is a continuum of stationary equilibrium strategies: any pair $\{p^{**}, p^*\}$ such that $p^* \in [p_L, p_H]$, $p^{**} \in [1-p_H, 1-p_L]$, where $\{p_L, p_H\}$ is defined in Proposition 3, defines a trigger strategy that is an equilibrium strategy.

The sufficient conditions for the existence of a continuum of equilibrium strategies are simple: the cost of information should be smaller than some value, traders should sufficiently care about the short-term profits, and the occurrence of exogenously informed agents should not be too high compared to that of the traders for whom information is endogenous.

Remarks

The case of high information cost

When c is sufficiently large (but not too large), the solution $\tilde{p}_L$ of $\tilde{W}(\tilde{p}_L, 0) = c$ may be smaller than $\bar{p}$ and Propositions 2 and 3 may not apply. In this case, there may be strategic substitutability and a unique equilibrium strategy. If p is greater than some value p^*, there is no information acquisition. If the price decreases from p^*, information investment increases gradually, possibly up to its maximum $\bar{\beta}$: the strategy for an information agent is to randomize with an increasing probability to acquire information as the price decreases. (If the price becomes lower than 1/2, investment in information decreases). The detailed analysis of this case is not the main focus in this paper and is left aside.

Heterogenous costs of information

We have assumed for simplicity that all agents have the same cost of information, but the equilibrium properties are robust when there is some cost heterogeneity. Suppose that an information agent can acquire information at the fixed cost c which is an increasing function $c(\beta)$ for $\beta \in [0, \bar{\beta}]$, with $c(0) > 0$. The distribution of costs is represented by a density function over β on the interval $[0, \bar{\beta}]$. An information agent is now characterized by a random draw from this distribution.

If there is strategic complementarity, Proposition 3 holds with a minor alteration: p_L and p_H are defined by $W(p_L, 0) = c(0)$ and $W(p_H, \bar{\beta}) = c(\bar{\beta})$. If there is strategic substitution, the equilibrium strategy becomes deterministic with a threshold value c^*. The randomization is carried by the draw of the agent from the cost distribution.

Convergence

The public belief is equal to the price of the asset and is a bounded martingale, hence it converges. If the probability of an exogenously informed agent α is strictly positive, the price converges to the value of the fundamental. If there is no exogenously informed agent and $\alpha = 0$, the price does not converge to the true value because the value of information would tend to zero while the cost of information is strictly positive.

Endogenous hedging

The assumption of exogenous noise traders can be replaced by traders who hedge against an exogenous source of individual income that is correlated with the fundamental. Adapting the model of Dow (2004), one can assume that hedgers trade one unit of the asset and are differentiated by the marginal utility of income in the good and the bad state. The cost of hedging increases with the bid-ask spread. When information agents buy more information, the widening of the spread crowds out some hedgers out of trade. This effect increases the information content and the variability of the prices and therefore the value of information. It reinforces the strategic complementarity and the range of parameter values for a continuum of equilibria[11].

5 Imperfect information and equilibrium uniqueness

The property of multiple equilibria is indicative of potentially large changes of the evolution of the price, but it is not entirely satisfactory. It is not clear how agents coordinate on the strategy $\{p^{**}, p^*\}$ since there is a continuum of such values. Furthermore, one should check that the property is robust to a perturbation.

In this section, we introduce an observation noise, which can be vanishingly small, on the history of prices. The game is then dominance solvable under some mi-

[11] In Dow (2004), the endogenous hedging may be sufficient to generate a discrete set of multiple equilibria.

nor additional assumption specified below: there is a unique strategy that survives the iterated elimination of dominated strategies and is therefore Strongly Rational-Expectations Equilibrium (SREE), (Guesnerie, 2002). That strategy is one of the trigger strategies analyzed in the previous section.

The setting is similar to the one-period global game of Carlsson and Van Damme (1993), with a notable difference however: since the optimal strategy in any period t depends on the strategy in period $t+1$, the eductive argument which eliminates strategies has to be applied backwards through time for all periods.

By assumption, the information agent who comes to the market in period t knows imperfectly the last transaction price p_{t-1}: his private information is the signal

$$s_t = p_{t-1} + \epsilon_t, \tag{14}$$

where ϵ_t is independently drawn from a distribution with support $[-\sigma, \sigma]$. The analysis holds for any non degenerate distribution of ϵ_t, but to simplify ϵ has a uniform distribution. The prior distribution on p_{t-1} is common knowledge and without loss of generality, it is assumed to be uniform[14].

Market-makers have perfect information as befits their role. (A noisy observation on their part would probably not change the results). The other parameters of the model are the same as in Section 4 without observation noise, and are such that Proposition 4 holds.

If an information agent after observing his signal s_t does not invest in information, he stays out of the market and does not trade because of the bid-ask spread, as in the case with no observation noise. If he invests at the cost c, he learns the exact value of the fundamental θ, and he trades whatever the equilibrium prices. A strategy is now a measurable function of the signal s and the level of the investment β of any information agent who may be called to the market in the same period.

The strategy β is rationally anticipated by the market-maker. Since he knows the price p_{t-1}, he can compute the distribution of private signal and using the probability of facing an informed agent, he sets the bid and ask to maximize his expected profit from trade, as in the models of the previous sections.

[14]When σ is arbitrarily small, the density of the prior p_{t-1} is nearly uniform for a given s_t. The important assumption is that the prior of an information agent has a support that includes an open interval that includes the interval $[1-p_H, p_H]$.

We will not restrict the strategy to be a trigger strategy, but the payoff with a trigger strategy will be a useful tool. In a trigger strategy, an information agent invests if his signal is in some interval $(\hat{s}', \hat{s})$. We will focus on behavior of agents near the value $\hat{s}$, which will be shown to be in the interval (p_L, p_H). (Without loss of generality, we may assume that $\hat{s}' = 1 - \hat{s}$). For σ sufficiently small, the level of investment is equal to

$$\beta(p, \hat{s}) = \overline{\beta} Min\Big(Max(\frac{\hat{s} - p + \sigma}{2\sigma}, 0), \Big). \tag{15}$$

In the model with perfect information, we have used the payoff function $W(p, \beta)$ with the zero-one expectations that in the period after a price rise (at the ask), there is no information investment, whereas after a transaction at the bid, investment is at the maximum $\overline{\beta}$. A similar function will play an important role here. For an information agent with signal s and zero-one expectations about the next period investment, assuming that the market-maker anticipates the strategy $\hat{s}$ from any customer, the payoff of information is equal to the function

$$J_\sigma(s, \hat{s}) = (1 - \delta)\tilde{J}_\sigma(s, \hat{s}) + \delta K_\sigma(s, \hat{s}), \tag{16}$$

with

$$\begin{aligned} \tilde{J}_\sigma(s, \hat{s}) &= \mu(s) \int_{s-\sigma}^{s+\sigma} (1 - p^+)\pi^+ \Delta(p^+, \pi^+)\phi(p|s)dp \\ &\quad + (1 - \mu(s)) \int_{s-\sigma}^{s+\sigma} p^- \pi^- \Delta(p^-, \pi^-)\phi(p|s)dp, \qquad (17) \\ K_\sigma(s, \hat{s}) &= \int_{s-\sigma}^{s+\sigma} \Big(\mu(s)(1 - p^+) + (1 - \mu(s))p^-\Big)\phi(p|s)dp, \end{aligned}$$

where $p^+ = p^+(p, \alpha + \beta)$, $p^- = p^-(p, \alpha + \beta)$, $\pi^+ = \alpha$, $\pi^- = \alpha + \overline{\beta}$, β is given by (15), $\mu(s)$ is the probability that $\theta = 1$ conditional on the signal s, and $\phi(p|s)$ is the density of p conditional on s. The functions $\tilde{J}_\sigma$ and K_σ and are continuous and have continuous partial derivatives.

Using the uniform distributions on p_{t-1} and of the signal s, the previous expressions can be rewritten

$$\begin{aligned} \tilde{J}_\sigma(s, \hat{s}) &= s\frac{1}{2\sigma} \int_{s-\sigma}^{s+\sigma} (1 - p^+)\pi^+ \Delta(p^+, \alpha)dp + (1 - s)\frac{1}{2\sigma} \int_{s-\sigma}^{s+\sigma} p^- \pi^- \Delta(p^-, \alpha + \overline{\beta})dp, \\ K_\sigma(s, \hat{s}) &= s\frac{1}{2\sigma} \int_{s-\sigma}^{s+\sigma} (1 - p^+)dp + (1 - s)\frac{1}{2\sigma} \int_{s-\sigma}^{s+\sigma} p^- dp. \end{aligned} \tag{18}$$

Vanishingly small observation noise
We will have to consider the case $\hat{s} = s$ in an equilibrium. After some elementary manipulations[15], we find

$$\underset{\sigma \to 0}{Lim}\ J_\sigma(s, s) = \frac{1}{\overline{\beta}} \int_0^{\overline{\beta}} W(s, \beta) d\beta. \tag{19}$$

Because of the differentiability of the Bayesian functions p^+ and p^- on $[0, 1]$,

$$\underset{\sigma \to 0}{Lim}\ \frac{dJ_\sigma(s, s)}{ds} = \frac{1}{\overline{\beta}} \int_0^{\overline{\beta}} \frac{\partial W(s, \beta)}{\partial s} d\beta. \tag{20}$$

These equations show that for σ arbitrarily small, the function $J_\sigma(s, s)$ is approximated by an average of the functions $W(s, \beta)$. The next result follows from the properties of $W(p, \beta)$ in Propositions 3 and 4.

Lemma 3
Under the assumptions of Proposition 4, there exists $\hat{\sigma}$ such that if $\sigma < \hat{\sigma}$, the equation $J_\sigma(s, s) = c$ has a unique solution s^* on the interval $[p_L, 1]$. Furthermore, $s^* \in (p_L, p_H)$, $J_\sigma(s, s) < c$ for $s > s^*$, and if $\sigma \to 0$ then $s^* \to S^*$ which is defined by

$$\frac{1}{\overline{\beta}} \int_0^{\overline{\beta}} W(S^*, \beta) d\beta = c.$$

The function $J_\sigma(s, s)$ replaces the function $W(p, \beta)$ that was used with perfect information. It is illustrated in Figure 3. The iterative dominance is stated in the next result and proven in the appendix.

[15] Using $p = s - \sigma(2\beta/\overline{\beta} - 1)$ from (15), we have

$$\tilde{J}_\sigma(s, s) = \frac{s}{\overline{\beta}} \int_0^{\overline{\beta}} (1 - p^+)\pi^+ \Delta(p^+, \pi^+) d\beta + \frac{1 - s}{\overline{\beta}} \int_0^{\overline{\beta}} p^- \pi^- \Delta(p^-, \pi^-) d\beta,$$

with

$$p^+ = p^+(s - \sigma(2\frac{\beta}{\overline{\beta}} - 1), \beta), \qquad p^- = p^-(s - \sigma(2\frac{\beta}{\overline{\beta}} - 1), \beta).$$

Recall that with perfect information, the short-term payoff of information (with $\delta \approx 0$) is given in (13) and (10):

$$\tilde{W}(p, \beta) = p(1 - p^+(p, \pi))\alpha\Delta(p^+; \alpha) + (1 - p)p^-(p, \pi)(\alpha + \overline{\beta})\Delta(p^-; \alpha + \overline{\beta}),$$

with $\pi = \alpha + \beta$. Equation (19) follows with the expression of J_σ in (16) and K_σ in (18)

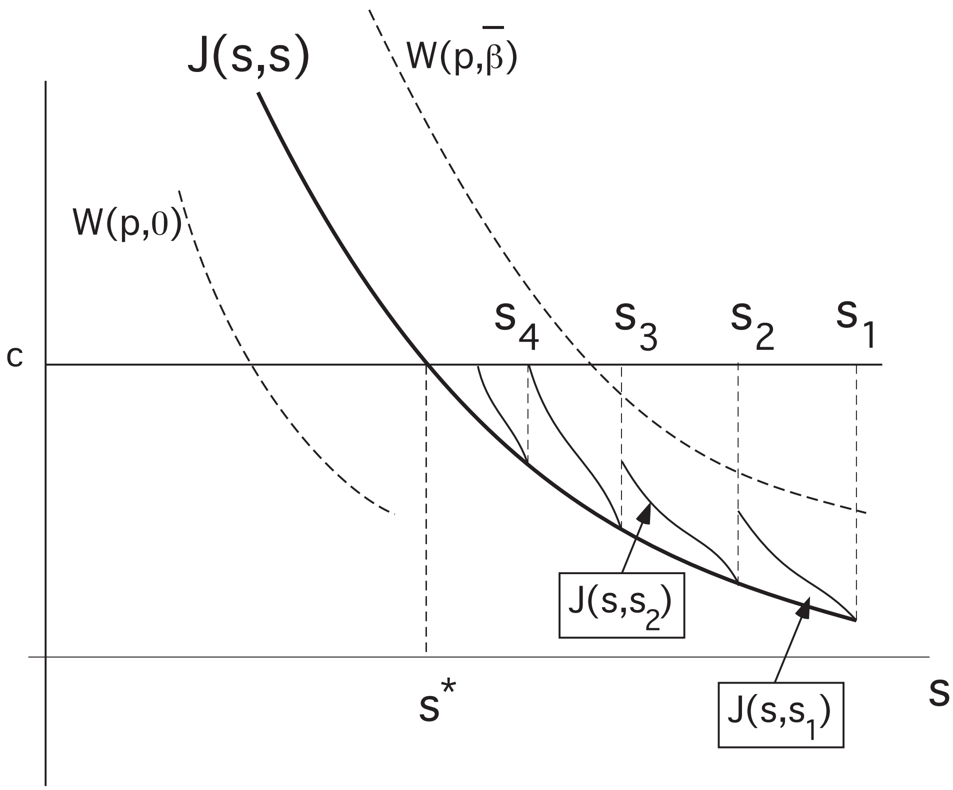

Figure 3: Function $J(s,s)$ and iterated dominance

Proposition 5

Assume c and δ such that Proposition 4 holds, and $\alpha > 0$. There exists $\bar{\sigma}$ such that if $\sigma < \bar{\sigma}$,

(i) investment is iteratively dominated for any $s > s^*$ where $s^* \in (p_L, p_H)$ is defined by $J_\sigma(s^*, s^*) = c$. If $\sigma \to 0$, then $s^* \to S^*$ such that

$$\frac{1}{\overline{\beta}} \int_0^{\overline{\beta}} W(S^*, \beta) d\beta = c;$$

(ii) there exists a value $\hat{c} \leq \bar{c}$ (that was defined in Proposition 3), such that if $c < \hat{c}$, then for $s \in (1 - s^*, s^*)$, not to invest in information is iteratively dominated. In this case the strategy to invest if and only $s \in (1 - s^*, s^*)$ is a SREE. (It is the only to survive the iterated elimination of dominated strategies).

The proposition introduces two minor assumptions: the restriction $\alpha > 0$ ensures that the variation of the price after a transaction has a strictly positive lower-bound. This lower bound ensures that zero-one expectations apply to an agent near a threshold value s^*: if he buys is is sure that for σ sufficiently small, if he buys (sells) then all information agents in the next period will have a signal strictly higher (lower) than

s^* and by definition of s^* will not invest (will invest) in information. The restriction on the cost c in Part (ii) is used to ensure that information investment is dominant if the price is near 1/2. This restriction can be removed if we assume that agents use a trigger strategy in some period.

6 Conclusion

We began by showing that the interaction of short-term trades and endogenous information generated strategic complementarities, and that these complementarities are sufficiently strong to generate a continuum of equilibria when agents have a common knowledge on the last transaction price. There is no contradiction between the results without and with observation noise however, and in my view, the property of multiple equilibria is important only because it exhibits discontinuities in agents' behavior. The multiplicity of equilibria is not robust to perturbation with an observation noise, but the discontinuity in behavior is robust.

Suppose for example that the price is initially near 1 while the fundamental is equal to 0. Eventually, the price must decrease (since it must converge to the truth) and enter a region where there are multiple equilibrium strategies. Under perfect information, in each of these strategies, endogenous investment is either nil or at the maximum. Following Proposition 4, investment must rise suddenly, but the result only states that this jump must occur not later than when the price falls below p_L. Propositions 5 and 6 state that the jump occurs when the price crosses the interval $s^* - \hat{\sigma}, s^* + \hat{\sigma}$ which is arbitrarily small. The model thus exhibits regime of "frenzies" of information gathering.

There is a linear relation in the model between the probability of a trade and the level of endogenous information. Hence, in the equilibrium, there is a positive relation between the volume of trade and the information that is generated by the market. Information frenzy is equivalent to trade frenzy.

A next step would be to consider small random changes or cycles of the fundamental[16]. One may anticipate that the present results will be extended and that in an equilibrium, there would be random swtiches between two regimes with sharply different levels of trade and information.

[16]David (1997) analyzes the learning process in a financial market when the state switches randomly between discrete values and agents have exogenous private information.

7 Appendix: proofs

Lemma 2

The value function V is defined in (9) which is repeated here:

$$V = (1-\delta)\tilde{V} + \delta\Big[p_{t-1}(1-p_t^+) + (1-p_{t-1})p_t^-\Big],$$

We will omit the time subscripts since there is no ambiguity. In the second term of this expression, p_t^+ and p_t^- are given by the Bayesian equations (7), and simple algebra shows that this term is a decreasing function of p if $p_{t-1} > 1/2$. Focusing now on the first term, from (10),

$$\tilde{V}(p_{t-1},\pi) = p_{t-1}(1-p_t^+)\pi_{t+1}^+\Delta(p_t^+,\pi_{t+1}^+) + (1-p_{t-1})p_t^-\pi_{t+1}^-\Delta(p_t^-,\pi_{t+1}^-).$$

The bid p_t^+ and p_t^- are increasing function of p_{t-1}. Since $p_t^- > 1/2$ under the assumption in the lemma, the spreads in period $t+1$, $\Delta(p_t^+,\pi_{t+1}^+)$ and $\Delta(p_t^-,\pi_{t+1}^-)$ are decreasing functions of p_{t-1}.

A small exercise shows that $p_{t-1}(1-p_t^+)$ is decreasing in p_{t-1} if $p_{t-1} > 1/2$ and that $(1-p_{t-1})p_t^-$ is decreasing in p_{t-1} if $p_t^- > 1-p_{t-1}$ which is satisfied by the assumption of the lemma, ($p_t^- > 1/2 > 1-p_{t-1}$). The last part of the Lemma follows from the symmetry in equation (12).

Proposition 2

The first part of the proposition was proven in Lemma 2. Recall the definition of $W(p,\beta)$ in (13):

$$W(p,\beta) = (1-\delta)\tilde{W}(p,\pi) + \delta L(p,\pi), \qquad \text{with} \qquad \pi = \alpha+\beta. \tag{21}$$

We first prove that the short-term component of the information value exhibits strategic complementarity, $\partial\tilde{W}(p,\pi)/\partial\pi > 0$, and then show that the long-term component can be ignored if δ is sufficiently small. The short-term component is given in (10) for any period t and is equal to

$$\tilde{V}_t = p_{t-1}\frac{p_t^+(1-p_t^+)^2(\pi_{t+1}^+)(\pi_{t+1}^+ + 2\pi^0)}{D_1(p_t^+,\pi_{t+1}^+)D_2(p_t^+,\pi_{t+1}^+)} + (1-p_{t-1})\frac{(p_t^-)^2(1-p_t^-)(\pi_{t+1}^-)(\pi_{t+1}^- + 2\pi^0)}{D_1(p_t^-,\pi_{t+1}^-)D_2(p_t^-,\pi_{t+1}^-)}.$$

We use the expressions of the ask and the bid in (7), set $\pi = \alpha+\beta$, $\pi_{t+1}^+ = \alpha$, $\pi_{t+1}^- = \alpha+\overline{\beta}$ because of the definition of $W(p,\beta)$, replace p_{t-1} by p, and omit the time subscript because there is no ambiguity:

$$\tilde{W}(p,\pi) = p^2(1-p)^2(\pi^0)^2\mathcal{H}(p,\pi), \qquad \text{with} \tag{22}$$

$$\mathcal{H}(p,\pi) = \frac{(\pi^0+\pi)\alpha(\alpha+2\pi^0)}{D_1^3(p,\pi)D_1(p^+,\alpha)D_2(p^+,\alpha)} + \frac{(\pi^0+\pi)(\alpha+\overline{\beta})(\alpha+\overline{\beta}+2\pi^0)}{D_2^3(p,\pi)D_1(p^-,\alpha+\overline{\beta})D_2(p^-,\alpha+\overline{\beta})}, \tag{23}$$

where $D_1(p,\pi) = \pi^0 + \pi p$ and $D_2 = \pi^0 + \pi(1-p)$ are the denominators in the equations (7) of the ask and the bid.

Let a be the first term in (23). Its derivative with respect to π is

$$a'_\pi = a\Big(\frac{1}{\pi^0+\pi} - \frac{3p}{\pi^0+\pi p} - \frac{\partial p^+}{\partial \pi}\big(\frac{\partial D_1}{\partial p^+}\frac{1}{D_1} + \frac{\partial D_2}{\partial p^+}\frac{1}{D_2}\big)\Big).$$

This expression is negative if $p > 1/2$, as befits the intuition: good news in period t increase the level of confidence and decrease the variability of the price in the next period. Taking the limit as $p \to 1$ and using $D_1(p,\pi) \to \pi^0+\pi$, $D_2(p,\pi) \to \pi^0$,

$$a'_\pi \to -\frac{2\alpha(\alpha+2\pi^0)}{(\pi^0+\pi)^3(\pi^0+\alpha)\pi^0}.$$

Likewise for the second term b in (23),

$$b'_\pi = b\Big(\frac{1}{\pi^0+\pi} - \frac{3(1-p)}{\pi^0+\pi(1-p)} - \frac{\partial p^-}{\partial \pi}\big(\frac{\partial D_1}{\partial p^-}\frac{1}{D_1} + \frac{\partial D_2}{\partial p^-}\frac{1}{D_2}\big)\Big) \to \frac{(\alpha+\bar{\beta})(\alpha+\overline{\beta}+2\pi^0)}{(\pi^0)^4(\pi^0+\alpha+\overline{\beta})}.$$

Combining the two previous expressions, if p tends to 1, then $\mathcal{H}_\pi(p,\pi) = a'_\pi + b'_\pi$ tends uniformly with respect to π to a limit $\lambda(\pi)$. Since $\pi \geq \alpha$,

$$\lambda(\pi) > \frac{(\alpha+\bar{\beta})(\alpha+\overline{\beta}+2\pi^0)}{(\pi^0)^4(\pi^0+\alpha+\overline{\beta})} - \frac{2\alpha(\alpha+2\pi^0)}{(\pi^0+\alpha)^4\pi^0}.$$

If $\alpha < \overline{\beta}/3$, there exists $\lambda_0 > 0$. Hence, there exists $\bar{p}$ such that if $p > \bar{p}$, then for all $p > \bar{p}$, $\mathcal{H}'_\pi > \lambda_0/2 > 0$. Because of (22), a similar inequality applies to $\tilde{W}'_\pi$.

Proposition 3

From the text that precedes Proposition 3, we have the following result.

Lemma 4

There exists $\bar{c}$ such that if $c < \bar{c}$, then the equation $\tilde{W}(\tilde{\phi}(\beta),\beta) = c$ has a unique solution for $\beta \in [0,\bar{\beta}]$, $\tilde{\phi}(\beta) \in [1/2,1]$ and we have the following properties:

(i) $\tilde{W}(p,\beta) > c$, *for* $p \in [1/2, \tilde{\phi}(\beta))$, $\tilde{W}(p,\beta) < c$ *for* $p \in (\tilde{\phi}(\beta), 1)$.

(ii) $\dfrac{\partial \tilde{W}(p,\beta)}{\partial \beta} > 0$ *if* $p \geq \tilde{\phi}(0)$.

Define $\tilde{p}_L = \phi(0)$, and $\tilde{p}_H = \tilde{\phi}(\bar{\beta})$. The functions $\tilde{W}(p, \beta)$ and $L(p, \beta)$ and their derivatives are continuous for $(p, \beta) \in (1/2, 1) \times [0, \bar{\beta}]$, therefore uniformly continuous on any compact subset of $(1/2, 1) \times [0, \bar{\beta}]$. There is an open interval $(\hat{p}', \hat{p})$ containing $[\tilde{p}_L, \tilde{p}_H]$ such that following Proposition 2, $\partial \tilde{W}(p, \beta)/\partial p$ has a strictly negative upper-bound and $\partial \tilde{W}(p, \beta)/\partial \beta$ has a strictly positive lower-bound for $\hat{p}' \leq p \leq \hat{p}$ and $0 \leq \beta \leq \bar{\beta}$. One can choose $\bar{\delta}$ such that (i) applies to the function W and (ii) applies also if p is not in an arbitrarily small interval that contains 1.

Proposition 4

Suppose first $p_{t-1} > p^*$. By definition of the strategy p^*, no information agent invests in period t. Consider the payoff of a deviating agent who invests. If after paying the cost c, he learns that $\theta = 1$, then he buys at $p_t^+ > p_{t-1} > p^*$. By the definition of the strategy p^*, no agent invests in the next period and $\pi_{t+1}^+ = 0$. We do not need to be concerned by the outcome if he learns that $\theta = 0$ because of the strategic complementarity from period $t + 1$ to period t: from 1, the payoff of investment in period t is not greater than if $\beta_{t+1} = \overline{\beta}$. The payoff of investment in period t is therefore bounded above by the payoff under zero-one expectations, $W(p_{t-1}, 0) < W(p_L, 0) = c$, using Proposition 3 and the definition of p_L in that proposition.

Suppose now that $1/2 \leq p_{t-1} < p^*$: any information agent invests in period t. We consider again a deviating agent who invests. If he learns that $\theta = 0$, he trades at the bid $p^-(p_{t-1}, \overline{\beta})$. Using the property of $\bar{p}$, $1/2 < p^-(p_{t-1}, \overline{\beta})$, and $\pi_{t+1}^- = \alpha + \overline{\beta}$. Using again Lemma 1, the payoff of investment in period t is now bounded *below* by the payoff under zero-one expectations, $W(p, \beta)$, which is strictly greater than c.

Proposition 5

To prove the result, one begins with a region of dominance. If s is sufficiently close to 1, for small σ the agent is sure that the price is close to 1 and the value of information is lower than the cost. There is a value s_1 such that any agent with a signal $s \in (s_1, 1)$ does not invest: information investment is dominated. The interval $(s_1, 1)$ is now extended by iterations to the left as illustrated in Figure 3. The next lemma which has a simple proof[17] will enable us to use zero-one expectations around the threshold

[17]Because $\alpha > 0$, the absolute value of the jump of the price after a buy or a sale is bounded below by a strictly positive number, say γ. It is sufficient to take $\hat{\sigma}_1 < \gamma/3$ and $\eta < \gamma/3$.

signal of an iteration.

Lemma 4
There exist $\hat{\sigma}_1$ and $\eta > 0$ such that if $\sigma < \hat{\sigma}_1$, then for any s_0 and s with $1/2 < s_0 < 1$, and $s \in (1 - s_0, s_0)$, an agent with signal s is sure that if he buys, (sells), the signal of any agent in the next period will be greater than $s + \eta$, (smaller than $s - \eta$).

Choose an arbitrary period T (which will be large in a sense defined below). In that period, information investment is dominated for $s \in (s_0, 1)$. Consider now an agent in period $T - 1$ with a signal $s = s_0$. Because of Lemma 4, that agent is sure that if he buys, any information agent in the next period has a signal higher than s_1 and therefore does not invest. If he sells, the price falls. Our agent may not know what investment will be in the next period T but because of the strategic complementarity from period T to period $T - 1$ (Lemma 1), he may assume that there is investment after a price fall in order to have an upper-bound on his value of investment. The value of information is therefore not greater than under zero-one expectations, and the payoff of information is bounded by the function $J(s, s_0)$.

Since J_σ is continuous and $J_\sigma(s_0, s_0) < c$, there is a value s_1 such that for any $s \in (s_1, s_0)$, $J(s, s_0) < c$ and $s_1 > s_0 - \eta$ where η is defined in Lemma 4. Since $J_\sigma(s, s_0) < c$, investment is dominated for an agent with signal $s \in (s_1, s_0)$ in period $T - 1$.

Repeating the argument for any period $T - k \geq 1$ and taking T arbitrarily large, we construct a sequence $\{s_k\}_{k \geq 1}$ that is decreasing and bounded below by s^*, and therefore converges. If it converges to $\hat{s}^* > s^*$, a small exercise which uses the continuity of $J_\sigma(s, s)$ shows that the differences $s_{k-1} - s_k$ are bounded below by a strictly positive number which brings a contradiction.

Hence, for any $s > s^*$, the information-investment is dominated after a finite number of iterations. This proves Part (i).

For Part (ii), we need to find an interval of "medium values" of the private signals such that information investment is a dominating strategy. Define M such that

$$M(p) = Min_{\beta \in [0, \bar{\beta}]} \Big(MaxV(p, \alpha + \beta, \alpha, \alpha) \Big). \tag{24}$$

Because of the continuity of the function V which is increasing with respect to its last two arguments (Lemma 1), $M(1/2) > 0$.

There exists $\bar{c}_1$ such that $\bar{c}_1 < M(p)$ for $p \in (1/2 - \eta', 1/2 + \eta')$ with some value of $\eta' > 0$. Take $\hat{c}$ in Proposition 5 to be the minimum of $\bar{c}_1$ and the value $\bar{c}$ defined in Proposition 3. Then there exists $\underline{s}_0 > 1/2$ such that an agent with signal $s \in [1-\underline{s}_0, \underline{s}_0]$ is sure that the value of information is higher than the cost c independently of the strategies of others. One then uses an iterative argument as in Part (ii) to generate an increasing sequence $\{\underline{s}_k\}_{k\geq 1}$ that converges to s^*, such that information investment is a dominating strategy for any interval $[1/2, \underline{s}_k)$.

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