EXERCISE 1.1. (Simple probability computation, searching for a wreck)

An airplane carrying "two blackboxes" crashes into the sea. It is estimated that each box survives (emits a detectable signal) with probability s. After the crash, a detector is passed over the area of the crash. (We assume that we are sure that the wreck is in the area). Previous tests have shown that if a box survives, its signal is captured by the detector with probability q.

1. Determine algebraically he probability p_D that the detector gets a signal. What is the numerical value of p_D for $s = 0.8$ and $q = 0.9$?

2. Assume that there are two distinct spots, A and B, where the wreck could be. Each has a *prior* probability of 1/2. A detector is flown over the areas. Because of conditions on the sea floor, it is estimated that if the wreck is in A, the detector finds it with probability 0.9 while if the wreck is in B, the probability of detection is only 0.5. The search actually produces no detection. What are the *ex post* probabilities for finding the wreck in A and B?

EXERCISE 1.2. (non symmetric binary signal)

There are two states of nature, θ_0 and θ_1 and a binary signal such that $P(s = \theta_i|\theta_i) = q_i$. Note that q_1 and q_0 are not equal.

1. Let $q_1 = 3/4$ and $q_0 = 1/4$. Does the signal provide information? In general what is the condition for the signal to be informative?

2. Find the condition on q_1 and q_0 such that $s = 1$ is good news about the state θ_1.

EXERCISE 1.3. (Bayes' rule with a continuum of states)

Assume that an agent undertakes a project which succeeds with probability θ, (fails with probability $1 - \theta$), where θ is drawn from a uniform distribution on $(0, 1)$.

1. Determine the *ex post* distribution of θ for the agent after the failure of the project.

2. Assume that the project is repeated and fails n consecutive times. The outcomes are independent with the same probability θ. Determine an algebraic expression for the density of θ of this agent. Discuss intuitively the property of this density.

EXERCISE 1.4. (composite binary signal)

There are two states of nature, 1 and 0. A person receives a binary signal on the state in two steps:

- First a precision, q, is drawn randomly such that $P(q = q_1) = \pi$, $P(q = q_2) = 1 - \pi$, where $1/2 \leq q_1 < q_2 < 1$ and π, q_1, q_2 are fixed and known to the agent. The agent does not know the value of of q.

- Second, the agent receives the symmetric binary signal s that is defined by $P(s = \theta|\theta) = q$.

Show that this process is equivalent to receiving a symmetric binary signal with precision r and compute r as a function of the previous parameters.

EXERCISE 1.5. (composite binary signal)

There is a continuum of agents with total mass one. Each agent is indexed with a parameter $q \in (0.5, 1)$ and the density of q is uniform on the interval $(0.5, 1)$.

There are two states of natures, 1 and 0. The prior distribution is that both states have the same probability 1/2. Each agent indexed by q receives a symmetric binary signal with precision q. Hence, you may assume that if $\theta = 1$, within a small interval around q, a fraction q of agents receives the signal 1 and a fraction $1 - q$ receives the signal 0.

1. This process determines a distribution of private beliefs: for each state, there is a distribution of agents according to their belief, that is, their probability of the state 1. That distribution is characterized by a density function. Determine the density functions of the distribution of beliefs for each of the two states 1 and 0.
2. Are the distributions of beliefs bounded or unbounded?
3. (optional) Determine the density functions of the beliefs measured by the LLRs (Log likelihood ratios).

EXERCISE 1.6. (distribution of signals

We start from the "canonical' model of social learning. There are two possible state of nature, characterized by a value θ that is either θ_0 or θ_1, with $\theta_1 > \theta_0$. The probability that $\theta = \theta_1$ (the high state) at the beginning of the process, for all agents (public information), is μ_1.

Each agent i is taking an action x_i in round i, and in round i only such that x_i is equal to 0 (no investment) or 1 (investment). The payoff of no investment is 0. The payoff of investment is $\theta - c$ where c is a fixed cost parameter strictly between θ_0 and θ_1. Each agent takes the action that maximizes the expected payoff given his available information. Recall that actions are publicly observable and that private signals are not.

Without loss of generality, we can assume that $\theta_0 = 0, \theta_1 = 1, c = 1/2$. (Feel free to keep the general notation if you think that it adds generality to the argument).

The private signals are different from those in the canonical model. Each agent i is endowed with a private signal s_i that is symmetric and binary such that $P(s_i = \theta|\theta) = q_i$. The value of q_i is drawn from a uniform distribution on the segment $[1/2, 1]$. Each agent knows his q_i and s_i. The random drawings of q_i and s_i are independent, conditional on the state of nature.

1. Show that a cascade cannot take place in this model. You do not need algebra. An intuitive argument is sufficient.

2. Are there herds in this model? (short but justified answer).

3. (more technical and some of you may skip that question) Let μ be the public belief on the high state in some round. Suppose that the agent in that round invests (taking action 1). Determine the public belief μ' after the observation of that action.

4. (requires MATLAB) Simulate the previous model for N round, with $N = 200$ or some other value. Represent the evolution of the public belief and comment on the results. You may run the simulation a few times.

5. Assume now that each individual signal is observable. Compare through simulations, the convergence of the public belief in this case with the previous one. Comment.