Chapter 7

Coordination with sequential information

Sequential information

You have a tip that the booming stock market may collapse, or that a pharmaceutical firm that is about to make a discovery, or that your bank is in trouble, or that your country may devaluate its currency with respect to others. The payoff of your response depends on your information relative to that of others. If you are the first to know about a stock market crash, you may enjoy the ride for a while. But if it turns out that you were one of the last to know, the crash may fall on you. Likewise, if the bank provides convenient services of returns to you, you may take the risk of waiting for a while before pulling your deposit out.

When the payoff of your action depends on the difference between your information and that of others, an important issue is **when** you receive your information relative to others. A central issue in this chapter is that people do not receive information at the same time and they do not know when others receive their information. Suppose that an event takes place randomly and that all people become informed about that event within a time interval T. When you become informed, you can say "I know" the event E, but you cannot say "I know that everyone knows" E. However, if you wait for an amount of time T, you can say "I know that everyone knows" R." You cannot say "I know that everyone knows that everyone knows". To say this, you have to wait for 2. The more times you want to repeat "that everyone knows," the longer you have to wait. But, at this is essential, no matter how long you wait, you repeat "that everyone knows" only a finite number of times. There is never *common knowledge* about the event E.

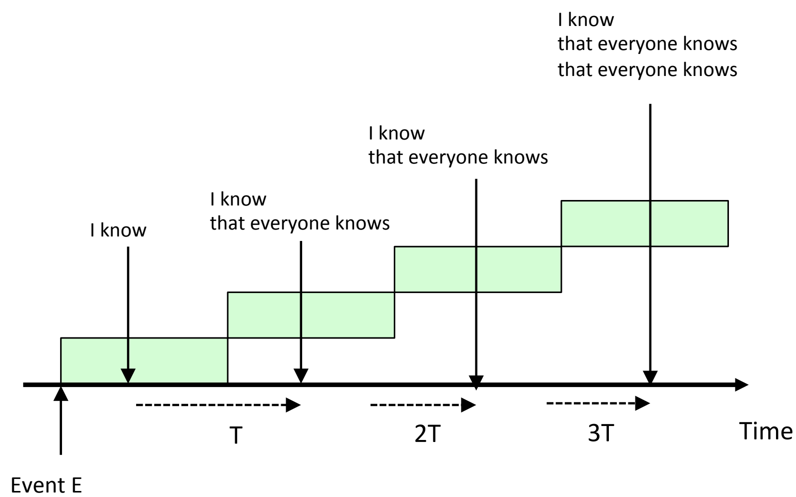


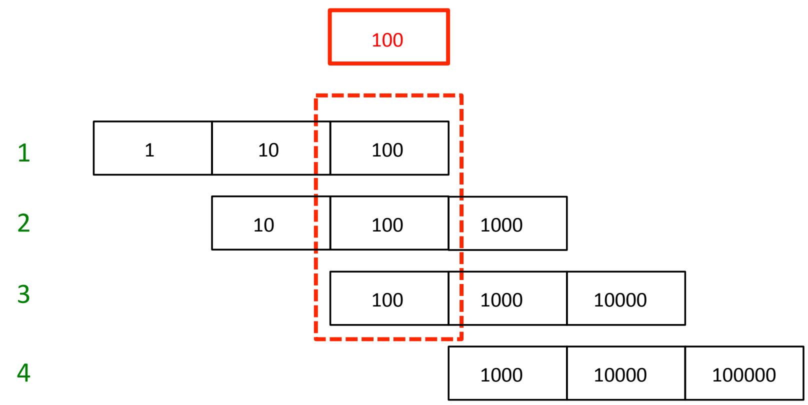
Figure 7.1: Lack of common knowledge

Abreu and Brunnermeier (2003) have provided a framework of analysis of the process of *sequential information*, in which agents do not know where they stand in the queue of information. When they learn about the news, they speculate rationally that not everyone is informed and they delay their action. But each agent knowing that others delay, will delay more. The "ride" may go on while all agents are informed about the news, all know that all are informed, all know that all know that,... The model of Abreu and Brunnermeier has been used by Rochon (2006) for the analysis of the end of a fixed exchange rate regime. That model is, without loss of rigor, much simpler. It is adapted in Section *** to present a canonical model that is illustrated by a fictitious bank run. The models of Abreu and Brunnermeier and of Rochon are discussed in the other sections of the chapter.

7.1 A game with sequential information

Play the following game with three people. Consider the four possible sequences for the price of an asset in Figure 7.2. You as the master of the game choose randomly one of the sequences and the players do not know which sequence has been drawn. Suppose it is sequence 2. Write on three identical cards, the numbers 10, 100, 1000, give a card to each player who cannot reveal to other the content of his card.

There is a player with the card 100. This player does not whether he is has the low, the middle or the high price.



If you face a price of 100, where are you in the sequence, first, middle or last?

Figure 7.2: Sequential information

The players are now informed that the number in the card represents the price of an asset and that each player has a wealth of 1$. Given a randomly chosen sequence, say sequence # 2, player "10" is offered the opportunity to invest his wealth in the asset, that is to buy 1/10 of the asset. In the next round, if another player buys, then the asset is worth 100 and the value of the first player's investment is multiplied by 10. If there is no buyer, the first player looses the investment.

For another player to buy in the next round, two conditions must be satisfied. First, there must be another player–there is one in sequence # 2–, and that player must decide to buy. His trade is subject to the same conditions as the first player: in this case he would buy 1/100 of the asset and sell it at 1000 if there is a buyer in the next round.

What is a rational equilibrium in this game?

- If a player gets the card "100,000", given the four possibilities in Figure 7.2, he is sure to be the last one and therefore does not buy.
- If a player gets the card "1,000", he knows that he cannot sell in the next round and therefore he does not buy.
- By backward induction, no rational player should buy.

We could extend the sequences of price to any sequence $\{10^k, 10^{k+1}, 10^{k+2}\}$, with $k \leq K$.

As long as K is finite, and there is a *cap* on the prices, the backward induction argument shows that the only equilibrium with rational players is that no one buys the asset.

In the game with rational players, every player assumes that other players are rational and these other players assume that other players are rational, etc... There is common knowledge about the rational behavior.

Exercise

Suppose now that in the game that is represented in Figure 7.2. Recall that a price sequence is chosen randomly and that the probabilities of the four sequences are the same. Suppose in addition that there is a mix of non rational players in the sense that there is a probability $1/2$ that one of the three players, and only one is buying the asset in his round, for any price on his card. You are a rational player in this game who is risk neutral. Assume that you have the price 10,000.

1. Show that if you buy, your expected payoff is greater than1.
2. What is your strategy for prices lower than 10,000?

The results of Moinas and Pouget

The previous game has been constructed by Moinas and Pouget. Figure 7.3 presents their summary of experimental results for different values of K.

A game with unbounded prices

Assume now the sequence of prices $\{10^k, 10^{k+1}, 10^{k+2}\}$ is chosen such that for any k, conditional on no sequence before, the probability of starting a sequence at 10^k is $\lambda < 1$.

1. Compute for any price, the probability to be the last one in the sequence.
2. Assuming that the probability of an irrational player buying at any price is equal to α, determine the expected payoff of buying.

7.2 Introduction

On the floor of the stock exchange, the Dow Jones suddenly falls by 1000 points: all the agents on the floor see the event of the crash. All agents can say, "I know that the other agents know there is a crash." Furthermore, each can say "I know that all the agents know

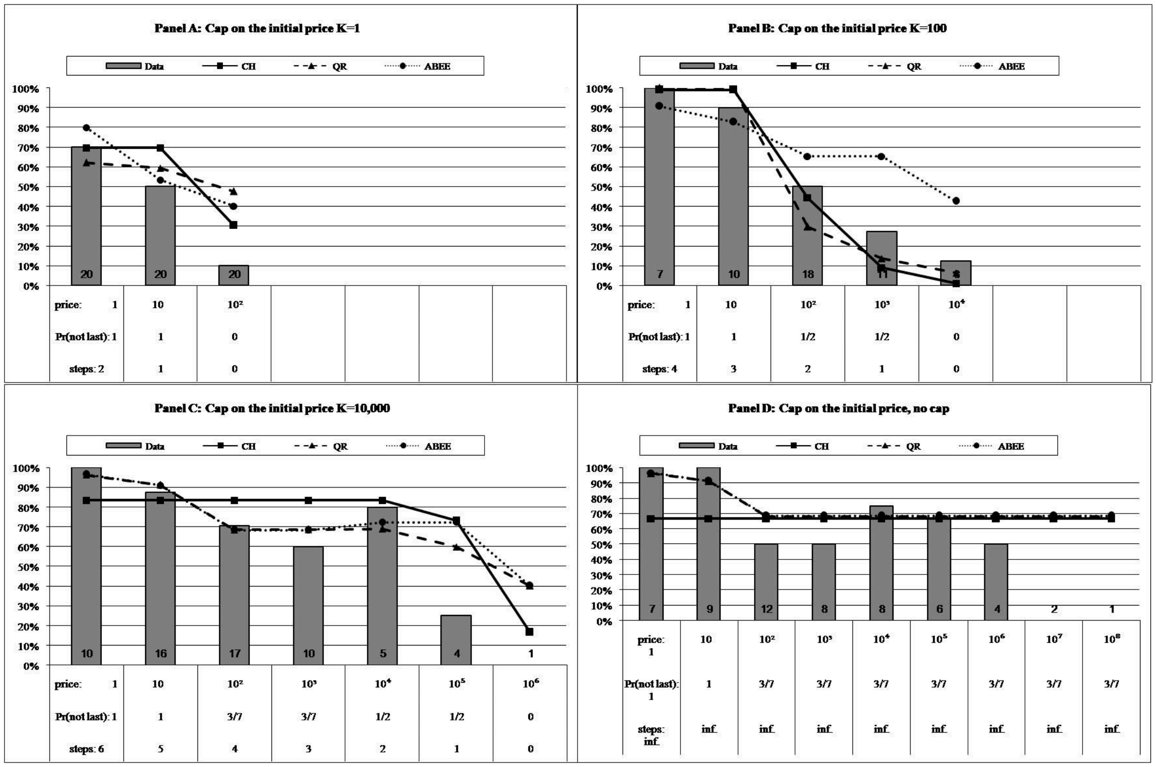


Figure 7.3: Experimental results of Moinas and Pouget

that all the agents that there is crash." Everyone can repeat an arbitrary number of times "all the agents know that...". There is common knowledge about the crash.

Consider the reverse case of a bubble of the stock market, that is a continuing rise without a stream of good news.[1] Why do people (like Newton) stay in a bubble (with an investment) when they think that there is a bubble and later on a crash. One plausible explanation is that they think that there are better informed than other people and that they will be able to cash in and get out before others do so and precipitate a crash. In such a situation the awareness of a bubble is not a common knowledge.

[1]Shelves and shelves of books and articles have been written on bubbles, and this definition here should serve as a tool for thinking about common knowledge.

7.3 Bubbles and crashes

Bubbles and crashes have existed since the very beginning of financial markets, in the 17th century Netherlands with tulip maniae, in France (1719) with the spectacular failure of the Law system, in England (1720) with the South Sea Bubble. First, let us review bubbles under common knowledge.

7.3.1 Definitions: rational and irrational bubbles

There does not seem to be a clear unique definition of a bubble. In common language, the "dot-com" or the housing bubble is one type of bubble. Such a bubble seems also to be associated with unsustainability and crashes. A simple idea about a "bubble asset" is that if provides no real good or income but it has a resale value and people hold it *only* for the resale value. (Any asset has a price that depends on the expected resale value). That is a general definition but it does not necessarily capture the previous idea of unsustainability and crash. One asset that fits the previous definition is money in the OG model of Samuelson (1958). Money provides no dividend income and people hold it only for its future value. Quite a few recent papers take that definition. In some equilibria of the Samuelson model, the price of money may expand, shrink or fluctuate, but the most interest equilibrium is with a stable price. That type of bubble obviously does not capture the type of bubbles with crashes that were mentioned initially. Money can have a price with no crash in the Samuelson model. In the models of the dot.com type, the crash seems to be a necessary outcome, with an uncertain timing.

Let us consider an economy where there is an asset that provides a rate of return r per unit of time (which is continuous). The value of r is constant but does not need to be positive. Assume that the return of the asset is in real income, as capital (which may have a negative return if it depreciates). That assumption is actually not required for the discussion but it is useful. Assume now that there is another asset, called the "bubble", in a fixed quantity (which can be normalized to one). The bubble pays no dividend and its price (per unit) is p_t.

There is no uncertainty in the economy. Assume first that p_t is a continuous function of time (no jump). The rate of return on the bubble is $\dot{p}_t/p_t$. By no-arbitrage, this rate must be equal to r:

$$\dot{p}_t = rp_t. \tag{7.1}$$

Hence,

$$p_t = p_0 e^{rt}. \tag{7.2}$$

Assume now that the rate of growth of the economy (output, or population plus productivity) is constant and equal to g. (One could also assume that it varies but is bounded by $\bar{g} \geq 0$). If $g > r$, then the equation (7.2) is admissible. The bubble can go on forever without its value exceeding that of output. If $r > g$, the previous equation cannot hold forever. This very simple observation helps to separate the types of bubbles that were discussed previously.

1. In the Samuelson model (there is no capital, but that is irrelevant here), the rate of return on money does not exceed the growth rate of the economy. There can be an exponential growth, at the same rate as the economy: if population growth at the rate n and the (nominal) quantity of money is fixed, there is an equilibrium with a constant real quantity and deflation at minus the population growth rate. If there is capital and the rate of return on capital, r, exceeds the population growth rate, n, then the accumulation of capital is inefficient (excessive), and money (the bubble) can have a value. Actually, the introduction of money (with a return n) may bring the economy back to an efficient allocation. In this type of bubble, there is no necessary crash. There can be growth with perfect foresight! This type of bubble has sometimes be called a *rational bubble*. However, that is not the type of bubble that people usually have in mind. Historic bubbles do not have growth rates of a few percent annually with no expectation of a crash.

2. If $r > n$, then the equation (7.2) is not sustainable. The rate of growth of the bubble price is such that if the bubble goes on, its value will eventually dwarf the economy. Suppose that the economy is stationary. In that case there is a ceiling on the price of the bubble. But if the price reaches the ceiling, it cannot grow more and it is that growth that sustains the price. Without growth, the price is not sustainable. Then it is not sustainable before it reaches the ceiling. By backward induction, the (positive) price of the bubble is not sustainable at the beginning of time. This type of bubble may be called an *irrational bubble*. We now focus on this type of bubble.

7.3.2 Boundedness and rational backward induction

In order to include the possibility of a bubble crash, one may consider the simple representation in the model of Blanchard (1979) where a crash occurs according to a Poisson process with parameter λ. The no-arbitrage equation (7.1) is now replaced by[2]

$$rp_t = \dot{p}_t - \lambda p_t. \tag{7.3}$$

The probability that the bubble goes on until time t is now $e^{-\lambda t}$, and it tends to zero. The bubble ends surely with a crash to the fundamental value which is zero. However, as long

[2]This equation can also be obtained by a discrete model with vanishingly short period length dt.

as the bubble goes on, its price is determined by the equation

$$p_t = p_o e^{(r+\lambda)t},$$

and the expected value of the price in period t is $p_0 e^{rt}$ exactly as in equation (7.2), which is not surprising: the discounted value of the price is a martingale. The previous remark on the effect of a ceiling on the possibility of a bubble is even more relevant for this model since the bubble grows faster to compensate for the risk of the crash.

We now turn to the model without common knowledge of Abreu and Brunnermeier (2003) who developed the method of sequential information that was presented in the previous section. That model is still subject to criticisms against rational bubbles (e.g. the ceiling problem), but it shows how a bubble can go while all agents are aware that it is a bubble but do not have common knowledge on what others think.

7.4 Bubble without common knowledge in the model of Abreu and Brunnermeier

One representation of this information process is due to Abreu and Brunnermeier (**), hereafter AB, and it is illustrated in Figure 7.12. A bubble, or any event (in the figure that is the beginning of an outflow of reserves at a bank) occurs at the time θ. Since people are imperfectly aware of that event, it must occur as a random event. In the AB model, the event is generated by a Poisson process with one occurrence of the event: time is continuous and conditional on not have occurred at time t, it occurs in the time interval $(t, t + dt)$, where $dt$is small, with probability λdt where λ is some fixed parameter in the model. That implies that if time starts at 0, the probability that the event has **not** taken place at time t is $e^{-\lambda t}$. The distribution of the time of the occurrence of this unique event is exponential.

The time of the event, θ, plays the same role at the state of nature in previous chapter. How do people become informed about that state of nature? For simplicity, we assume that there a large mass of people, a continuum of total mass 1 and that people become gradually informed after the time θ. Beginning at time θ, there is a constant flow of newly informed people: for each interval of time dt, the mass of newly informed people is equal to σdt where σ is a fixed parameter. This process takes place until all people are informed. The allocation of the timing for information is random. When a person becomes informed, she is told “a bubble is on”, but she does not know whether she was among the first to know or among the last. That property is the central assumption in the AB model. People don’t know where they stand in the sequence. As in the previous Bayesian models, people know the setting and the parameters of the model.

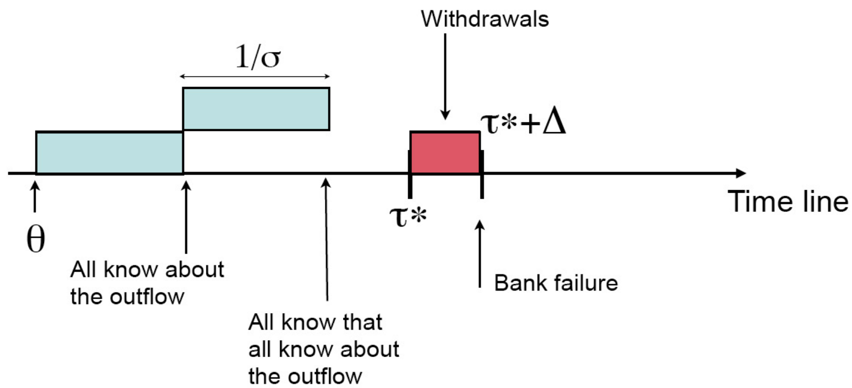


Figure 7.4: Time line of events

From this setting, the time it takes for all people to be informed is $1/\sigma$. At time $\theta+1/\sigma$, all persons are informed. The person that has been informed at time θ, seeing that the time $1/\sigma$ has passed since she received the information can say "I know that all know about the event." But the person who is informed at time $\theta + 1/\sigma$ is just newly informed and cannot say that sentence. It is only at the instant $\theta + 2/\sigma$ that she can say the sentence. At instant, all the agents can say "I know that all know about the event." And so on...

Riding the bubble and the eventual crash

In the AB model, the bubble is on, say, a stock price, and people stay for a while in the bubble after being informed. They sell their stock (everyone has a fixed amount of the stock) after some time interval τ. Remember that any person's information is only the time t when she was informed (not θ!). Hence, the only strategy is with respect to the time of information t. Everyone uses the same lag τ between information and selling. Therefore, beginning at time $\theta + \tau$ when the first one informed sells, there is a constant flow of sales. That is represented in the figure by the red rectangle. When the stock of sales reaches some critical level, there is a sudden crash. If the critical mass is M, the time interval of the sales is Δ such that $\sigma\Delta = M$.

The AB model is illustrated by Figure 7.5. There is an asset that pays no dividend, for simplicity. (Dividends could be paid after the resolution of the bubble). By the no-arbitrage condition, the price of this asset should grow at the risk-free rate r. It is assumed here that in some time interval $[0, \theta]$, the fundamental price of the asset grows at the rate $g > r$. During this regime, there is no bubble. The instant θ which occurs according to a Poisson process, with one occurrence only, with parameter λ, determines the beginning of the bubble. From that instant on, the fundamental value of the asset grows that the rate r, while the price of the asset keeps growing at the same rate g under some conditions specified below. The price of the asset is observed at any instant by all agents.

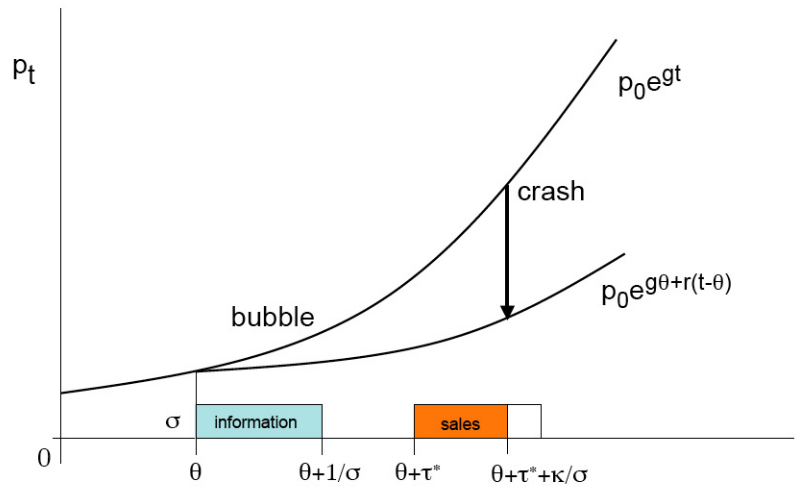


Figure 7.5: The sequence of bubble, information, sales and crash

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As mentioned previously, there is a continuum of agents of mass one. At the time θ, agents become gradually informed that a bubble has started. They receive their information according to the same process as in Section **??** with a constant flow σ. Each agent holds one (infinitesimal) unit of the asset and can sell at any time.

Two conditions have to be met for the bubble to keep going at time $t > \theta$.

- First, $t < \theta + T$ for some value T. The justification is that after θ the gap between the market the fundamental price grows exponentially. When this gap becomes too wide, a crash takes place, exogenously.
- Second, the amount of sales of the assets by agents cannot exceed a value $\kappa < 1$ which is exogenous. Note that sales may take place during the bubble, but as long as the cumulated sales do not reach κ, the price keeps rising at the steady rate g. When the critical mass of sales, κ, is reached, the last cent of sales precipitates the crash.

Remarks

The paper of Abreu and Brunnermeier is the seminal paper for this chapter, but (i) it is a very difficult reading, unnecessarily so; (ii) the application of the method to an asset bubble is not appropriate, for at least three reasons.

- First, in the model, the bubble price is untouched by the sales of informed agents until the last cent that reaches the critical mass for a crash and then the price collapses to the fundamental. Clearly, the price may be affected by informed sales, but it is impossible to put this in the model, at least in its present form.

- Second, the initial regime at a growth rate higher than that of the market is *ad hoc* and needs some explanation. Prospects of higher future profits should be factored in the market price immediately. A regime of price increases at higher than market rate can only fit with a sequence of surprises with good news. There is no such interpretation in the model.

- Finally, we have seen that rational bubble under common knowledge cannot exist because of the ceiling on the growth of the price. The same objection applies to the model of Abreu and Brunnermeier if the growth rate g is higher than the growth rate of the economy, as it should be to have some relevance with actual bubble. Hence the model of Abreu and Brunnermeier cannot give an explanation of bubbles that is compatible with rationality.

7.5 An experiment (Moinas and Pouget)

The central feature in the AB setting is that agent do not know their position in a sequence of information. That feature is investigated by Moinas and Pouget in a experimental setting. They construct a game where any agent is placed randomly in a sequence of three people, with equal probabilities to be the first, second and third. The agent has one dollar and has the option to use it to buy a share of an asset at a price q that is taken randomly from a sequence $\{q_n = 10^n\}$ with geometric probabilities $P(n = j) = (1/2)^{j+1}$.

If the agent buys at the price P , then there are two possibilities.

- If he is not the last in his group of 3, he will be matched (through a computer) with the next agent in the queue and the transaction price will be $10q$. In this case, the next agent is proposed a similar option and so on.

- If the agent is the last one (with probability 1/3), then there is no opportunity to sell and the agent has lost one dollar.

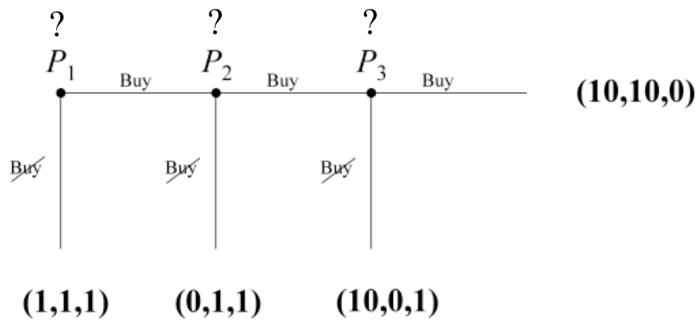


Figure 7.6: Time line of events

Payoff

The payoff of buying is

$$V = (1 - P(\text{last})) \times P(\text{new trader buy}) \times 10, \tag{7.4}$$

where P denotes a probability. He should buy if $V > 1$.

The cap

There is a cap K on the first price (and therefore a cap $100K$ on the last price), where K may be infinite or finite. If there is a cap, by backward induction there is no rational bubble: no agent should buy.

7.6 Financial Crisis Dynamics

Suppose that in a world with no deposit insurance, you learn that the bank you relied on for a special relation is in trouble. Would you withdraw immediately your deposits? Probably not: you don't expect an eventual failure to take place immediately and may want to "enjoy the ride" for a little longer. Likewise when the stock market is going up at a fast rate and you know that a "market correction" will take place. Or in a country with a fixed exchange rate like Argentina recently, people wait to transfer their assets abroad. In all these cases, the crash is caused or accelerated by the actions of individuals (withdrawing deposits, selling stocks or currency) that exhibit strategic complementarity. These actions have an effect when they are coordinated, but each individual upon learning that a crash may take place will think "I am learning about this event, but it probably will take some time before a large number of people become aware of the shock and even more time before they take action". The information on the event that a crash may take place is not conveyed (albeit imperfectly) to all agents simultaneously. Some agents learn early, others later.

Abreu and Brunnermeier (2003) have highlighted that as a consequence of *sequential information*, agents do not know where they stand in the queue of information. When they learn about the news, they speculate rationally that not everyone is informed and they delay their action. But each agent knowing that others delay, will delay more. The "ride" may go on while all agents are informed about the news, all know that all are informed, all know that all know that,.... However, one can repeat the segment "all agents know that" only a finite number of times.

The model of Abreu and Brunnermeier has been used by Rochon (2006) for the analysis of the end of a fixed exchange rate regime. That model is, without loss of rigor, much simpler. It is adapted here to present a canonical model that is illustrated by a fictitious bank run. The models of Abreu and Brunnermeier and of Rochon are discussed in the other sections of the chapter.

A canonical model of a banking crisis (or fixed exchange rate regime crisis)

We consider a bank that fixes the exchange rate between its deposits and a currency. That may be a commercial bank (with the domestic currency), or a central bank (with a foreign currency). To fix the narrative, take the case of a central bank with a regime of fixed exchange rate between the domestic currency, pesos, and the foreign currency (dollars) is fixed and normalized to 1. (Think of Argentina before December 2001). The regime is possible because the central bank trades pesos for dollars at that exchange rate. An exchange rate crisis arises when the demand for dollars in exchange of pesos exceeds the reserves of the central bank (actual or possibly borrowed). Pesos are like deposits at the central bank. An exchange rate crisis is like a run on the central bank. When the central bank has no more reserves (or before that), it changes the exchange rate and devalues the peso with respect to the dollar. The issue and the analysis is the same for a run on a commercial bank that maintains a fixed exchange rate of one between its deposits and the currency.

Time is continuous[3]. At the beginning of time, the bank has reserves $\bar{R} > 1$. There is a continuum of agents of mass one. These agents are more active than other depositors in the sense that they maximize their instantaneous return and can withdraw their deposit at any time. Each agent is wealth constrained and that constraint is normalized such that each agent has a deposit between 0 and 1. Because the agent maximizes the instantaneous rate of return, his action will be at the corner and equal to 1 or 0. We can therefore assume that each agent has a deposit of 1 or 0. The market rate of return is zero and in order to attract the agents deposits, the bank is paying a return of r per unit of time (which is continuous) on deposits[4].

At some random time θ, there is a regime switch: the bank's assets begin to decrease through an *exogenous* constant flow a. As in all the models with imperfect information, we begin with the case of perfect information where all agents observe θ and each other.

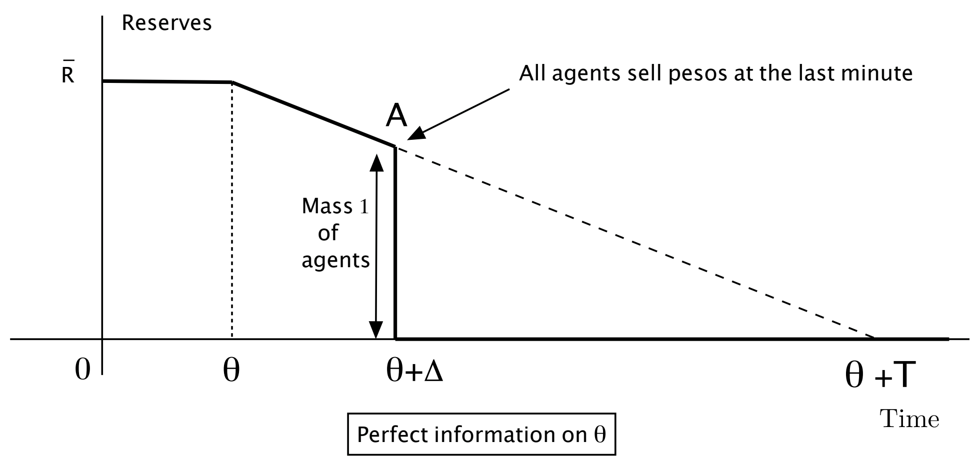
Common knowledge

As often in this kind of discussion, we begin with the benchmark of perfect information: when θ occurs, that value is to be seen perfectly by all agents. The dynamics of the reserves are represented in Figure 7.7 where $\bar{R} = aT = 1 + a\Delta$.

Agents know θ and can predict the future. Since the reserves exceed 1 for a while, they can confidently keep the return r on pesos for an interval of time $\Delta = (\bar{R} - 1)/a$. At

[3] Continuous time is a must in models of sequential information.

[4] In the context of a central bank, the domestic rate of return is higher than the international rate.



$\bar{R} = aT = 1 + a\Delta$.

Figure 7.7: Time profiles of reserves with perfect information

the time $\theta + \Delta$, the reserves of the central bank are exactly equal to 1. (They have been depleted from $\bar{R}$ to that level solely by the exogenous outflow). At that instant, all agents demand dollars for pesos (or deposits). The bank has no reserves after that and devalues immediately after the rush. Any delay beyond Δ is strictly suboptimal for an agent. The model with perfect information is a benchmark. It does not take into account what is essential in a speculative attack, timing and information. Note that this model is a stylization of the model of speculative attack of Krugman (1979). In that model, currency crisis occur according to perfect foresight and there is actually no devaluation.

No common knowledge and sequential private signals

As in most models of Bayesian learning, the structure of information has two parts, first the generation of the state of nature θ, second the individual information on θ.

1. The exogenous run (the event θ) starts according to a Poisson process with parameter λ. That assumption provides stationarity. If the instantaneous probability of the event at time t, conditional on no run before, would depend on t, the time an agent is informed would provide him some information on θ.

2. When θ occurs, speculators are informed sequentially. Beginning at the time θ, there is a flow of newly informed agents that is constant per unit of time, and equal to $1/\sigma$

(where σ is a parameter), until all agents are informed. It takes a time interval of length σ for all agents to be informed. The time of information (after θ) of an agent is determined randomly. The information is only *"the exogenous run is on"*. Agents **do not know** the time θ when the run has started. Knowing θ could be equivalent to perfect information. The assumption, which is due to Abreu and Brunnermeier (2003) is a clever way to model the sequential information: each agent does not know where he stands in the queue of informed agents. He could be one of the first to be informed or one of the last. Nothing that he knows provides information on that ranking. That is one of the very nice features of the assumption.

An overview of the equilibrium

Given the two previous assumptions, an agent can only base his strategy on the time he was informed. His strategy is his delay between the time he is informed and the time he withdraw his deposit and gets dollars. Because of the Poisson distribution of θ, the information received at any time t is the same as the information at time t'. One can therefore assume that all agents have the same delay, and that the equilibrium (to be defined) is symmetric.

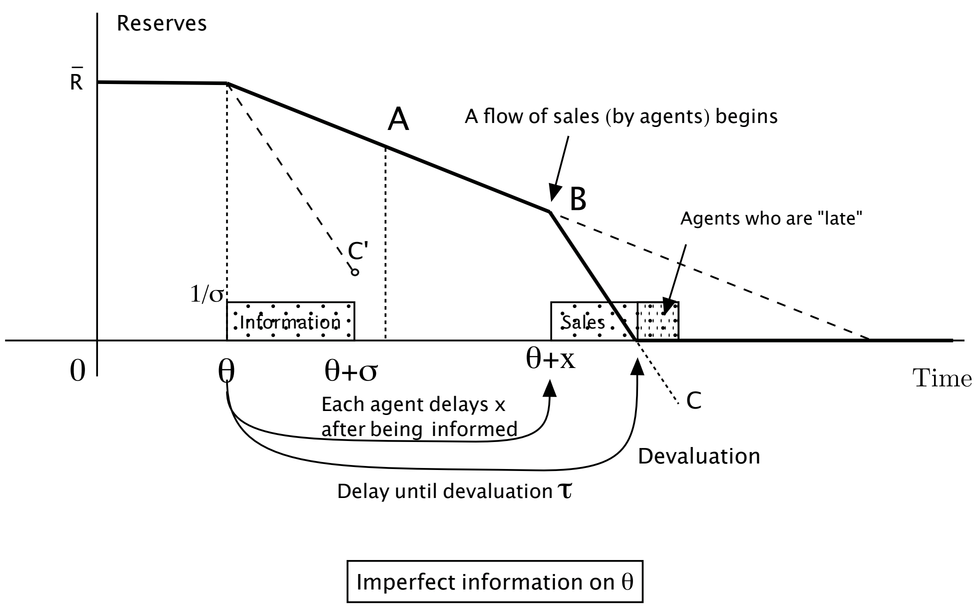


Figure 7.8: Time profiles of reserves with imperfect information

Assume that all agents but one have a delay x. Given x, there is an optimal delay $y = \mathcal{R}(x)$ for our deviating agent. The shape of the reaction function $\mathcal{R}(x)$ is represented in Figure

7.9 and it follows from the following remarks:

- The function $\mathcal{R}(x)$ is continous.
- $\mathcal{R}(T) < T$. If all agents delay T, then they may as well delay forever since the exogenous drain exhaust reserves after a length of time T. The deviating agent should delay strictly less than T. If not devaluation has occurred at $T - \epsilon$, the agent knows that it is extremely likely the devaluation will take place within ϵ (and also that he was one of the first to be informed). He should sell the currency to take advantage of the devaluation[5]
- $0 < \mathcal{R}(x) < 1$. If other agents increase their delay, then the deviating agent can delay longer, but not one for one because the exogenous drain reduces the reserves over time. There is strategic complementarity between the agents.
- The inequality $\mathcal{R}(0) > 0$ means that if other agents sell pesos as soon as they are informed, then it still pays to delay for some strictly positive time. That property will depend on the parameter assumptions. We can guess that it will be true if the initial level of reserves $\bar{R}$ is sufficiently large. For example, the worst possible event for our deviating agent is that he is the last informed. In this case, all other agents have sold. Reserves are strictly positive in which case it pays to wait a little if
$$1 + \sigma a < \bar{R}. \tag{7.5}$$
On the left hand side we have the sales of the agents plus the exogenous drain during the time it takes to inform all agents. On Figure 7.8, the point C' represents the level of reserves at when the last agent is informed and all other agents have been informed and bought dollars.

The previous properties could be obtained by intuition, without algebra. We could do more and see how the unique equilibrium could be a *Strongly Rationalizable Equilibrium*: any delay different from the equilibrium delay can be eliminated in a finite number of steps by the process of iterative elimination of dominated strategies. We will have that discussion after the solution of the model.

The equilibrium without common knowledge

The solution of the equilibrium goes through the determination of the reaction function. Assume that all agents sell the domestic currency (or withdraw their deposit) with a delay

[5] To be picky, the expected capital gain per unit of time if he is in dollars goes to infinity near T while the expected capital gain in pesos stays at r.

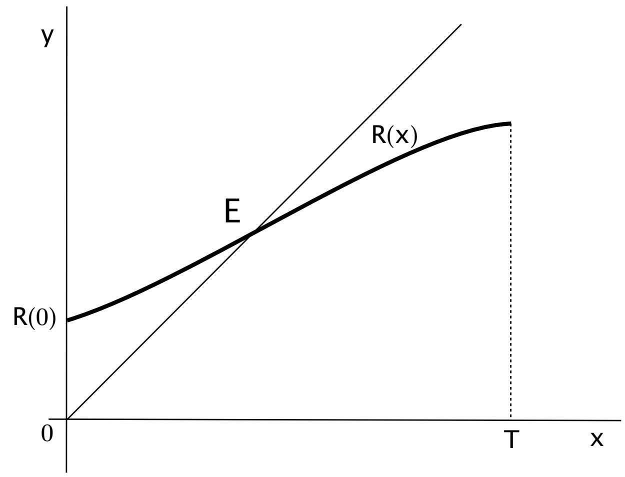


Figure 7.9: Reaction function

x after the instant they become informed. We can compute the delay of the crash, *i.e.*, the time interval τ between the event θ and the crash. It is a function $\tau(x)$ which is, intuitively, increasing in x. We can now compute this function now, using the mechanics of the model.

The delay of the crash

There is a crash when the total purchase of dollars equals the initial reserves $\bar{R}$, that is when $\dfrac{\bar{R}}{T}\tau + \dfrac{\tau - x}{\sigma} = \bar{R}$, which is equivalent to

$$\tau(x) = \frac{x + \sigma\bar{R}}{1 + \sigma\dfrac{\bar{R}}{T}}. \tag{7.6}$$

The function $\tau(x)$ is linear in x (in this particular model). Since agents can compute the model, the value of τ is common knowledge.

This step is similar to the first step in the one period global game where the threshold value θ^* is computed as a function of the agents' strategy s^*, where τ and x play the roles of θ^* and s^*.

The optimal strategy and the reaction function

Consider an agent who delays $y \neq x$, and knows that other agents delay x. We now compute the optimal response to the strategy x of others. For our agent, the instantaneous probability of a devaluation (crash) at the time y after his information is a function $\pi(y;x)$. It is intuitively an increasing function of y and a decreasing function of x. (Think about it). The optimal response, the reaction function (x), will be given by the no-arbitrage equation

$$\beta\pi(\mathcal{R}(x);x) = r. \tag{7.7}$$

If $\pi(\cdot;\cdot)$ is increasing in its first argument, when y is smaller (greater) than $\mathcal{R}(x)$, the return on peso or deposit net of the probability of a devaluation is greater (smaller) than on dollars or the currency.

We therefore have to determine $\pi(y;x)$ which is the probability per unit of time of a crash in the next small time interval, conditional on no crash so far. When all agents delay x, the flow of purchases of dollars is identical to the flow of information translated in time by the delay, x, as shown in Figure 7.8. We have already computed the delay $\tau(x)$, or τ for short, between the event θ and the crash.

When an agent becomes informed at time t, the support of his belief on θ is the interval $(t-\sigma,t)$. The next argument is in two steps.

1. If he delays y, and a crash has not occurred yet, the support of his belief is the interval $I=[t+y-\tau,t]$, of length $\tau-y$. Indeed, θ cannot be left of that interval, because a crash would have occurred before in that case (τ is the lag between θ and the crash), and the information at time t that θ has occurred means that $\theta\leq t$. On that support, by Bayesian inference, the density of θ at the instant $t'\in I$ for an agent informed at time t is $g(t';t)=Ae^{-\lambda(t'-t)}$, where A is a constant. This property follows from the Poisson distribution of θ and the uniform distribution of the private information. The value of A is such that the integral of the density is equal to 1. Hence,

$$A = \frac{\lambda}{1-e^{-\lambda(\tau-y)}}. \tag{7.8}$$

2. If the crash has not occurred yet, and it about to happen in the next time interval dt, then the time interval between now, $t+y$ and θ must be τ, by definition of τ. Hence, θ must be at the lower bound of the support of the distribution $[t+y-\tau,t]$. The probability of an imminent crash, in the next time interval dt, is therefore Adt. Using (7.8), the instantaneous probability of an imminent crash is

$$\pi(y;x) = \frac{\lambda}{1-e^{-\lambda(\tau(x)-y)}},$$

The graph of the function $\pi(y;x)$ for given x is represented in Figure 7.10.

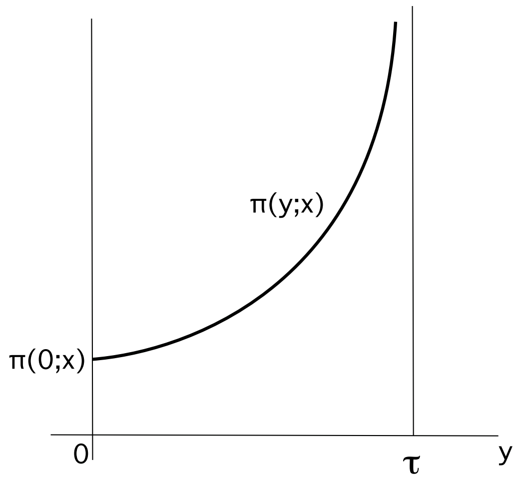


Figure 7.10: Instantaneous probability of a crash

As we guessed by intuition, for fixed delay of others, x, the instantaneous probability increases with one's delay y. When y approaches τ, that probability goes to infinity. Indeed, as y increases and a crash has not occurred yet, the informed agents know that there are fewer and fewer informed agents who have not sold earlier. As mentioned previously, if you wait τ, the length of time between θ and the crash, you know that (i) you have become informed immediately after θ and (ii) the crash is about to happen with probability one.

The no arbitrage equation (7.7) can now be written[6]

$$\pi(y;x) = \frac{\lambda}{1-e^{-\lambda(\tau(x)-y)}} = \frac{r}{\beta}. \tag{7.9}$$

We make the following assumption.

Assumption $\lambda < r/\beta$.

From (7.9),

$$y = \tau(x) + \frac{1}{\lambda}Log\Big(1-\frac{\beta\lambda}{r}\Big),$$

and using (7.6), the reaction function is

$$y = \mathcal{R}(x) = \frac{x+\sigma\bar{R}}{1+\sigma\dfrac{\bar{R}}{T}} + \frac{1}{\lambda}Log\Big(1-\frac{\beta\lambda}{r}\Big). \tag{7.10}$$

[6]Note how the instantaneous rate of return on deposits is strictly positive when agent delays less than y.

The reaction function is linear with a positive slope that is less than one, as anticipated. There is a unique meaningful solution if and only if $\mathcal{R}(0) > 0$. Using $T = \bar{R}/a$, that condition is equivalent to

$$\bar{R} > \frac{1+\sigma a}{\sigma\lambda} Log\Big(\frac{r}{r-\beta\lambda}\Big). \tag{7.11}$$

If λ is small, that condition is approximated by

$$\bar{R} > (\frac{1}{\sigma} + a)\frac{\beta}{r}.$$

We must have λ not too large and the initial reserves $\bar{R}$ sufficiently large, β not too large and r sufficiently large. All this conditions are intuitive. (Exercise your intuition).

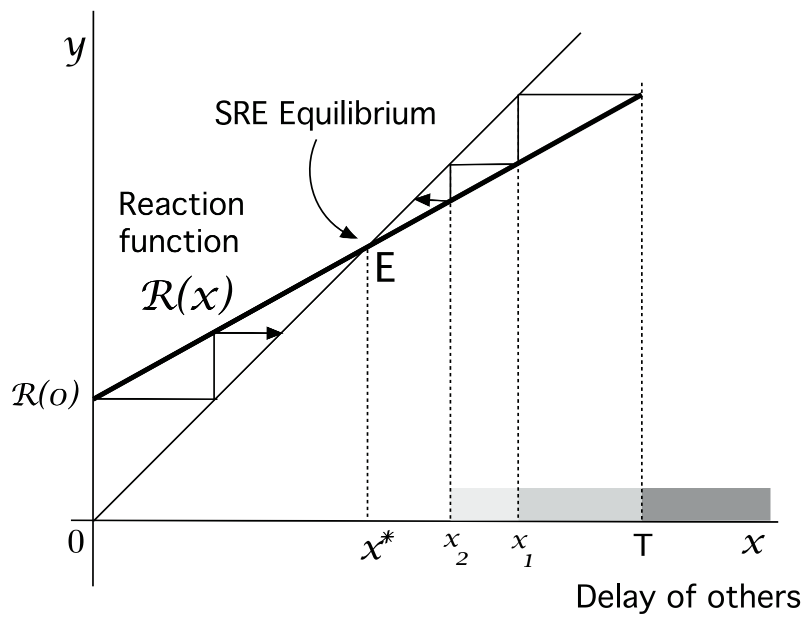


Figure 7.11: Reaction function on delays

The symmetric Nash-equilibrium is characterized by the solution

$$x^*(\beta) = \frac{1}{a}\left(\bar{R} - \frac{1+a\sigma}{\sigma\lambda} Log\Big(\frac{r}{r-\beta\lambda}\Big)\right). \tag{7.12}$$

When λ is vanishingly small, equation (7.12) becomes

$$x^* \sim \frac{1}{a}\Big(\bar{R} - \frac{\beta(a+\sigma)}{r}\Big). \tag{7.13}$$

Discussion: the Strongly Rationalizable Equilibrium

The strategy x^* with $x^* = \mathcal{R}(x^*)$ defines a property that is stronger than a Nash-equilibrium. It is a *Strongly Rationalizable Equilibrium.* As illustrated in Figure 7.11, no agent delays more than T after being informed since the exogenous drain exhausts the reserves after T. Knowing that no agent delays more than $x_0 = T$, any delay more than $x_1 = \mathcal{R}(x_0)$ is strictly dominated (see the figure). By iteration, the sequence x_k converges to x^*.

On the "other side", no agent withdraws his deposit before being aware of the shock. This is proven when the probability of a shock, λ, is small. (We skipped that step, but you can think about it...). If delay is positive, any delay smaller than $\mathcal{R}(0)$ is dominated, and so on. We have a increasing sequence that converges to the same limit, x^* that is a SRE.

Awareness about the outflow shock

When the shock occurs at time θ and the outflow begins, all agents are aware of the shock after a time interval σ. But at the end of that time interval, no one can say that he knows that all other agents are aware of the shock. It is only after a time interval of 2σ that all agents know that all agents know. And so on... The process is illustrated in Figure 7.12. The expression "all agents know that" can be repeated only a finite number of time. How many times depends on the value of $\tau^* + \Delta$ which is the time lag between the shock and the bank failure. There is no common knowledge about the event θ.

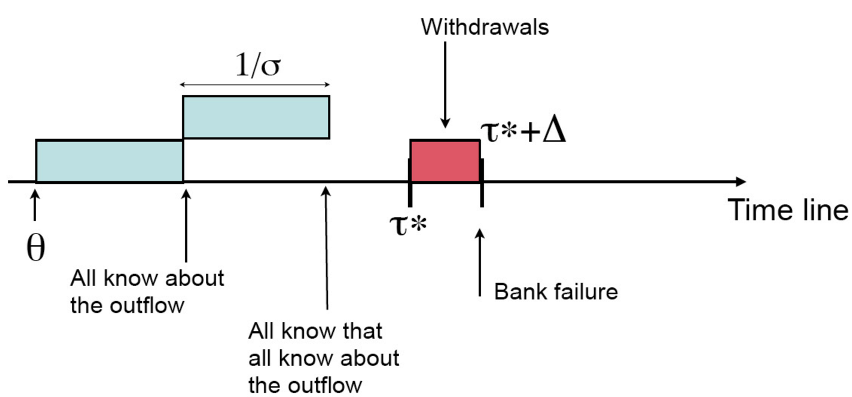


Figure 7.12: Time line of events

7.7 APPENDIX

7.7.1 Currency crisis and endogenous devaluation

The model of the text is used to determine an endogenous rate of devaluation after the collapse of a fixed exchange rate regime.

A central bank operates a regime of fixed exchange rate for its currency against a foreign currency very much like a commercial bank that provides a fixed parity for its deposits against the currency of the central bank. When the reserves a depleted, the regime has to be abandoned. In Krugman (1979), agents have perfect foresight and therefore common knowledge. When the peg is abandoned, domestic inflation jumps up at the start of a new regime. The demand for domestic money jumps down. Because of perfect foresight, there is no devaluation and the variable of adjustment is the stock of money: there is a run on the currency at the time of the regime switch. Such a property is counterfactual: currency crises are associated with devaluation and agents end up with balances they wish they had exchanged before the crises.

In Rochon (2006), agents become gradually aware a drain on the reserves of the central bank takes place. They cannot observe the reserves of the central bank but they observe the central bank operating the regime of fixed exchange rate. Since agents are surprised in a model of sequential information, the model generates "excess balances" at the time of the regime switch and a discrete devaluation. Furthermore, the rate of this devaluation is endogenous to the model.

It is assumed that in the first regime, domestic money (deposits at the central bank, to simplify) is demanded for transactions and speculation (with a positive domestic interest rate and zero foreign rate) while after the peg is abandoned, there is only a demand for transactions with the simple quantitative form $M/P = 1$, where k is a parameter. During the peg, $P = 1$. Just after the switch the price is P' and the rate of devaluation is $\beta = P' - 1$.

Let K_0 the holding of pesos of the speculator at the beginning of time and K is their holdings at the time of regime switch (devaluation). The quantity of money at that time is $K + D$ where D is the demand for transactions. After the peg is abandoned, the demand of speculators is 0, and the real quantity of money demanded is $(D+K)/P' = k$. The rate of devaluation is therefore

$$\beta = \frac{D+K}{k} - 1 = \frac{K}{k}. \tag{7.14}$$

The rate of devaluation is proportional to the amount of balances K by the speculators who are caught by surprise at the time of devaluation and have not sold their balance yet. Using the notation of Section **??**,

$$K = K_0 - \frac{1}{\sigma}(\tau - x^*). \tag{7.15}$$

Using (7.6) and $\bar{R}/T = a$,

$$\beta(x^*) = \frac{1}{k}\Big(K_0 - \frac{\bar{R}}{1 + a\sigma} + \frac{a}{1 + a\sigma}x^*\Big). \tag{7.16}$$

The endogenous rate of devaluation is an increasing function of the delay τ^*: if speculators delay their currency sales, they are caught by the devaluation with higher balances and the a greater devaluation rate is required for the adjustment to the new real balances.

Closing the model

Deposits holders lose a fraction β of their real holdings. The value of delay x^* was given in equation (7.12). The graphs of the function $\beta(x^*)$ in (7.16) and of the delay function in (7.12) are represented in Figure 7.13. There is a unique intersection.

`τ should be replaced by x in the figure`

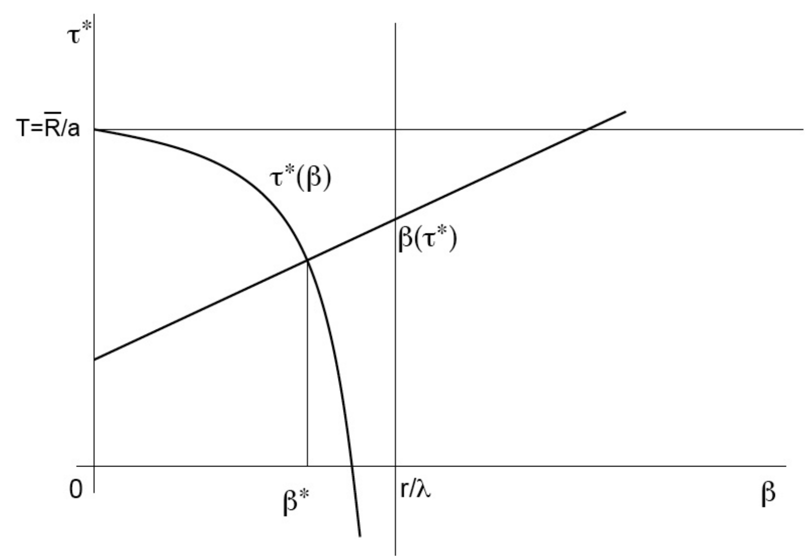


Figure 7.13: The endogenous devaluation rate

The model exhibits the following properties.

- The endogenous rate of devaluation decreases when the rate of information σ increases.
- When the rate of information tends to infinity, the equilibrium value of the endogenous rate of devaluation tends to zero. At the limit, there is a rush out of the currency just before the regime switch. We have the result of Krugman (1979) under perfect information.
- It is shown in Rochon (2006) that when the central bank chooses randomly the minimum level of reserves to defend the peg and agents have imperfect information on that randomly chosen policy, a small randomness does not alter the mean time length of the peg. However, if agents do not know the exact policy of the central bank (but only its probability distribution), then the mean time length is extended. There is thus an argument for not publicizing the central bank policy.

7.7.2 A formal analysis of the AB model

Equilibrium

The analysis parallels that of Section **??**. Let $\bar{\tau}$ be the delay between the instants when an agent is informed and when he sells his asset, and $\hat{\tau}$ the delay between θ and the instant of the crash. There are two types of crashes, endogenous and exogenous, depending on the parameters of the model.

Endogenous crash

An endogenous crash is determined by the conditions

$$\begin{cases} \hat{\tau} - \tau^* = \Delta = \dfrac{\kappa}{\sigma}, \\ rp = gp - \beta(\hat{\tau})p\dfrac{\lambda}{1 - e^{-\lambda\Delta}}, \quad \text{or} \quad g - r = \beta(\hat{\tau})\dfrac{\lambda}{1 - e^{-\lambda\Delta}}. \end{cases}$$

Such a crash takes place when the solution $\hat{\tau} = \tau^* + \Delta$ is smaller than the time lag T of an endogenous crash.

Exogenous crash

Such a crash is triggered by the critical gap between the bubble and the fundamental price. Because both prices move exponentially at different rates, the gap determines the interval of time T from the onset of the bubble. Agents determine their delay strategy τ^* by comparing the diffential rate of return of the bubble and the instantaneous probability that a time T has lapsed since the time θ. This arbitrage equation is the second equation below. The solution τ^* must be such that the mass of sales at time T is lower than the critical mass κ that would trigger an endogenous crash. That is the meaning of the first equation.

$$\begin{cases} T - \tau^* < \dfrac{\kappa}{\sigma}, \\ g - r = \beta(T)\dfrac{\lambda}{1 - e^{-\lambda(T-\tau^*)}}. \end{cases}$$

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