Chapter 5

Delays

Does the waiting game end with a bang or a whimper?

Each agent chooses when to invest (if at all) and observes the number of investments by others in each period. That number provides a signal on the private information of other agents about the state of nature. The waiting game has in general multiple equilibria. An equilibrium depends on the intertemporal arbitrage between the opportunity cost of delay and the value of the information that is gained from more observations. The informational externality generates strategic substitutabilities and complementarities. Multiple equilibria appear which exhibit a rush of activity or delays, and generate a low or high amount of information. The convergence of beliefs and the occurrence of herds are analyzed under a variety of assumptions about the boundedness of the distribution of private beliefs, the number of agents, the existence of an observation noise, the length of the periods, and the discreteness of investment decisions.

In 1993, the US economy was in a shaky recovery from the previous recession. The optimism after some good news was dampened by a few bad news, raised again by other news, and so on. In the trough of the business cycle, each agent is waiting for some "good news" about an upswing. What kind of news? Some count occupancy rates in the first class section of airplanes. Others weigh the newspapers to evaluate the volume of ads. Housing starts, expenditures on durables are standard indicators to watch. The news are the actions of other agents. Everyone could be waiting because everyone is waiting in an "economics of wait and see" (Sylvia Nasar, 1993).

In order to focus on the problem of how a recession may be protracted by the waiting game for more information, we have to take a step back from the intricacies of the real world and the numerous channels of information. In this chapter, agents learn from the observation of the choices of action taken by others but not from the payoffs of these actions. This assumption is made to simplify the analysis. It is also justified in the context of the business cycle where lags between the initiation of an investment process and its payoff can be long (at least a year or two). The structure of the model is thus the same as in Chapter 3 but each agent can make his investment in any period: he has one option to make a fixed size investment. The central issue is when to exercise the option, if at all.

When the value of the investment is strictly positive, delay is costly because the present value of the payoff is reduced by the discount factor. The *opportunity cost of delay* for one period is the product of the net payoff of investment and the discount rate. Delay enables an agent to observe others' actions and infer some information on the state of nature. These observations may generate good or bad news. Define the bad news as an event such that the agent regrets *ex post* an irreversible investment which he has made, and would pay a price to undo it (if it were possible). The expected value of this payment in the next period after observing the current period's aggregate investment, is the option value of delay. The key issue which commands all results in this chapter is the trade-off, in equilibrium, between the opportunity cost and the option value of delay.

Consider the model of Chapter *** with two states of nature and assume that agents can choose the timing of their investment. If all beliefs (probability of the good state) are below the cost of investment, the only equilibrium is with no investment and there is a herd as in the BHW model. If all beliefs are higher than the cost of investment, there is an equilibrium in which all agents invest with no delay. This behavior is like a herd with investment in the BHW model and it is an equilibrium since nothing is learned by delaying. The herds in the BHW model with exogenous timing are equilibria in the model with endogenous timing.

However, the model with endogenous timing may have other equilibria with an arbitrage between the option value and the opportunity cost of delay. For a general distribution of private beliefs, the margin of arbitrage may occur at different points of the distribution. Generically, there are at least two equilibrium points, one in the upper tail of the distribution and another in the lower tail. In the first equilibrium, only the most optimistic agents invest; in the second, only the most pessimistic delay. The two equilibria in which most agents delay or rush, respectively, are not symmetric because of the arbitrage mechanism. In the first, the information conveyed by the aggregate activity must be large in order to keep the agents at the high margin of beliefs (with a high opportunity cost) from investing. In the second, both the opportunity cost of relatively pessimistic agents and the information conveyed by the aggregate activity are low. In the particular case of a bounded distribution, the rush where few agents delay may be replaced by the corner solution where no agent delays.

Multiple equilibria are evidence of strategic complementarities (Cooper and John, 1988). These complementarities arise here only because of informational externalities. There is no payoff externality. As in other models with strategic complementarities, multiple equilibria may provide a support for sudden switches of regime with large fluctuations of economic activity (Chamley, 1999).

The main ideas of the chapter are presented in Section 5 with a simple two-agent model based on Chamley and Gale (1994). The unique equilibrium is computed explicitly.

The general model with heterogeneous beliefs is presented in Section 5. It is the full extension of the BHW model to endogenous timing. Heterogeneous beliefs is a plausible assumption *per se* and it generates non random strategies. The model has a number of players independent of the state of nature and generalizes Chamley and Gale (1994) who assume identical beliefs. In the model with identical beliefs, the endowment of an option is the private signal and the number of players thus depends on the state of nature. This case is particularly relevant when the number of players is large.

When private beliefs are not identical, the analysis of the symmetric sub-game perfect Bayesian equilibria (PBE) turns out to be simple due to an intuitive property which is related to the arbitrage condition: an agent never invests before another who is more optimistic. Therefore, the agent with the highest belief among those who delay must be the “first” to invest in the next period if there is any investment in that period (since he has the highest belief then). All equilibria where the arbitrage condition applies can be described as sequences of two-period equilibria.

Some properties of the model are presented in Section **??**. Extensions will be discussed in the next chapter. When the public belief is a range (μ^*, μ^{**}), the level of investment in each period is a random variable and the probability of no investment is strictly positive. If there is no investment, the game stops with a herd and no investment takes place in any subsequent period. Hence the game lasts a number of periods which is at most equal to the number of players in the game. If the period length tends to zero, the game ends in a vanishingly short time. Since an agent can always delay until the end of the game, and the cost of delay tends to zero with the length of the period, the information generated by the game also tends to zero with the period length: another effect of arbitrage.

The game is illustrated in Section **??** by an example with two agents with normally distributed private signals (unbounded), which highlights the mechanism of strategic complementarity. When the time period is sufficiently short, there cannot be multiple equilibria, under some specific conditions. The presence of time lags between observation and action is thus necessary for the existence of multiple equilibria.

The case of a large number of agents (Section **??**) is interesting and illustrates the power of the arbitrage argument. When the number of agents tends to infinity, the distribution of the levels of investment tends to a Poisson distribution with a parameter which depends on the public belief, and on the discount rate. This implies that as long as the public belief μ is in the interval (μ^*, μ^{**}), the level of investment is a random variable which is small compared to the number of agents. The public belief evolves randomly until it exits the interval: if $\mu < \mu^*$, investment goes from a small random amount to nil forever; if $\mu > \mu^{**}$, all remaining agents invest with no further delay. The game ends with a whimper or a bang.

The Appendix presents two extensions of the model which show the robustness of the results: (i) with a very large number of agents (a continuum) and an observation noise, there are multiple equilibria as in the model with two agents; the equilibrium with high aggregate activity generates an amount of information which is significantly smaller than the equilibrium with low activity and delays; (ii) multiple equilibria also appear when individual investments are non-discrete.

`The simple model is another example of how to start the analysis of general issues as presented in the introduction. One should stylize as much as possible. The investigation of robustness and extensions will be easier once the base model is firmly understood.`

5.1 The simplest model

There are two players and time is divided in periods. There are two states of nature, $\theta \in \{0, 1\}$. In state 0, only one of two players (chosen randomly with equal probability) has one option to make an investment of a fixed size in any period. In state 1, both players have one option. To have an option is private information and is not observable by the other agent. Here, the private signal of the agent is the option. The number of players in the game depends on the state of nature[1]. As an illustration, the opportunities for productive investment may be more numerous when the state of the economy is good.

[1] One could also think that the cost of investment is very high for one or zero agent thus preventing the investment. Recall that in the BHW model, the number of players does not depend on the state of nature.

For an agent with an option, the payoff of investment in period t is

$$U = \delta^{t-1}(E[\theta] - c), \quad \text{with} \quad 0 < c < 1,$$

where E is the expectation conditional on the information of the agent and δ is the discount factor, $0 < \delta < 1$.

All agents in the game have the same private information (their own option), and observe the same history. They have the same belief (probability of state $\theta = 1$). Let μ_t be the belief of an agent at the beginning of period t. The belief in the first period is given[2] and satisfies the next assumption in order to avoid trivialities.

Assumption 5.1. $\quad 0 < \mu - c < \delta\mu(1 - c).$

Agents play a game in each period and the strategy of an agent is his probability of investment. We look for a symmetric perfect Bayesian equilibrium (PBE): each agent knows the strategy z of the other agent (it is the same as his own); he anticipates rationally to receive a random amount of information at the end of each period and that the subgame which begins next period with a belief updated by Bayes' rule has an equilibrium.

Let z be the probability of investment in the first period by an agent with an option. Such an agent will be called a player. We prove that there is a unique symmetric equilibrium with $0 < z < 1$.

- $z = 1$ cannot be an equilibrium. If $z = 1$, both agents "come out" with probability one, the number of players and therefore the state is revealed perfectly at the end of the period. If an agent deviates from the strategy $z = 1$ and delays (with $z = 0$), he can invest in the second period if and only if the true state is good. The expected payoff of this delay strategy is $\delta\mu(1-c)$: in the first period, the good state is revealed with probability μ in which case he earns $1-c$. The discount factor is applied because the investment is made in the second period. The payoff of no delay is $\mu - c$, and it is smaller by Assumption 5.1. The strategy $z = 1$ cannot define a PBE. Note that the interpretation of the right-hand side inequality is now clear: the payoff of investment, $\mu - c$, should be smaller than the payoff of delay with perfect information in the next period.
- $z = 0$ cannot be an equilibrium either. The argument is a bit more involved and proceeds by contradiction. If $z = 0$, there is no investment in the first period for

[2] One could assume that agents know that nature chooses state $\theta = 1$ with probability μ_0. In this case, by Bayes' rule, $\mu = 2\mu_0/(1 + \mu_0)$.

any state, no information and therefore the same game holds at the beginning of period 2, with the same belief μ. Indefinite delay cannot be an equilibrium strategy because it would generate a zero payoff which is strictly smaller than the payoff of no delay, $\mu - c > 0$ (Assumption 5.1). Let T be the first period in which there is some investment with positive probability. Since $z = 0$, $T \geq 2$. In period T, the current value of the payoff of investment is $\mu - c > 0$ because nothing has been learned before. The present value of this payoff is strictly smaller than the payoff of immediate investment, $\mu - c$. Hence, $T \geq 2$ is impossible and $z = 0$ cannot be an equilibrium strategy.

The necessity of investment in every period

We have shown that in an equilibrium, agents randomize with $0 < z < 1$. The level of total investment is a random variable. We will see that the higher the level of investment, the higher the updated belief after the observation of the investment. In this simple model, one investment is sufficient to reveal to the other player (if there is one), that the state is good. No investment in the first period is bad news. Would anyone invest in the second period after this bad news? The answer is no, and the argument is interesting.

If anyone delays in the first period and expects to invest in the second period after the worst possible news (zero investment), his payoff in the subgame of period 2 is the same as that of investing for sure in period 2. (He invests if he observes one investment). That payoff, $\delta(\mu - c)$, is inferior to the payoff of immediate investment because of the discount. The player cannot invest after observing no investment. Hence, *if there is no investment in the first period, there is no investment in any period after*. We will see in this chapter that this property applies in more general models. The argument shows that: (i) if there is no investment, the *ex post* belief of any agent must be smaller than the cost of investment c; (ii) since agents randomize in the first period, the event of no investment has a positive probability. There is a positive probability of an incorrect herd.

Using the previous argument, we can compute the payoff of delay. If an agent delays, he invests in period 2 if and only if he sees an investment (by the other agent) in period 1, in which case he is sure that the state is good and his second period payoff is $1 - c$. The probability of observing an investment in the first period is μz, (the product of the probability that there is another agent and that he invests). The payoff of delay (computed at the time of the decision) is therefore $\delta \mu z(1 - c)$.

Arbitrage and the existence of a unique PBE

Since $0 < z < 1$, agents randomize their investment in the first period and are indifferent between no delay and delay. This arbitrage condition between the value of investment and the value of the option to invest is essential in this chapter and is defined by

$$\mu - c = \delta\mu z(1-c). \tag{5.1}$$

By Assumption 5.1, this equation in z has a unique solution in the interval $(0,1)$. The analysis of the solution may be summarized as follows: first, the arbitrage condition is necessary if a PBE exists; second, the existence of a unique PBE follows from the arbitrage condition by construction of the equilibrium strategy. This method will be used in the general model.

Interpretation of the arbitrage condition

A simple manipulation shows that the arbitrage equation can be restated as

$$\begin{aligned} \frac{1-\delta}{\delta}(\mu - c) &= \Big(\mu z(1-c) - (\mu - c)\Big) \\ &= P(x=0|\mu)\Big(c - P(\theta_1|x=0,\mu)\Big) \end{aligned} \tag{5.2}$$

where $P(x=0|\mu)$ is the probability for an agent with belief μ that the other agent does not invest in period 1, *i.e.* the probability of bad news. The term $\mu - c$ has the dimension of a stock, as the net present value of an investment. The left-hand side is the *opportunity cost of delay*: it is the value of investment multiplied by the interest rate between consecutive periods. (If $\delta = 1/(1+r)$, then $(1-\delta)/\delta = r$). The right-hand side will be called the *information value of delay*. It provides the measurement of the value of information obtained from a delay. To interpret it, note that the term $P(\theta_1|x=0,\mu)$ is the value of an investment after the bad news in the first period. If an agent could reverse his decision to invest in the first period (and get the cost back), the associated value of this action would be $c - P(\theta_1|x=0,\mu)$. The option value of delay is the expected "regret value" of undoing the investment when the agent wishes he could do so. The next properties follow from the arbitrage condition.

In an equilibrium, the cost of delay is equal to the information value of delay---the expected regret value. This arbitrage is the linchpin of all equilibria in this chapter.

Information and time discount

The power of the signal which is obtained by delay increases with the probability of investment z in the strategy. If $z = 0$, there is no information. If $z = 1$, there is perfect information.

The discount factor is related to the length of the period, τ, by $\delta = e^{-\rho\tau}$, with ρ the discount rate per unit of time. If δ varies, the arbitrage equation (5.1) shows that the product δz is constant. A shorter period (higher δ) means that the equilibrium must generate less information at the end of the first period: the opportunity cost of delay is smaller and by arbitrage, the information value of delay decreases. Since this information varies with z, the value of z decreases. From Assumption 5.1, $0 < z < 1$ only if δ is in the interval $[\delta^*, 1)$, with $\delta^* = (\mu - c)/(\mu(1-c))$.

If $\delta \to \delta^*$, then $z \to 1$. If $\delta \leq \delta^*$, then $z = 1$ and the state is revealed at the end of the first period. Because this information comes late (with a low δ), agents do not wait for it.

If $\delta \to 1$ and the period length is vanishingly short, information comes in quickly but there is a positive probability that it is wrong. The equilibrium strategy z tends to δ^*. If the state is good, with probability $(1-\delta^*)^2 > 0$ both agents delay and end up thinking that the probability of the good state is smaller than c and that investment is not profitable. There is a trade-off between the period length and the quality of information which is revealed by the observation of others. This trade-off is generated by the arbitrage condition. The opportunity cost of delay is smaller if the period length is smaller. Hence the value of the information gained by delay must also be smaller.

A remarkable property is that the waiting game lasts one period, independently of the discount factor. If the period is vanishingly short, the game ends in a vanishingly short time, but the amount of information which is released is also vanishingly short. In this simple model with identical players, the value of the game does not depend on the endogenous information which is generated in the game since it is equal to the payoff of immediate investment. However, when agents have different private informations, the length of the period affects welfare (as shown in the next chapter).

Investment level and optimism

In the arbitrage equation (5.1), the probability of investment and the expected value of investment are increasing functions of the belief μ: a higher μ entails a higher opportunity cost and by arbitrage a higher option value of delay. The higher information requires that players "come out of the wood" with a higher probability z. This mechanism is different from the arbitrage mechanism in the q-theory of Tobin which operates on the margin between the financial value μ and an adjustment cost.

Observation noise and investment

Suppose that the investment of an agent is observed with a noise: if an investment is made, the other agent sees it with probability $1-\gamma$ and sees nothing with probability γ, (γ small). The arbitrage operates beautifully: the information for a delaying agent is unaffected by the noise because it must be equal to the opportunity cost which is independent of the noise. Agents compensate for the noise in the equilibrium by increasing the probability of investment (Exercise 5.8).

Large number of agents

Suppose that in the good state there are N agents with an option to invest and that in the bad state there is only one agent with such an option. These values are chosen to simplify the game: one investment reveals that the state is good and no investment stops the game. For any N which can be arbitrarily large, the game lasts only one period, in equilibrium, and the probability of investment of each agent in the first period tends to zero if $N \to \infty$. Furthermore, the probability of no investment, conditional on the good state, tends to a positive number. The intuition is simple. If the probability of investment by a player remains higher than some value $\alpha > 0$, its action (investment or no investment) is an signal on the state with a non vanishing precision. If $N \to \infty$, delay provides a sample of observations of arbitrarily large size and perfect information asymptotically. This is impossible because it would contradict the arbitrage with the opportunity cost of delay which is independent of N. The equilibrium is analyzed in Exercise 5.10.

Strategic substitutability

Suppose an agent increases his probability of investment from an equilibrium value z. The option value (in the right-hand side of (5.1) or (5.2)) increases. Delay becomes strictly better and the optimal response is to reduce the probability of investment to zero: there is strategic substitutability between agents. In a more general model (next section) this property is not satisfied and multiple equilibria may arise.

Non symmetric equilibrium

Assume there are two agents, A and B, who can see each other but cannot see whether the other has an option to invest. It is common knowledge that agent B always delays in the first period and does not invest ever if he sees no investment in the first period.

Agent A does not get any information by delaying: his optimal strategy is to invest with no delay, if he has an option. Given this strategy of agent A, agent B gets perfect information at the end of period 1 and his strategy is optimal. The equilibrium generates perfect information after one period. Furthermore, if the state is good, both agents invest. If the period length is vanishingly short, the value of the game is $\mu - c$ for agent A, and $\mu(1 - c)$ for agent B which is strictly higher than in the symmetric equilibrium. If agents could "allocate the asymmetry" randomly before knowing whether they have an option, they would be better off *ex ante*.

5.2 A general model with heterogeneous beliefs

The structure of the model extends the canonical model in Section **??** by allowing each agent to make his fixed size investment in any period of his choice. There are N agents each with one option to make one irreversible investment of a fixed size. Time is divided in periods and the payoff of exercising an option in period t is $\delta^{t-1}(\theta - c)$ with δ the discount factor, $0 < \delta \leq 1$, and c the cost of investment, $0 < c < 1$. The payoff from never investing is zero. Investment can be interpreted as an irreversible switch from one activity to another[3].

The rest of the model is the same as in the beginning of Section **??**. The productivity parameter θ which is not observable is set randomly by nature once and for all before the first period and takes one of two values: $\theta_0 < \theta_1$. Without loss of generality, these values are normalized at $\theta_1 = 1$ for the "good" state, and $\theta_0 = 0$ for the "bad" state. As in Section **??**, each agent is endowed at the beginning of time with a private belief which is drawn from a distribution with *c.d.f.* $F_1^\theta(\mu)$ depending on the state of nature θ. For simplicity and without loss of generality, it will be assumed that the cumulative distribution functions have derivatives[4]. The support of the distribution of beliefs is an interval $(\underline{\mu}_1, \bar{\mu}_1)$ where the bounds may be infinite and are independent of θ. The densities of private beliefs satisfy the Proportional Property (**??**). Hence, the cumulative distribution functions satisfy the property of first order stochastic dominance: for any $\mu \in (\underline{\mu}_1, \bar{\mu}_1)$, $F_1^1(\mu) < F_1^0(\mu)$.

After the beginning of time, learning is endogenous. In period t, an agent knows his private belief and the history $h_t = (x_1, \ldots, x_{t-1})$, where x_k is the number of investments in period k.

The only decision variable of an agent is the period in which he invests. (This period is postponed to infinity if he never invests). We will consider only symmetric equilibria. A strategy in period t is defined by the *investment set* $I_t(h_t)$ of beliefs of all investing agents: an agent with belief μ_t in period t invests in that period (assuming he still has an option) if and only if $\mu_t \in I_t(h_t)$. In an equilibrium, the set of agents which are indifferent between investment and delay will be of measure zero and is ignored. Agents will not use random strategies.

As in the previous chapters, Bayesian agents use the observation of the number of investments, x_t, to update the distribution of beliefs F_t^θ into the distribution in the next period F_{t+1}^θ. Each agent (who has an option) chooses a strategy which maximizes his expected

[3] The case where the switch involves the termination of an investment process (as in Caplin and Leahy, 1994) is isomorphic.

[4] The characterization of equilibria with atomistic distributions is more technical since equilibrium strategies may be random (*e.g.*, Chamley and Gale, 1994).

payoff, given his information and the equilibrium strategy of all agents for any future date and future history. For any period t and history h_t, each agent computes the value of his option if he delays and plays in the subgame which begins in the next period $t+1$. Delaying is optimal if and only if that value is at least equal[5] to the payoff of investing in period t. All equilibria analyzed here are symmetric subgame perfect Bayesian equilibria (PBE).

As in the model with exogenous timing (Section **??**), a belief can be expressed by the Log likelihood ratio (LLR) between the two states, $\lambda = Log(\mu/(1-\mu))$ which is updated between periods t and $t+1$ by Bayes' rule

$$\begin{aligned} \lambda_{t+1} &= \lambda_t + \zeta_t, \quad \text{where} \quad \zeta_t = Log\Big(\frac{P(x_t \mid I_t, \theta_1)}{P(x_t \mid I_t, \theta_0)}\Big), \\ \text{and } P(x_t \mid I_t, \theta) &= \frac{n_t!}{x_t!(n_t - x_t)!}\pi_\theta^{x_t}(1-\pi_\theta)^{n_t - x_t}, \;\; \pi_\theta = P(\lambda_t \in I_t \mid \theta). \end{aligned} \tag{5.3}$$

All agents update their individual LLR by adding the *same* value ζ_t. Given a state θ, the distribution of beliefs measured in LLRs in period t is generated by a translation of the initial distribution by a random variable ζ_t.

5.2.1 Characterization and existence of equilibria

The incentive for delay is to get more information from the observation of others. Agents who are relatively more optimistic have more to loose and less to gain from delaying: the discount factor applies to a relatively high expected payoff while the probability of bad news to be learned after a delay is relatively small. This fundamental property of the model restricts the equilibrium strategies to the class of *monotone strategies*. By definition, an agent with a monotone strategy in period t invests if and only if his belief μ_t is greater than some value μ_t^*. The next result, which is proven in the appendix, shows that equilibrium strategies must be monotone.

LEMMA 5.1. (monotone strategies) *In any arbitrary period t of a PBE, if the payoff of delay for an agent with belief μ_t is at least equal to the payoff of no delay, any agent with belief $\mu_t' < \mu_t$ strictly prefers to delay. Equilibrium strategies are monotone and defined by a value μ_t^*: agents who delay in period t have a belief $\mu_t \leq \mu_t^*$.*

Until the end of the chapter, strategies will be defined by their minimum belief for investment, μ_t^*. Since no agent would invest with a negative payoff, $\mu_t^* \geq c$. The support of the distribution of μ in period t is denoted by $(\underline{\mu}_t, \bar{\mu}_t)$. If all agents delay in period t, one can define the equilibrium strategy as $\mu_t^* = \bar{\mu}_t$.

[5] By assumption, an indifferent agent delays. This tie breaking rule applies with probability zero and is inconsequential.

The existence of a non trivial equilibrium in the subgame which begins in period t depends on the payoff of the most optimistic agent[6], $\bar{\mu}_t - c$. First, if $\bar{\mu}_t \leq c$, no agent has a positive payoff and there is no investment whatever the state θ. Nothing is learned in period t (with probability one), or in any period after. The game stops. Second, if $\bar{\mu}_t > c$, the next result (which parallels a property for identical beliefs in Chamley and Gale, 1994) shows that in a PBE, the probability of some investment is strictly positive. The intuition of the proof, which is given in the appendix, begins with the remark that a permanent delay is not optimal for agents with beliefs strictly greater than c (since it would yield a payoff of zero). Let T be the first period after t in which some agents invest with positive probability. If $T > t$, the current value of their payoff would be the same as in period t (nothing is learned between t and T). Because of the discount factor $\delta < 1$, the present value of delay would be strictly smaller than immediate investment which is a contradiction.

LEMMA 5.2. (condition for positive investment) *In any period t of a PBE:*

(i) if $c < \bar{\mu}_t$ (the cost of investment is below the upper-bound of beliefs), then any equilibrium strategy μ_t^* is such that $c \leq \mu_t^* < \bar{\mu}_t$; if there is at least one remaining player, the probability of at least one investment in period t is strictly positive;

(ii) if $\bar{\mu}_t \leq c$ (the cost of investment is above the upper-bound of beliefs), then with probability one there is no investment for any period $\tau \geq t$.

The decision to invest is a decision whether to delay or not. In evaluating the payoff of delay, an agent should take into account the strategies of the other agents in all future periods. This could be in general a very difficult exercise. Fortunately, the property of monotone strategies simplifies greatly the structure of equilibria. A key step is the next result which shows that any equilibrium is a sequence of two-period equilibria each of which can be determined separately.

LEMMA 5.3. (one-step property) *If the equilibrium strategy μ_t^* of a PBE in period t is an interior solution ($\underline{\mu}_t < \mu_t^* < \bar{\mu}_t$), then an agent with belief μ_t^* is indifferent between investing in period t and delaying to make a final decision (investing or not) in period $t+1$.*

Proof Since the Bayesian updating rules are continuous in μ, the payoffs of immediate investment and of delay for any agent are continuous functions of his belief μ. Therefore, an agent with belief μ_t^* in period t is indifferent between investment and delay. By definition

[6] Recall that such an agent may not actually exist in the realized distribution of beliefs.

of μ_t^*, if he delays he has the highest level of belief among all players remaining in the game in period $t+1$, *i.e.*, his belief is $\bar{\mu}_{t+1}$. In period $t+1$ there are two possibilities: (i) if $\bar{\mu}_{t+1} > c$, then from Lemma 5.2, $\mu_{t+1}^* < \bar{\mu}_{t+1}$ and a player with belief $\bar{\mu}_{t+1}$ invests in period $t+1$; (ii) if $\bar{\mu}_{t+1} \leq c$, then from Lemma 5.2 again, nothing is learned after period t; a player with belief $\bar{\mu}_{t+1}$ may invest (if $\bar{\mu}_{t+1} = c$), but his payoff is the same as that of delaying for ever. □

In an equilibrium, an agent with belief μ compares the payoff of immediate investment, $\mu - c$, with that of *delay for exactly one period*, $W(\mu, \mu^*)$, where μ^* is the strategy of others. (For simplicity we omit the time subscript and other arguments such as the number of players and the *c.d.f.* F^θ). From Lemma 5.3 and the Bayesian formulae (5.3) with $\pi^\theta = 1 - F^\theta(\mu^*)$, the function W is well defined. An interior equilibrium strategy must be solution of the arbitrage equation between the payoff of immediate investment and of delay:

$$\mu^* - c = W(\mu^*, \mu^*).$$

The next result shows that this equation has a solution if the cost c is interior to the support of the distribution of beliefs.

LEMMA 5.4. *In any period, if the cost c is in the support of the distribution of beliefs, i.e., $\underline{\mu} < c < \bar{\mu}$, then there exists $\mu^* > c$ such that $\mu^* - c = W(\mu^*, \mu^*)$: an agent with belief μ^* is indifferent between investment and delay.*

Proof Choose $\mu^* = \bar{\mu}$: there is no investment and therefore no learning during the period. Hence, $W(\bar{\mu}, \bar{\mu}) = (1-\delta)(\bar{\mu} - c) < \bar{\mu} - c$. Choose now $\mu^* = c$. With strictly positive probability, an agent with belief c observes $n-1$ investments in which case his belief is higher (n is the number of remaining players). Hence, $W(c, c) > 0$. Since the function W is continuous, the equation $\mu^* - c = W(\mu^*, \mu^*)$ has at least one solution in the interval $(c, \bar{\mu})$. □

The previous lemmata provide characterizations of equilibria (PBE). These characterizations enable us to construct all PBE by forward induction and to show existence.

THEOREM 5.1. *In any period t where the support of private beliefs is the interval $(\underline{\mu}_t, \bar{\mu}_t)$:*

(i) if $\bar{\mu}_t \leq c$, then there is a unique PBE with no agent investing in period t or after;

(ii) if $\underline{\mu}_t < c < \bar{\mu}_t$, then there is at least one PBE with strategy $\mu_t^* \in (c, \bar{\mu}_t)$;

(iii) if $c \leq \underline{\mu}_t$, then there is a PBE with $\mu_t^* = \underline{\mu}_t$ in which all remaining players invest in period t.

In case (ii) and (iii) there may be multiple equilibria. The equilibrium strategies $\mu_t^* \in (\underline{\mu}_t, \bar{\mu}_t)$ *are identical to the solutions of the arbitrage equation*

$$\mu^* - c = W(\mu^*, \mu^*), \tag{5.4}$$

where $W(\mu, \mu^*)$ *is the payoff of an agent with belief* μ *who delays for one period exactly while other agents use the strategy* μ^*.

The only part which needs a comment is (ii). From Lemma 5.4, there exists μ_t^* such that $c < \mu_t^*$ and $\mu^* - c = W(\mu^*, \mu^*)$. From Lemma 5.1, any agent with belief $\mu_t > \mu_t^*$ strictly prefers not to delay and any agent with belief $\mu_t < \mu_t^*$ strictly prefers to delay. (Otherwise, by Lemma 5.1 an agent with belief μ_t^* would strictly prefer to delay which contradicts the definition of μ_t^*). The strategy μ_t^* determines the random outcome x_t in period t and the distributions F_{t+1}^{θ} for the next period, and so on.

5.3 Properties

5.3.1 Arbitrage

Let us reconsider the trade-off between investment and delay. For the sake of simplicity, we omit the time subscript whenever there is no ambiguity. If an agent with belief μ delays for one period, he foregoes the implicit one-period rent on his investment which is the difference between investing for sure now and investing for sure next period, $(1-\delta)(\mu - c)$; he gains the possibility of "undoing" the investment after bad news at the end of the current period (the possibility of not investing). The expected value of this possibility is the option value of delay. The following result, proven in the appendix, shows that the belief μ^* of a marginal agent is defined by the equality between the opportunity cost and the option value of delay.

PROPOSITION 5.1. (arbitrage) *Let* μ^* *be an equilibrium strategy in a game with* $n \geq 2$ *remaining players,* $\underline{\mu} < \mu^* < \bar{\mu}$. *Then* μ^* *is solution of the arbitrage equation between the opportunity cost and the option value of delay*

$$(1-\delta)(\mu^* - c) = \delta Q(\mu^*, \mu^*), \qquad \textit{with}$$

$$Q(\mu, \mu^*) = \sum_{k=0}^{n-1} P(x = k | \mu, \mu^*, F^{\theta}, n) Max\Big(c - P(\theta = \theta_1 | x = k; \mu, \mu^*, F^{\theta}, n), 0\Big), \tag{5.5}$$

where x *is the number of investments by other agents in the period.*

The function $Q(\mu, \mu^*)$ is a "regret function" which applies to an agent with belief μ. It depends on the strategy μ^* of the other agents and on the *c.d.f.*s F^θ at the beginning of the period. Since the gain of "undoing" an investment is c minus the value of the investment after the bad news, the regret function $Q(\mu, \mu^*)$ is the expected value of the amount the agent would be prepared to pay to undo his investment at the beginning of next period.

At the end of that period, each agent updates his LLR according to the Bayesian formula (5.3) with $\pi_\theta = 1 - F^\theta(\mu_t^*)$. A simple exercise shows that the updated LLR is an increasing function of the level of investment in period t and that the lowest value of investment $x_t = 0$ generates the lowest level of belief at the end of the period. Can the game go on after the worst news of no investment? From Proposition 5.1, we can deduce immediately that the answer is no. If the agent would invest after the worst news, the value of $Q(\mu^*, \mu^*)$ would be equal to zero and would therefore be strictly smaller than $\mu^* - c$ which contradicts the arbitrage equation (5.5).

PROPOSITION 5.2. (the case of worst news) *In any period t of a PBE for which the equilibrium strategy μ_t^* is interior to the support $(\underline{\mu}_t, \bar{\mu}_t)$, if $x_t = 0$, then $\bar{\mu}_{t+1} \leq c$ and the game stops at the end of period t with no further investment in any subsequent period.*

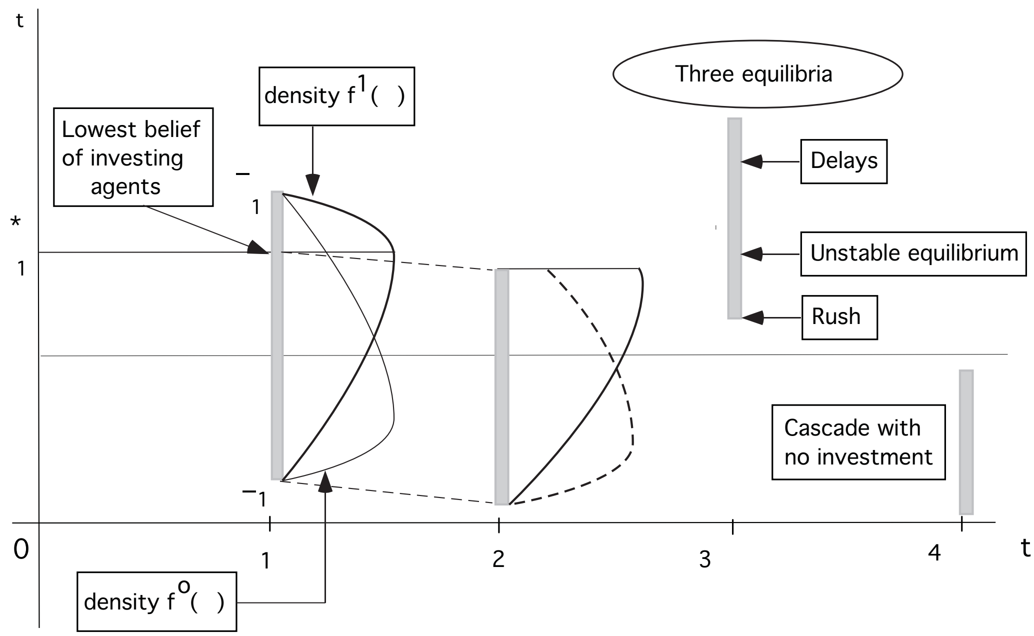
The result shows that a game with N players lasts at most N periods. If the period length τ is vanishingly short, the game ends in a vanishingly short time. This case is analyzed in Section **??**.

5.3.2 Representation of beliefs

An example of the evolution of beliefs is illustrated in Figure 5.1. The reader may compare with the equivalent Figure **??** in the case of exogenous timing. Beliefs are measured by the LLR and are bounded, by assumption. The support of their distribution at the beginning of a period is represented by a segment. Suppose that the state is bad: $\theta = 0$. At the beginning of period 1, the private beliefs of the N players are the realizations of N independent drawings from a distribution with density $f^0(\cdot)$ which is represented by a continuous curve. (The density in state $\theta = 1$ is represented by a dotted curve).

In period 1, agents with a belief above λ_1^* exercise their option to invest. The number of investments, x_1, is the number of agents with belief above λ_1^*, which is random according to the process described in the previous paragraph.

Each agent who delays knows that x_1 is generated by the sum of $N - 1$ independent binary



The number of investments in a period t depends on the number of agents with a belief higher than λ_t^*. At the end of a period, the updated distributions in the two states are truncated, translated and rescaled. Period 3 (in which the representation of the densities is omitted) corresponds to a case with three equilibria. In period 4, there is no investment since all beliefs are smaller than the cost of investment.

Figure 5.1: An example of evolution of beliefs

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variables equal to 1 with a probability π^θ that depends on θ: $\pi^\theta = 1 - F^\theta(\lambda_1^*)$. The probability is represented in Figure 5.1 by the lightly shaded area if $\theta = 0$ and the darker area if $\theta = 1$.

From the updating rule (5.3), the distribution of LLRs in period 2 is a translation of the distribution of the LLRs in period 1, truncated at λ_1^*, and rescaled (to have a total measure of one): $\lambda_1^* - \underline{\lambda}_1 = \bar{\lambda}_2 - \underline{\lambda}_2$. An agent with LLR equal to λ_1^* in period 1 and who delays has the highest belief in period 2. The news at the end of period 1 depend on the random number of agents with beliefs above λ_1. In Figure 5.1, the observation of the number of investments in period 1 is bad news: the agent with highest belief has a lower belief in period 2 compared to period 1.

There are two critical values for the LLR in each period: (i) an agent who has a LLR below the break-even value $\gamma = Log(c/(1-c))$ does not invest; (ii) no agent who has an LLR above some value λ^{**} delays. The value λ^{**} is defined such that if $\lambda > \lambda^{**}$, the payoff of no delay is higher than that of delay with perfect information one period later. Since the latter yields $\delta\mu(1-c)$ to an agent with belief μ, we have

$$\lambda^{**} = Log\Big(\frac{\mu^{**}}{1-\mu^{**}}\Big), \quad \text{with} \quad \mu^{**} - c = \delta\mu^{**}(1-c). \tag{5.6}$$

Note that λ^{**} (or μ^{**}) depends essentially on the discount rate. If the discount rate is vanishingly small, the opportunity cost of delay is vanishingly small and only the super-optimists should invest: if $\delta \to 1$, then $\lambda^{**} \to \infty$.

5.3.3 Herds: a comparison with exogenous sequences

Case (iii) in Theorem 5.1 is represented in period 3 of Figure 5.1. The lower bound of the distribution of beliefs is higher than the cost of investment, with $\underline{\lambda}_3 > \gamma = Log(c/(1-c))$. There is an equilibrium called a rush, in which no agent delays. In that equilibrium, nothing is learned by delay since the number of investments is equal to the number of remaining players, whatever the state of nature. This outcome occurs here with endogenous delay under the same condition as the "cascade" or herd of BHW, in which all agents invest, regardless of their private signal[7].

For the distribution of beliefs in period 3, there may be another equilibrium with an interior solution λ_3^* to the arbitrage equation (5.4). Since agents with the lowest LLR $\underline{\lambda}_3$ strictly prefer to invest if all others do, there may be multiple equilibria with arbitrage, some of them unstable. This issue is reexamined in the next subsection.

For the case of period 4, all beliefs are below the break-even point: $\bar{\lambda}_4 < \gamma$. No investment takes place in period 4 or after. This equilibrium appears also in the BHW model with exogenous timing, as a cascade with no investment. From Proposition 5.2, this equilibrium occurs with positive probability if agents coordinate on the equilibrium λ_3^* in period 3.

The present model integrates the findings of the BHW model in the setting with endogenous timing. We could anticipate that the herds of the BHW model with exogenous timing are also equilibria when timing is endogenous because they generate no information and therefore no incentive for delay.

`The cascades of the BHW model are also equilibria when timing is endogenous.`

A rush where all agents invest with no delay can take place only if the distribution of beliefs

[7] In the BHW model, distributions are atomistic, but the argument is identical.

(LLR) is bounded below. However, if beliefs are unbounded, the structure of equilibria is very similar to that in Figure 5.1. In a generic sense, there are multiple equilibria and one of them may be similar to a rush. This issue is examined in an example with two agents and Gaussian signals. The Gaussian property is a standard representation of unbounded beliefs.

EXERCISES

Solutions to exercises on delays

Ex 1:
$$P(\theta = 1|x = 0) = \frac{P(x = 0|\theta = 1)P(\theta = 1)}{P(x = 0)} = \frac{1 - z)\mu}{(1 - z)\mu + 1 - \mu}.$$

The value of z is solution of
$$\mu - c = \delta\mu z(1 - c).$$

We can replace z in the first equation by its value in the second equation.

When δ tends to 1 the value of z decreases. We can and we should answer the question without the previous answer, using the arbitrage equation (1.2) in the text. At the limit, the value of $P(\theta = 1|x = 0$ tends to c. Verify your intuition.

Ex 2: The solution is $\mu - c = \delta\mu z(1 - \gamma)(1 - c)$. The value of z is endogenous to γ. The product $z(1 - \gamma)$ is constant. When γ iincreases, the value of z increases to compensate for the loss of "visibility" of investment. Of course, z is bounded by 1. Therefore, there is a maximum value γ^* for which the argument is true.

Ex 3: The arbitrage equation becomes $\mu - c = \delta\mu z(1 - c + \tau)$. The subsidy increases the incentive to delay and therefore reduces z. The welfare of agent(s) is *unchanged* because it is equal to $\mu - c$ and independent of τ.

Ex 4: In the equilibrium, $\mu - c = \delta\mu(1 - (1 - z)^{N-1}(1 - c)$. When N increases, the term $(1 - z)^{N-1}$ is constant. In the good state, the probability of a herd with no investment (following no investment in the first period) is $(1 - z)^N = (1 - z)(1 - z)^{N-1}$, which is constant. (Here we have the power N because we have the point of view of an outside observer and not the point of view of a player who decides to delay and see what other players (if any) do.

The other exercises are related to material that was not covered in the class.

EXERCISE 5.1.

Consider the model of Section 5. Determine the belief (probability of the good state) after the bad news of no investment. Determine the limit of this value when $\delta \to 1$.

EXERCISE 5.2. Observation noise

Consider the model of Section 5 with observation noise. Assume that if an agent invests, he is seen as investing with probability $1-\gamma$ and not investing with probability γ, where γ is small. Determine the equilibrium strategy. Show that for some interval $\gamma \in [0, \gamma^*)$ with $\gamma^* > 0$, the probability of the revelation of the good state and the probability of an incorrect herd are independent of γ.

EXERCISE 5.3.

Consider the simple model of delay in Section 5 where there are two possible states 1 and 0. In state 1, there are two agents each with an option to make an investment equal to 1 at the cost $c < 1$. In state 0, there is only one such agent. The gross payoff of investment is θ. The discount factor is $\delta < 1$ and the initial probability of state 1 is μ such that $0 < \mu - c < \mu\delta(1-c)$.

1. A government proposes a policy which lowers the cost of investment, through a subsidy τ which is assumed to be small. Unfortunately, due to lags, the policy lowers the cost of investment by a small amount in the *second* period, and only in the second period. This policy is fully anticipated in the first period. Analyze the impact of this policy on the equilibrium and the welfare of agents.

2. Suppose that in addition (in each state) one more agent with an option to invest (and discount factor δ), and a belief (probability of the good state) $\underline{\mu} < c$. How is your previous answer modified?

EXERCISE 5.4.

Consider the model of Section 5 with N players in the good state and one player in the bad state. Solve for the symmetric equilibrium. Show that the probability of a herd with no investment converges to $\pi^* > 0$ if $N \to \infty$. Analyze the probability of investment by any agent as $N \to \infty$.

EXERCISE 5.5.

Show that there is strategic substitutability at an equilibrium with the strategy μ^* if

$$\mu^* > \frac{\sqrt{c/(1-c)}}{1+\sqrt{c/(1-c)}}.$$

EXERCISE 5.6.

In the model of Section **??**, assume $n \to \infty$ and the period length converges to zero, ($\delta \to 1$), at a rate slower than n. Assume that not all agents invest in the equilibrium (there is no rush).

1. Determine the payoff of an agent with private belief μ as a function of μ, $\bar{\mu}$ and c.
2. Is there a measurement of the externality of information which an agent with private belief μ receives from the agents in the upper tail of the distribution of beliefs?

EXERCISES

EXERCISE 5.7. (Vanishingly small period)

Consider the model of Section 5. Determine the limit of the belief (probability of he good state) after the bad news of no investment, μ^-, when $\delta \to 1$, without computing this value. Explain the result. Do agents learn something, asymptotically. Show that you can also compute μ^-.

EXERCISE 5.8. (Observation noise)

Consider the simple model of delay in this chapter with two agents and two possible states. We now introduce an observation noise. Assume that if a person invests, she is seen as investing with probability $1 - \gamma$ and not investing with probability γ, where γ is small. Determine the equilibrium strategy. Show that for some interval $\gamma \in [0, \gamma^*)$ with $\gamma^* > 0$, the probability of the revelation of the good state and the probability of an incorrect herd are independent of γ.

EXERCISE 5.9. (An investment subsidy)

Consider the simple model of delay in Section 5 where there are two possible states 1 and 0. In state 1, there are two agents each with an option to make an investment equal to 1 at the cost $c < 1$. In state 0, there is only one such agent. The gross payoff of investment is $\theta = 1$. The discount factor is $\delta < 1$ and the initial probability of state 1 is μ such that $0 < \mu - c < \mu\delta(1 - c)$.

1. A government proposes a policy which lowers the cost of investment, through a subsidy τ which is assumed to be small. Unfortunately, due to lags, the policy lowers the cost of investment by a small amount in the *second* period, and only in the second period. This policy is fully anticipated in the first period. Analyze the impact of this policy on the equilibrium and the welfare of agents.

2. Suppose that in addition (in each state) one more agent with an option to invest (and discount factor δ), and a belief (probability of the good state) $\underline{\mu} < c$. How is your previous answer modified?

EXERCISE 5.10. (delay with a large number of agents)

Consider the simple model of this chapter with N players in the good state and one player in the bad state. Solve for the symmetric equilibrium. Show that the probability of a herd with no investment converges to $\pi^* > 0$ if $N \rightarrow \infty$. Analyze the probability of investment by any agent as $N \rightarrow \infty$.

EXERCISE 5.11.

Show that there is strategic substitutability at an equilibrium with the strategy μ^* if

$$\mu^* > \frac{\sqrt{c/(1-c)}}{1+\sqrt{c/(1-c)}}.$$

EXERCISE 5.12.

In the model of Section **??**, assume $n \rightarrow \infty$ and the period length converges to zero, ($\delta \rightarrow 1$), at a rate slower than n. Assume that not all agents invest in the equilibrium (there is no rush).

1. Determine the payoff of an agent with private belief μ as a function of μ, $\bar{\mu}$ and c.
2. Is there a measurement of the externality of information which an agent with private belief μ receives from the agents in the upper tail of the distribution of beliefs?