Ec 517

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Solutions to exercises on delays

Ex 1:

$$P(\theta = 1|x = 0) = \frac{P(x = 0|\theta = 1)P(\theta = 1)}{P(x = 0)} = \frac{1 - z)\mu}{(1 - z)\mu + 1 - \mu}.$$

The value of z is solution of

$$\mu - c = \delta\mu z(1 - c).$$

We can replace z in the first equation by its value in the second equation.

When δ tends to 1 the value of z decreases. We can and we should answer the question without the previous answer, using the arbitrage equation (1.2) in the text. At the limit, the value of $P(\theta = 1|x = 0$ tends to c. Verify your intuition.

Ex 2: The solution is $\mu - c = \delta\mu z(1 - \gamma)(1 - c)$. The value of z is endogenous to γ. The product $z(1 - \gamma)$ is constant. When γ iincreases, the value of z increases to compensate for the loss of "visibility" of investment. Of course, z is bounded by 1. Therefore, there is a maximum value γ^* for which the argument is true.

Ex 3: The arbitrage equation becomes $\mu - c = \delta\mu z(1 - c + \tau)$. The subsidy increases the incentive to delay and therefore reduces z. The welfare of agent(s) is *unchanged* because it is equal to $\mu - c$ and independent of τ.

Ex 4: In the equilibrium, $\mu - c = \delta\mu(1 - (1 - z)^{N-1}(1 - c)$. When N increases, the term $(1 - z)^{N-1}$ is constant. In the good state, the probability of a herd with no investment (following no investment in the first period) is $(1 - z)^N = (1 - z)(1 - z)^{N-1}$, which is constant. (Here we have the power N because we have the point of view of an outside observer and not the point of view of a player who decides to delay and see what other players (if any) do.

The other exercises are related to material that was not covered in the class.

Assignment 3 with answers

There are two projects, X and Y. The payoff of X is equal to 0 or $2+a$, with equal prior probabilities. a is small and positive. The payoff of Y is equal to $1-a$ or $1+a$ with equal prior probabilities. (The projects' return are not random and they are independent). As in the BHW model, agents, in an exogenous sequence, have the choice between two projects. Each individual has a SBS s on the project X such that $P(s=1|x=2+a)=q$ with $p>1/2$. Agents have a symmetric binary signal, H or L, of precision q on the payoff of action 1. There is no signal on the project Y. By assumption a is positive and sufficiently small (whatever will be needed to answer questions).

1. Assume that the outcome of a project that is implemented cannot be observed by subsequent individuals. Show that if the first agent has a signal H, a cascade starts in period 2. Show that if the first two signals are L and L, then a cascade starts in period 3.

2. Compare the previous model with the standard BHW model with two states for a project, payoffs 1 and 0, a SBS with precision p and a prior μ on the state 1 such that $1/2<\mu<p$. Determine the probabilities of an eventual cascade in project X and the probability of an eventual Y cascade.

3. Suppose that the first signal is H. Determine in that case, after the signal H becomes public, the long-run expected welfare of individuals, that is the average expected value of the payoff of X for agents.

4. Answer the same question after the signal L. Call that welfare U_0.

5. Using your previous answers, determine the *ex ante* long-run expected welfare of individuals.

6. Assume now that individuals can observe the payoff of project Y at the end of the first period in which it is undertaken.

   (a) Assume the first signal is L and the output of Y is high $(1+a)$. Determine the long-run welfare conditional on that event.

   (b) Assume that the first signal is L and the output of Y is low $(1-a)$. Show that the impact of the private signals on the individual actions are the same from that point on, as in the case where the output of Y could not be observed. Using your answer in question 2 to determine the expected long-run welfare from that point on. (Be careful on the welfare in a Y cascade).

7. Using your previous answers, determine the long-run expected welfare of individuals and show that it is smaller than in the previous question.

8. Assume now the payoff of Y is observable with one period delay. Analyze the model and show that the long-run social welfare is higher than under the condition of the previous question.

Answers

1. With no signal, project X has a higher payoff that project Y. Assuming the precision of the signal p is sufficiently high (otherwise a cascade on X starts right away), the action in period 1 reveals the signal H. In period, the impact of a signal L (that reduces the attractiveness of X) on a belief is nullified by the signal H of the first period. The belief is the same as with no signal and X has a higher payoff. A cascade starts after the first period.

2. Any sequence of signals with H and L generates a sequence of actions X or Y that is the same as in the BHW model when we replace H and L by the realization 1 and 0 of the private signals and the actions X and Y by the actions 1 and 0, respectively.

   We use the equivalence and these probabilities have been computed before in the semester. A cascade on X takes place when the difference between the number of H and L signals is equal to 1. Assume that the state is "high" (meaning here that X has a positive payoff). Let π_0 the probability of a X cascade when the difference between the total H and L signals is 0, and π_1 that probability when the difference is -1. By the previous calculations,

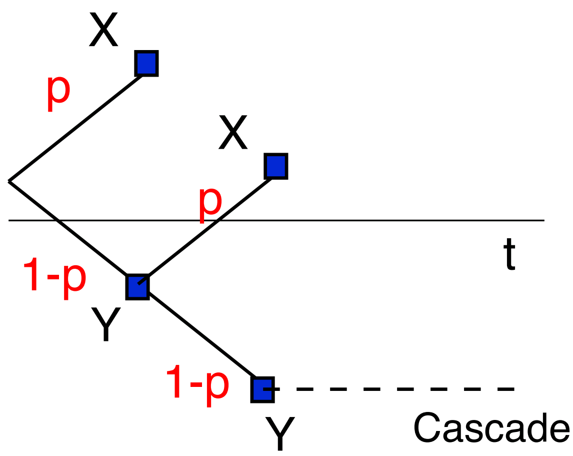
$$\pi_0 = p + (1-p)\pi_1, \qquad \pi_1 = p\pi_0, \qquad \pi_0 = \frac{p}{1-p(1-p)}.$$

   If the true state is low (payoff of X equal to 0), then these probabilities are

$$\pi_0' = p + (1-p)\pi_1', \qquad \pi_1' = p\pi_0', \qquad \pi_0' = \frac{1-p}{1-p(1-p)}.$$

   One can for the probabilities. Ex ante, both states are equally likely and the probability of a X cascade is $\pi = (\pi_0 + \pi_0')/2 = 1/2(1-p(1-p))$. The probability of a Y cascade is 1 minus the previous probability (since we always end in one of the two cascades).

3. In a X cascade the public belief (probability of positive payoff for X) is equal to p (because the prior belief is 1/2 and the precision of the signal is p. In a X cascade, private beliefs are different for the individuals with



The probabilities are in the good state. In the bad state, p and $1-p$ are switched.

Figure 1: Trajectories

H and L signals. However, the expected value of the belief after receiving a signal is also equal to the belief p before getting the signal (because of the martingale property). Therefore, the long-run expected payoff in that cascade is $p(2+a)$.

4. The expected payoff in a Y cascade is 1. If π is the probability to end in a X cascade, the *ex ante* expected payoff in the long run is therefore $U = 1 + \pi(p(2+a) - 1)$ or

$$U = \frac{1 + ap + 2p^2}{2(1 - p + p^2)}.$$

Note that if $a \approx 0$, $U \approx 1 + \pi(2p - 1)$.

5. If the first signal is H, the project is X and there is a cascade with payoff $p(2+a)$. The probability of that event is $1/2$. The contribution to welfare by that "branch" is therefore $p(1 + a/2)$.

   If the first signal is L (with ex ante probability $1/2$), then Y is undertaken. If its payoff is equal to $1+a$, there is a cascade at Y after that. The expected long-run payoff in that case is $(1+a)/2$. The probability of this event is $1/2$ and the contribution to total welfare is $(1+a)/4$.

   If its payoff is $1-a$, then in the second period, a bad signal leads to a Y cascade with payoff $1-a$ because at the point the expected payoff of X is lower than $1-a$: $(2+a)(1-p)^2/((1-p)^2+p^2) < 1-a$ if a is sufficiently small (using $p > 1/2$). That event has probability $((1-p)(1-p)+p^2)/2$ and the contribution to welfare is $((1-p)^2+p^2)(1-a)/2$.

A good signal in period 2 leads to X in period 2. After period 2, the dynamics are the same as in the case with no payoff observation with the same first belief μ. There is one difference however: in the Y cascade, the payoff is now $1 - a$ instead of 1 because the (low) output of Y has been revealed. The probability of that event is $(1/2)(1/2)((1-p)p + p(1-p)) = p(1-p)/2$. The contribution to welfare from that point on is $(p(1-p)/2)(\pi p(2+a) + (1-\pi)(1-a))$.

Adding up all the terms and taking a vanishingly small (we want to prove the result for a small), the long-run expected welfare is

$$V = p + \frac{1}{4} + \frac{(1-p)^2 + p^2}{2} + \frac{p(1-p)}{2}(2\pi p + (1-\pi)).$$

If $V(0) = p + \frac{1}{4} + (1-p)^2 + p^2 + 2\pi p + 1 - \pi < 1 + \pi(2p-1)$.

$V < U$ is equivalent to $(1+a)/2 < U$, which is true for sufficiently small a.

How come more information leads to a lower welfare gain in the long-run? More information here is more information for agent 2 who chooses to stick to project Y even if he has a good signal. By doing so, he deprives other agents, after period 2 of the information about his signal. That feature is at the core of the cascade property: agents make a selfish decision without regard to the information that is conveyed to others by their action. In the long-run, the average welfare of agents is lower.

6. In this case, if agent 1 chooses Y and agent 2 has a signal H, he undertakes X and thus reveals to others his signal.

    (a) If Y is revealed to pay $1 - a$, then the dynamics that start in period 3 are the same as with no output observation, with the same initial belief μ on the good state for project Y.