Cascades and Herds

Ec 717

October 28, 2024

The 2x2x2 Model

- Recall the fundamental model of social learning:
  - Agents learn about a state of nature ω.
  - They have private information, which is modeled as private signals $s_i$depending on ω.
  - They take payoff maximizing actions which reveal some information about s_i.

The 2x2x2 Model

- Recall the fundamental model of social learning:
  - Agents learn about a state of nature ω.
  - They have private information, which is modeled as private signals $s_i$depending on ω.
  - They take payoff maximizing actions which reveal some information about s_i.

-
  - What is the minimum number of states needed for a SL model ?
  - What is the minimum number of signal values in the canonical SL model?
  - What is the minimum number of possible actions?

The 2x2x2 Model

- Recall the fundamental model of social learning:
  - Agents learn about a state of nature ω.
  - They have private information, which is modeled as private signals $s_i$depending on ω.
  - They take payoff maximizing actions which reveal some information about s_i.
- - What is the minimum number of states needed for a SL model ?
  - What is the minimum number of signal values in the canonical SL model?
  - What is the minimum number of possible actions?
- The model
  - 2 states $\omega \in \{0, 1\}$.
  - 2 signal values $P(s = \omega) = q \in (0, 1)$. (symmetric binary signal, SBS)
  - $x \in 0, 1$, payoff $U = x(\omega - c)$, $\qquad c \in (0, 1.)$

Social learning in the 2x2x2 Model

- $\mu_t = P(\omega = 1 | h_t), \quad \tilde{\mu}_t = P(\omega = 1 | h_t, s_t)$. ($h_t$ history of past actions– CK).

- Bayes' rule is unwieldy. If possible, use the Log-likelihood ratio LR :
  For an agent with private signal $s_t = 1$,

$$\tilde{\ell}_t = \frac{\tilde{\mu}_t}{1 - \tilde{\mu}_t} = \frac{P(\omega_1 | s = 1)}{P(\omega_0 | s = 1)} = \frac{\frac{P(s = 1 | \omega_1) P(\omega_1)}{P(s)}}{\frac{P(s = 1 | \omega_0) P(\omega_0)}{P(s)}} = \ell_t \frac{P(s = 1 | \omega_1)}{P(s = 1 | \omega_0)} = \ell_t \frac{q}{1 - q}.$$

Social learning in the 2x2x2 Model

- $\mu_t = P(\omega = 1|h_t)$, $\tilde{\mu}_t = P(\omega = 1|h_t, s_t)$. ($h_t$ history of past actions– CK).

- Bayes' rule is unwieldy. If possible, use the Log-likelihood ratio LR :
  For an agent with private signal $s_t = 1$,

$$\tilde{\ell}_t = \frac{\tilde{\mu}_t}{1-\tilde{\mu}_t} = \frac{P(\omega_1|s=1)}{P(\omega_0|s=1)} = \frac{\frac{P(s=1|\omega_1)P(\omega_1)}{P(s)}}{\frac{P(s=1|\omega_0)P(\omega_0)}{P(s)}} = \ell_t \frac{P(s=1|\omega_1)}{P(s=1|\omega_0)} = \ell_t \frac{q}{1-q}.$$

- To have a linear formulation of Bayes' rule, use the Log-likelihood ratio LLR:
  $\lambda_t = Log(\frac{\mu_t}{1-\mu_t})$: $\tilde{\lambda}_t = \lambda_t \begin{cases} +a \text{ if } s_t = 1, \\ -a \text{ if } s_t = 0. \end{cases}$, with $a = Log\frac{q}{1-q}$.

Social learning in the 2x2x2 Model

- $\mu_t = P(\omega = 1|h_t)$, $\tilde{\mu}_t = P(\omega = 1|h_t, s_t)$. ($h_t$ history of past actions– CK).

- Bayes' rule is unwieldy. If possible, use the Log-likelihood ratio LR :
  For an agent with private signal $s_t = 1$,

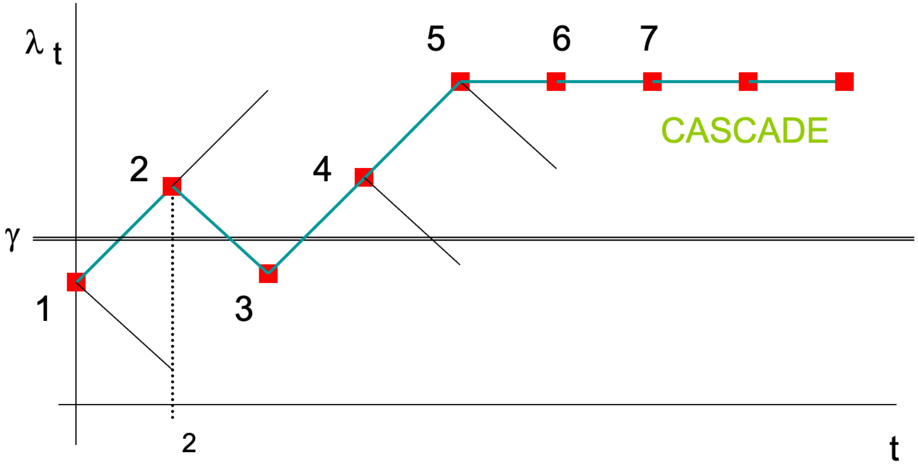
$$\tilde{\ell}_t = \frac{\tilde{\mu}_t}{1-\tilde{\mu}_t} = \frac{P(\omega_1|s=1)}{P(\omega_0|s=1)} = \frac{\frac{P(s=1|\omega_1)P(\omega_1)}{P(s)}}{\frac{P(s=1|\omega_0)P(\omega_0)}{P(s)}} = \ell_t \frac{P(s=1|\omega_1)}{P(s=1|\omega_0)} = \ell_t \frac{q}{1-q}.$$

- To have a linear formulation of Bayes' rule, use the Log-likelihood ratio LLR:

$$\lambda_t = Log(\frac{\mu_t}{1-\mu_t}): \quad \tilde{\lambda}_t = \lambda_t \begin{cases} +a \text{ if } s_t = 1, \\ -a \text{ if } s_t = 0. \end{cases}, \text{ with } a = Log\frac{q}{1-q}.$$

- Optimal action: $x_t = 1$ iff $\tilde{\mu}_t > c$, equivalent to $\tilde{\lambda}_t > \gamma = Log(c/(1-c))$.

Cascade representation



Properties of cascades

- Fast convergence.
- Limit belief can be wrong

Continuum of beliefs

- Individual beliefs (instead of signals) are distributed according to the *c.d.f.* $F^{\omega}(\mu)$
  First order stochastic dominance: if $\omega_1 > \omega_0$, $F^{\omega_0}(s) > F^{\omega_1}(s)$

Observations

States of Nature

	$x_t = 1$	$x_t = 0$
$\omega = \omega_1$	$1 - F_t^{\omega_1}(\gamma)$	$F_t^{\omega_1}(\gamma)$
$\omega = \omega_0$	$1 - F_t^{\omega_0}(\gamma)$	$F_t^{\omega_0}(\gamma)$

with $\gamma = Log\Big(\frac{c}{1-c}\Big)$.

- Social learning

$$\lambda_{t+1} = \lambda_t + \nu_t, \text{ with } \nu_t = Log\Big(\frac{P(x_t|\omega_1)}{P(x_t|\omega_0)}\Big). \quad (1)$$

Signals and private beliefs

- 2 states $\{\omega_0, \omega_1\}$ with equal probabilities. Private signals with distributions, cdf $F^{\omega}(s)$.

- Call p the probability of ω_1 for an agent with signal s. When an agent receives the signal s, by Bayes' rule, the likelihood ratio between the two states is

$$\frac{F'^{\omega_1}(s)}{F'^{\omega_0}(s)} = \frac{p}{1-p}. \qquad (2)$$

  In state ω, F^{ω} generates a distribution of signals, for which the belief is given by the previous equation.

- The reverse applies: belief is a signal. The distribution of this signals must satisfy the previous equation.

Representation of social learning with bounded private beliefs

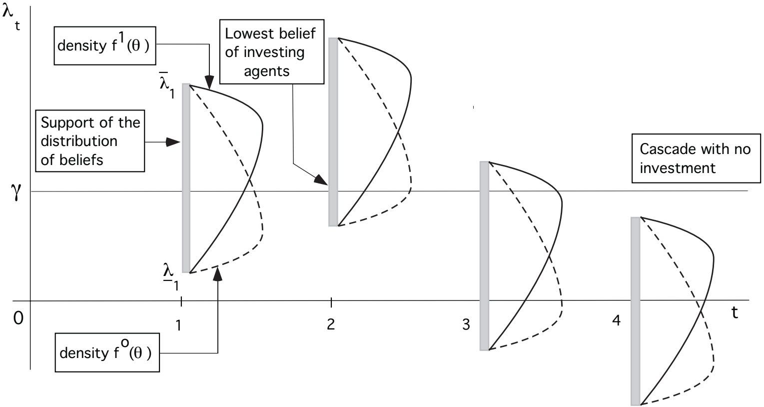
- $x_t = 1$ iff $\lambda_t + Log(\frac{s_t}{1-s_t}) > \gamma$, or $s_t > s_t^*$.

- After the observation of x_t, $\lambda_{t+1} = \lambda_t + Log(\dfrac{1 - F^1(s^*t)}{1 - F^0(s^*t)})$.

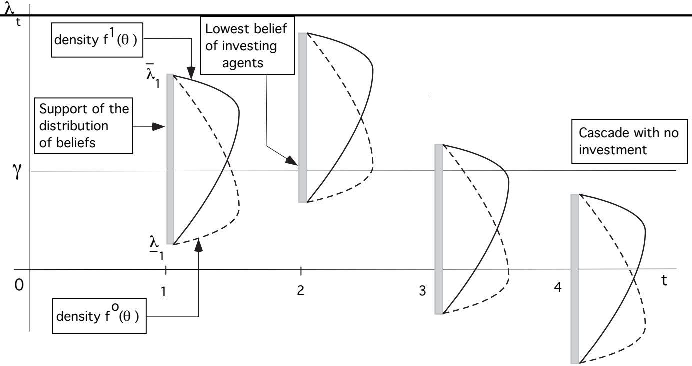
Representation of social learning with bounded private beliefs

- $x_t = 1$ iff $\lambda_t + Log(\frac{s_t}{1-s_t}) > \gamma$, or $s_t > s_t^*$.

- After the observation of x_t, $\lambda_{t+1} = \lambda_t + Log(\dfrac{1 - F^1(s^* t)}{1 - F^0(s^* t)})$.



Representation of social learning with bounded private beliefs



- A cascade takes place only when the interior of the support of private beliefs does not contain γ
- If the support is infinite, there is no cascade.

Cascades and bounded private beliefs

- Previous model: distribution of private beliefs with cdf $F^\omega(s)$ and density $f^\omega(s)$.
- Assume that $f(s) > 0$ for $s \in (a, b)$ with $0 < a < b < 1$ and $f(s) = 0$ otherwise. Private beliefs are bounded.
- Payoff: agent choose the state that is more likely (equivalent to $c = 1/2$).
- Update: $\frac{\tilde{\mu}}{1-\tilde{\mu}} = \frac{\mu}{1-\mu} \frac{}{1-s}$. Invest $(x = 1)$ if $s > 1 - \mu$.
- Cascade set with investment $a > 1 - \mu$ which is equivalent to $\mu > 1 - a$.
- Cascade set with no investment $\mu < 1 - b$.
- MCT $\Longrightarrow \mu_t \to \mu^*$.
- The limit μ^* cannot be in the interval $(1 - b, 1 - a)$.
- A cascade occurs with probability one.

Unbounded support of private beliefs

- There is no cascade.
- The belief converges to the truth: if $Lim P_t(\omega|\omega) = 1$.
- Argument: $Lim P_t(\omega|\omega) \neq 0$.

$$\frac{P(\omega_1|h_t)}{P(\omega_0|h_t)} = \frac{P(h_t|\omega_1)}{P(h_t|\omega_0)}.$$

Rational learning cannot be totally wrong, asymptotically

Proposition

Let $\Omega = \{\omega, \ldots, \omega_K\}$ *be the finite set of states of nature,* $\mu_t = \{\mu_t^1, \ldots, \mu_t^K\}$ *the probability assessment of a Bayesian agent in period* t*, and* $\mu_1^1 > 0$ *where* ω_1 *is the true state. Then for any* $\epsilon > 0$,

$$P(\mu_t < \epsilon) < \epsilon/\mu_1^1.$$

If $\bar{\mu}^1$ *is the limit value of* μ_t^1, $P(\bar{\mu}^1 = 0) = 0$.

Under Bayesian learning, if the subjective distribution on ω converges to a point, it must converge to the truth.

Proof

For any history h_t, $P(h_t|\omega = \omega_1) = P(\omega = \omega_1|h_t)\dfrac{P(h_t)}{P(\omega = \omega_1)}$.

Let H_t be the set of histories h_t such that $\mu_t^1 < \epsilon$. By definition,

$$P(h_t \in H_t|\omega = \omega_1) < \epsilon\frac{P(h_t \in H_t)}{P(\omega = \omega_1)} \leq \epsilon\frac{1}{P(\omega = \omega_1)}$$

Q.E.D.

Crashes and booms

- Model: Two states ω_0 and ω_1, $s_t = \omega + \epsilon_t$ with ω and ϵ Gaussian; $x_t \in \{0, 1\}$.
- Belief (LLR) of agent with signal s $$\lambda(s) = \lambda_t + \frac{\omega_1 - \omega_0}{\sigma_\epsilon^2}\Big(s - \frac{\omega_0 + \omega_1}{2}\Big).$$
- Cutoff for investment ($x_t = 1$): $$s > s^*(\lambda_t) = \frac{\omega_0 + \omega_1}{2} - \frac{\sigma_\epsilon^2}{\omega_1 - \omega_0}\lambda_t.$$
- Model with one agent. Discussion
- Model with a continuum of agent in each period: $X_t = 1 - F(s^*(\lambda_t) - \theta; \sigma_\epsilon)$.
- Observed aggregate activity $$Y_t = 1 - F(s^*(\lambda_t) - \theta; \sigma_\epsilon) + \eta_t.$$

Crashes and booms with a continuum of agents

- Observed aggregate activity $\quad Y_t = 1 - F(s^*(\lambda_t) - \theta; \sigma_\epsilon) + \eta_t.$